1 Problem

What are the frequencies of small transverse oscillations in a vertical plane of an inelastic string of length \( l \) and linear mass density \( \lambda \) whose upper point is fixed at a point in a uniform gravitational field of strength \( g \)?

Estimate the lowest oscillation frequency via Rayleigh’s energy method using, say, a trial waveform \( s(y) = lp - y^p \) for \( y \) measured upwards from the lower end of the string, where \( p \) is to be optimized.

2 Solution

The equilibrium state of the string is, of course, that it hangs vertically, with its lower end at \( y = 0 \) and its upper end at \( y = l \).

The tension in the string is

\[
T(y) = \lambda gy. \tag{1}
\]

The equation of motion for a transverse displacement \( s(y, t) \) in a vertical plane of a segment \( dy \) of the string is

\[
\lambda dx \ddot{s} = T(y + dy)s'(y + dy) - T(y)s'(y) = \frac{\partial Ts'}{\partial y} dy = \lambda g \frac{\partial (ys')}{\partial y} dy. \tag{2}
\]

For oscillations at angular frequency \( \omega \) of the form \( s(y, t) = s(y)e^{i\omega t} \), eq. (2) reduces to

\[
\frac{d(ys')}{dy} + \frac{\omega^2}{g} s = y \frac{d^2 s}{dy^2} + \frac{ds}{dy} + \frac{\omega^2}{g} s = 0. \tag{3}
\]

This is a form of Bessel’s equation of order zero, as can be seen using the substitution \( x = \sqrt{\frac{y}{g}} \), with which eq. (3) becomes

\[
x^2 \frac{d^2 s}{dx^2} + x \frac{ds}{dx} + \frac{4\omega^2}{g} x^2 s = 0, \tag{4}\]

whose solutions are

\[
s(y) = s_0 J_0(2\omega \sqrt{y/g}). \tag{5}\]

The condition that \( s(y = l) = 0 \) determine a series of frequencies of small oscillation,

\[
2\omega \sqrt{\frac{L}{g}} = 2.405, \ 5.520, \ 8.654, \ldots, \tag{6}\]
or

$$\omega = 1.202 \sqrt{\frac{g}{l}}, \ 2.760 \sqrt{\frac{g}{l}}, \ 4.318 \sqrt{\frac{g}{l}}, \ldots \quad (7)$$

Rayleigh notes that for a springlike system, \(\langle KE \rangle = \langle PE \rangle\) (virial theorem), so that a trial waveform with parameter \(p\) can be used to estimate the frequency \(\omega(p)\) using this constraint. Then the lowest frequency is obtained by minimizing \(\omega(p)\) with respect to the parameter \(p\).

We consider the form

$$s(y, t) = (l^p - y^p)e^{i\omega t}, \quad (8)$$

for which the time-average kinetic energy is

$$\langle KE \rangle = \left\langle \int_0^l \frac{\lambda s^2}{2} dy \right\rangle = \frac{\lambda \omega^2}{4} \int_0^l (l^p - y^p)^2 dy = \frac{\lambda \omega^2}{4} l^{2p+1} \left(1 - \frac{2}{p+1} + \frac{1}{2p+1}\right)$$

$$= \frac{\lambda \omega^2}{4} l^{2p+1} \frac{2p^2}{(p+1)(2p+1)}, \quad (9)$$

and the time-average potential energy (= work done in stretching the string) is

$$\langle PE \rangle = \left\langle \int_0^l T(\sqrt{1 + s'^2} - 1) dy \right\rangle \approx \left\langle \int_0^l \frac{T s'^2}{2} dy \right\rangle = \frac{\lambda g}{4} \int_0^l y(-py^{p-1})^2 dy = \frac{\lambda g}{4} l^{2p+1} \frac{2p^2}{2p+1}. \quad (10)$$

Equating the kinetic and potential energies, we have that

$$\omega^2(p) = \frac{g (p + 1)(2p + 1)}{l}. \quad (11)$$

The minimum frequency occurs for \(p = 1/\sqrt{2}\), which implies that its value is

$$\omega \approx \sqrt{\frac{\sqrt{2}}{l} \frac{1.707 \cdot 2.414}{2.828}} = 1.207 \sqrt{\frac{g}{l}}, \quad (12)$$

which compares well with the “exact” value of 1.202 \(\sqrt{g/l}\).

For additional discussion, see A.B. Western, Demonstration for observing \(J_0(x)\) on a resonant rotating vertical chain, Am. J. Phys. 48, 54 (1980),