The Electromagnetic Fields Outside a Wire That Carries a Linearly Rising Current

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(November 28, 1996)

1 Problem

A neutral wire along the \( z \)-axis carries current \( I \) that varies with time \( t \) according to

\[
I(t) = \begin{cases} 
0 & t \leq 0, \\
\alpha t & t > 0, \quad \alpha \text{ is a constant.} 
\end{cases}
\]  

(1)

Deduce the time-dependence of the electric and magnetic fields, \( E \) and \( B \), observed at a point \((r, \theta = 0, z = 0)\) in a cylindrical coordinate system about the wire. Use your expressions to discuss the fields in the two limiting cases that \( ct \gg r \) and \( ct = r + \epsilon \), where \( c \) is the speed of light and \( \epsilon \ll r \).

2 Solution

We follow the familiar method of first calculating the retarded potentials and then taking derivatives to find the fields. The retarded scalar and vector potentials \( V \) and \( A \) are given by

\[
V(x, t) = \int \frac{\rho(x', t - R/c) \, d^3x'}{R}, \quad \text{and} \quad A(x, t) = \frac{1}{c} \int \frac{J(x', t - R/c) \, d^3x'}{R},
\]  

(2)

in Gaussian units, where \( \rho \) and \( J \) are the charge and current densities, respectively, and \( R = |x - x'| \).

In the present case, we assume that the wire remains neutral when the current flows.\(^1\) Then, the scalar potential vanishes. For the vector potential, we see that only the component \( A_z \) will be nonzero. Also, \( J 
\]  

(3)

Altogether,

\[
A_z(r, 0, 0, t) = \frac{\alpha}{c} \int_{-z_0}^{z_0} \left( \frac{t}{\sqrt{r^2 + z^2}} - \frac{1}{c} \right) \, dz = \frac{\alpha}{c} \left( t \ln \frac{ct + z_0}{ct - z_0} - \frac{2z_0}{c} \right) = \frac{2\alpha}{c} \left( t \ln \frac{z_0 + ct}{r} - \frac{z_0}{c} \right).
\]  

(4)

\(^1\)We ignore the small departure from neutrality that scales as \( v^2/c^2 \) where \( v \approx 1 \text{ cm/sec} \) is the velocity of the conduction electrons.
[The two forms in eq. (4) arise depending on whether or not one notices that the integrand is even in \( z \).]

The magnetic field is obtained via \( \mathbf{B} = \nabla \times \mathbf{A} \). Since only \( A_z \) is nonzero, the only nonzero component of \( \mathbf{B} \) is (noting that \( \partial z_0/\partial r = -r/z_0 \))

\[
B_\phi = -\frac{\partial A_z}{\partial r} = \frac{2\alpha z_0}{cr}.
\] (5)

The only nonzero component of the electric field is

\[
E_z = -\frac{1}{c} \frac{\partial A_z}{\partial t} = -\frac{2\alpha}{c^2} \ln \frac{z_0 + ct}{r}.
\] (6)

For long times, \( ct \gg r \Rightarrow z_0 \approx ct \), and the fields become

\[
B_\phi \approx \frac{2\alpha t}{cr} = \frac{2I(t)}{cr} = B_0(t), \quad E_z \approx -\frac{2\alpha}{c^2} \ln \frac{2ct}{r} = -B_0 \frac{r}{ct} \ln \frac{2ct}{r} \ll B_0,
\] (7)

where \( B_0(t) = 2I(t)/cr \) is the instantaneous magnetic field corresponding to current \( I(t) \). That is, we recover the magnetostatic limit at large times.

For short times, \( ct = r + \epsilon \) with \( \epsilon \ll r \), after the fields first become nonzero we have

\[
z_0 = \sqrt{2r\epsilon + \epsilon^2} \approx \sqrt{2r\epsilon},
\] (8)

so

\[
B_\phi \approx \frac{2\alpha}{c^2} \sqrt{\frac{2\epsilon}{r}}, \quad \text{and} \quad E_z \approx -\frac{2\alpha}{c^2} \ln \frac{r + \epsilon + \sqrt{2r\epsilon}}{r} \approx -\frac{2\alpha}{c^2} \sqrt{\frac{2\epsilon}{r}} = -B_\phi.
\] (9)

In this regime, the fields have the character of radiation, with \( \mathbf{E} \) and \( \mathbf{B} \) of equal magnitude, mutually orthogonal, and both orthogonal to the line of sight to the closest point on the wire. (Because of the cylindrical geometry, the radiation fields do not have \( 1/r \) dependence, which holds instead for cylindrical static fields.)

In sum, the fields build up from zero only after time \( ct = r \). The initial fields propagate outwards at the speed of light and have the character of cylindrical waves. But at a fixed \( r \), the electric field dies out with time, and the magnetic field approaches the instantaneous magnetostatic field due to the current in the wire.\(^2,3\)

\(^2\)Of possible amusement is a direct calculation of the vector potential for the case of a constant current \( I_0 \). First, from Ampere’s law we know that \( B_\phi = 2I_0/cr = -\partial A_z/\partial r \), so we have that

\[
A_z = -\frac{2I_0}{c} \ln r + \text{const.}
\] (10)

If we use the integral form for the vector potential we have

\[
A_z(r, 0, 0) = \frac{1}{c} \int_{-\infty}^{\infty} \frac{I_0 dz}{\sqrt{r^2 + z^2}} = \frac{2I_0}{c} \int_0^\infty \frac{dz}{\sqrt{r^2 + z^2}} = \frac{2I_0}{c} \ln r + \lim_{z \to \infty} \ln(z + \sqrt{z^2 + r^2}).
\] (11)

Only by ignoring the large constant, which does not depend on \( r \) for a long wire, do we recover the “elementary” result.

\(^3\)For the related example of the fields associated with a linearly rising current in a solenoid, see [1].
References