Helmholtz and the Velocity Gauge

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(March 31, 2018; updated November 14, 2018)

In 1870, Helmholtz published an article [38] (see also [39]) that was one of the first to cite Maxwell's great paper of 1865 [33], and which also considered the electrodynamics of F. Neumann [17, 21], Weber [18, 22, 26, 27], Kirchhoff [28, 29] and C. Neumann [36, 37].[1] Here, we emphasize that Helmholtz used the electromagnetic potentials φ and $\mathbf{A} = (U, V, W)$ as a way of comparing these different visions, noting that these potentials can be subject to the (gauge) condition,[2] eq. (3^a), p. 80 of [38],

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{k}{c}\frac{\partial\varphi}{\partial t} \qquad \text{(Helmholtz)}, \tag{1}$$

in Gaussian units, where c is a constant determined from electro- and magnetostatic experiments [27] equal to the speed of light in vacuum, and k is arbitrary.[3,4] The gauge (1) considered by Helmholtz in 1870, well before the notion of a gauge was appreciated, is now called the α-Lorentz gauge [61, 64, 74] or the velocity gauge [70, 71, 78].

The electrodynamics of the Neumanns, Weber and Kirchhoff did not include concepts of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$,[5] which are related to the potentials in Maxwell's electrodynamics by

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{5}$$

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[1] F. Neumann, who introduced the concepts of self and mutual inductance in [17, 21], was the father of C. Neumann, who eventually became an advocate of Maxwell's theory [47].

[2] Helmholtz might have been alerted to the arbitrariness of the potentials by Maxwell's discussion in secs. 98-99, p. 500, of [33], where the transformations $\mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\chi$ and $\Psi' = \Psi + \partial\chi/\partial t$ (= Maxwell's φ) were presented in eqs. (75) and (77), with $\mathbf{A} = (F, G, H)$.

[3] Helmholtz introduced his parameter k in an expression for the interaction energy U_{12} of two closed circuits that carry electric currents I_1 and I_2, writing in his eq. (1),

$$U_{12} = \oint_1 \frac{I_1\, d\mathbf{l}_1}{c} \cdot \oint_2 I_2 \frac{(1+k)\, d\mathbf{l}_2 + (1-k)\, \hat{\mathbf{r}}\, (\hat{\mathbf{r}} \cdot d\mathbf{l}_2)}{2cr} = \oint_1 \frac{I_1\, d\mathbf{l}_1 \cdot \mathbf{A}_2}{c} = M_{12} I_1 I_2, \tag{2}$$

where $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, and the vector potential $\mathbf{A}$ was given by his eq. (1^a),

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{(1+k)\, \mathbf{J}(\mathbf{x}', t) + (1-k)\, \hat{\mathbf{r}}\, (\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{x}', t))}{2cr}\, d\mathrm{Vol}', \tag{3}$$

in terms of the current density $\mathbf{J}$ rather than current I; the mutual inductance M_{12} can be written as

$$M_{12} = \oint_1 \oint_2 \cdot \frac{(1+k)\, d\mathbf{l}_1 \cdot d\mathbf{l}_2 + (1-k)\, (\hat{\mathbf{r}} \cdot d\mathbf{l}_1)\, (\hat{\mathbf{r}} \cdot d\mathbf{l}_2)}{2c^2 r}. \tag{4}$$

The case of $k = 1$ in our eqs. (2)-(3) is consistent with sec. 11 of F. Neumann's paper [17], and while Helmholtz claimed that $k = -1$ corresponds to the theory of Weber [18, 22] (see pp. 100-102 of [67]). Our eq. (4) appeared as eq. (1) of [68].

[4] Discussions of the rest of Helmholtz' paper include chap. 4 of [52], p. 446 of [55], [57], pp. 160-166 of [58], pp. 297-301 of [62], sec. 3 of [66], and sec. IIB of [70]. See also [54, 60, 63, 65, 73].

[5] Kirchhoff, p. 199 of [28], expressed Ohm's law [14] in a form equivalent to $\mathbf{J} = \sigma(-\boldsymbol{\nabla}\varphi - \partial\mathbf{A}/\partial ct)$, where σ is the electrical conductivity, without relating this to an electric field $\mathbf{E}$, such that $\mathbf{J} = \sigma\mathbf{E}$.

and also obey the four Maxwell equations,[6]

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \tag{6}$$

Using eq. (5) in (6), we obtain second-order differential equations for the potentials,

$$\nabla^2\varphi + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) = -4\pi\rho, \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J}. \tag{7}$$

Then, using Helmholtz' condition (1), the wave equations for the potentials become (in Maxwell's theory),

$$\nabla^2\varphi - \frac{k}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho, \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \frac{1-k}{c^2}\frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla}\varphi) = -\frac{4\pi}{c}\mathbf{J} \qquad \text{(Helmholtz)}. \tag{8}$$

Thus, in the Helmholtz gauge (1) (and in Maxwell's theory), the scalar potential propagates with speed $c/\sqrt{k}$ where k is arbitrary (even negative!), while the vector potential has a piece that propagates at speed c, and another piece (related to $\boldsymbol{\nabla}\varphi$) that propagates at speed $c/\sqrt{k}$. Of course, the fields $\mathbf{E}$ and $\mathbf{B}$ propagate at speed c, as follows directly from the Maxwell equations (6), which can be combined into the wave equations,

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 4\pi\boldsymbol{\nabla}\rho + \frac{4\pi}{c^2}\frac{\partial\mathbf{J}}{\partial t}, \qquad \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{\nabla}\times\mathbf{J}. \tag{9}$$

When using eq. (1) to compute the fields from the potentials, the piece of $\mathbf{A}$ that propagates at speed $c/\sqrt{k}$ does not affect the propagation of $\mathbf{B}$, and its effect on the propagation of $\mathbf{E}$ is canceled by the contribution from $-\boldsymbol{\nabla}\varphi$.

Maxwell favored use of the Coulomb gauge, $k = 0$ in eqs. (1) and (8),[7] as was noted by Helmholtz in [38, 39].[8]

Helmholtz did not cite Lorenz' paper of 1867 [35] that corresponded to use of $k = 1$ in our eq. (1) (shortly before eq. (8) of [35]), in which gauge both the scalar and vector potentials propagate with velocity c. Instead, he claimed that F. Neumann assumed $k = 1$ (in our eqs. (2)-(3)), as remarked in footnote 3 above.

Already in 1857, Kirchhoff [28, 29] had used the velocity gauge, $k = -1$ in our eq. (1), which work was cited by Helmholtz but not identified with any choice of k.[9,10] However, Helmholtz attributed the use of $k = -1$ in our eqs. (2)-(3) to C. Neumann and Weber.

[6]In [33], the first and fourth Maxwell equations were expressed in terms of $\mathbf{D} = \epsilon\mathbf{E}$, ρ_{free}, $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{J}_{\text{free}}$, where ϵ and μ are the relative permittivity and permeability.

[7]The Maxwellian vector potential for magnetostatics in the Coulomb gauge corresponds to $k = 1$ in our eqs. (2)-(3).

[8]Surprisingly, Maxwell did not seem bothered by the "instantaneous" propagation of the scalar potential in the Coulomb gauge, having convinced himself that such behavior was not a type of *periodic phenomenon* such as light (see Appendix A.4.3 of [75]).

[9]It remains impressive that Kirchhoff [28, 29] was able to build on Weber's electrodynamics [18] to deduce a wave equation for the current and charge on current elements (conductors), finding the wavespeed to be c. In Kirchhoff's analysis, the electric charge density was zero, in which case the (Maxwell) equations (8) for the potentials with $k = -1$ are consistent with $\varphi = 0$, and with $\mathbf{A}$ (and $\mathbf{J}$, the source of $\mathbf{A}$) propagating at velocity c. However, as Weber's electrodynamics was based on action-at-a distance, and was not a field theory in the sense of Faraday, Weber and Kirchhoff did not infer that, since waves of electric current $\mathbf{J}$ in wires move at light speed, light must be an electromagnetic phenomenon.

[10]Riemnann mentioned the condition (1) with $k = -1$ in an 1861 lecture, p. 330 of [32].

While Maxwell's electrodynamics is independent of the choice of gauge (*i.e.*, choice of k),[11] the electrodynamics of the other Germans considered by Helmholtz was not. Note that use of the potentials in eq. (1) implies the second and third Maxwell equations of (6), but not the first and fourth. The first Maxwell equation could be considered as compatible with all the various electrodynamic theories of the mid 1800's, which were distinguished by their different proposals for extensions of Ampère's electrodynamics to include time-dependent phenomena. Maxwell's fourth equation of (6) contains the so-called displacement current $(1/4\pi)\,\partial\mathbf{E}/\partial t$, which was not consistent with any of the other theories cited.[12]

A Appendix: Some History of Potentials

Leibniz (1686) may have been the first to consider kinetic energy, which is 1/2 his *vis viva* mv^2.[13] He had some awareness that the kinetic energy of an object that falls from rest is proportional to the height through which it is has fallen, of spoke of a *vis mortua* of an object at rest.[14]

D. Bernoulli (1738) wrote of a *vis viva potentialis*, p. 229 of [3], whose loss by a falling weight generates a *vis viva*, and in the same sentence pointed out that kinetic energy (*visa viva*) is better written as $mv^2/2$. This appears to be the earliest conception of potential energy.[15,16]

Lagrange (1773), p. 348 of [5], may have been the first to note that Newton's $1/r^2$ gravitational (vector) force law, pp. 108 and 390 of [1], can be related to the gradient of a scalar force function V (of dimensions mass/distance for the force on a unit mass) as, in present notation,

$$\mathbf{F} = -\boldsymbol{\nabla} V \qquad \text{(Lagrange).} \tag{11}$$

Laplace (1799), p. 25 of [8], added that in regions free from source masses, the force function

[11] Electro- and magnetostatic potentials are independent of the choice of k within the velocity gauge, as can be seen from eq. (8). This may have led physicists in the 19^{th} century to consider the potentials as "real" physical elements of their theories, rather than as "merely" useful mathematical functions. These physicists would have been disconcerted by Gibb's remark (1896) [45] that one can define the scalar potential to be zero, in which case the vector potential in electro- and magnetostatics is given by

$$\mathbf{A}_{\rm Gibbs}(\mathbf{x},t) = -ct\int \frac{\rho(\mathbf{x}',t)\,\mathbf{r}}{r^3}\,d{\rm Vol}' + \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}',t)}{r}\,d{\rm Vol}'. \tag{10}$$

[12] For example, Helmholtz used Weber's theory to arrive at eq. (3), p. 80 of [38], which is our eq. (8) but without the term $-\partial^2\mathbf{A}/\partial(ct)^2$ (which derived from the displacement current).

[13] See, for example, sec. 96 of Maxwell's *Matter and Motion* [41]. For a review, see [59].

[14] Johann Bernoulli (1710), p. 525 of [2], may have been the first to consider the integral $\int F\,dx$, which we now identify with the work done by the force F, or the negative of the potential energy if F is a conservative force. However, Bernoulli merely used this integral in a calculation, without commenting as to its physical significance. See also [69].

[15] Bernoulli noted (1748), p. 359 of [4], that the kinetic energy of a particle subject to the gravitational force of a much more massive object is equal to $-\int F\,dx = K/r+$ const, but he did not use his term *vis potentialis* here.

[16] See sec. 97 of [41], as well as [44, 56, 72] and p. 112, pp. 116-123 of [77].

obeys

$$\nabla^2 V = 0 \qquad \text{(Laplace)}, \tag{12}$$

and Poisson (1813) clarified that the force function is related to the source volume density ρ according to, p 390 of [9],

$$\nabla^2 V = -4\pi\rho \qquad \text{(Poisson)}. \tag{13}$$

Coulomb (1785) noted that both static electric [6] and magnetic [7] forces between charges/poles obey a $1/r^2$ force law. Poisson (1821) applied Laplace's equation (with φ as the scalar function) to a theory of magnetism based on magnetic poles, p. 493 of [11], and then used Poisson's equation (1823) with this theory on p. 463 of [13]. The force function was applied to electrostatics by Green, p. 9 of [15], who called it the potential function.

Following Örsted's discovery (1820) [10] that electric currents exert forces on permanent magnets, Ampère (1821) [12] developed a theory that all magnetic effects are associated with electric currents. In separate elaborations of Ampère's theory, Neumann (1845) [17, 21] and Weber (1846) [18] discussed scalar interaction energies that they called potentials, from which electromotive forces[17,18] could be deduced by taking derivatives. Neumann's scalar potential (energy) for two closed currents I_1 and I_2, sec. 11 of [17], can be written (in Gaussian units) as

$$U_{\text{Neumann}} = \frac{I_1}{c}\oint_1 d\mathbf{l}_1 \cdot I_2 \oint_1 \frac{d\mathbf{l}_2}{c\, r_{12}} = \frac{I_1}{c}\oint_1 d\mathbf{l}_1 \cdot \mathbf{A}_2, \tag{14}$$

where c is the speed of light in vacuum ($= 1/\sqrt{\epsilon_0\mu_0}$ in SI units, with permittivity ϵ_0 and permeability μ_0 determined from static experiments), and

$$\mathbf{A} = \frac{I}{c}\oint \frac{d\mathbf{l}}{r} \tag{15}$$

is the vector potential associated with a (steady) current I that flows in a closed loop.

Neumann never factorized his potential into the second form of eq. (14), but is nonetheless generally credited with inventing the vector potential.

In contrast, Weber's scalar potential energy for a pair of moving charges e and e' separated by distance r had the form, p. 230 of [22],

$$U_{\text{Weber}} = \frac{ee'}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right). \tag{16}$$

In 1846, sec. I, p. 63 of [19], W. Thomson described the (vector) electrical force (α, β, γ) due to a unit charge at the origin "exerted at the point (x, y, z)" as $\mathbf{r}/r^3$, Thomson's eq. (I),

[17]The vector force on an electric charge or circuit was called an electromotive force, as also was the scalar quantity that we write as $\int \mathbf{E}\cdot d\mathbf{l}$ (and still call "the" electromotive force on occasion).

[18]Grassmann's contribution [16] should also be noted, that the force law between electric circuits can be expressed in a factorizable form, permitting identification of a magnetic field (not done by Grassmann), at the expense of abandoning Newton's third law of action and reaction for current elements (moving charges, which had not yet been studied in Grassmann's time, but first done by Rowland (1878) [42] while working with Helmholtz). For discussion of action and reaction in magnetic interactions, see, for example, [53].

without explicit statement that a charge exists at that point to experience the force.[19] He continued, sec. II, p. 63 of [19], with the example of a "point" magnetic dipole $\mathbf{m}$, whose (magnetic) scalar potential is $\Phi = \mathbf{m} \cdot \mathbf{r}/r^3$, noting that the magnetic force $\mathbf{B} = (X, Y, Z) = -\boldsymbol{\nabla}\Phi$ on a unit magnetic pole p can also be written as $\boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A} = (\alpha, \beta, \gamma) = \mathbf{m} \times \mathbf{r}/r^3$.[20] The vector $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, Thomson's eq. (II), is the earliest mathematical statement of a magnetic field (although Thomson did not use the term "field" until p. 179 of [24] (1850)).

The term potential energy was introduced by Rankine in 1853 [25].

We have already mentioned that the first statement of a gauge condition was by Kirchhoff (1857) [28, 29],

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{c}\frac{\partial V}{\partial t} \qquad \text{(Kirchhoff)}. \tag{17}$$

The first statement of the Coulomb-gauge condition was by Maxwell (1861), eq. (57), p. 290 of [31],

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \qquad \text{(Coulomb)}. \tag{18}$$

Lorenz stated his gauge condition in 1867, shortly before eq. (8) of [35], and gave expressions for the retarded potentials, following Riemann [34],

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t} \qquad \text{(Lorenz)}. \tag{19}$$

In 1870, Helmholtz introduced the velocity gauge, eq. (3^a) of [38],

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{k}{c}\frac{\partial V}{\partial t} \qquad \text{(velocity)}. \tag{20}$$

In 1896, Gibbs [45] introduced the gauge where the scalar potential is everywhere zero,

$$V = 0 \qquad \text{(Gibbs)}. \tag{21}$$

The arbitrariness of the potentials under gauge transformations was noted by Maxwell (1865) in secs. 98-99 of [33], but this was little appreciated for many years.[21] Some awareness of this arbitrariness was expressed by Larmor (1900), sec. 77, p. 121 of [46], and by Macdonald (1902), p. 12 of [48] (where the Lorenz-gauge condition was invoked). The general gauge transformation was mentioned by Lorentz (1904) on p. 157 of [49], along with a statement of his preference for the Lorenz gauge (without attribution, such that this gauge became known to many as the Lorentz gauge). The influence of Lorentz was reflected, for example, by the discussion of Schott (1912), p. 3 of [50].

The term gauge invariance was invented by Weyl, p. 100 of the English version of [51].

[19] In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term "electric field" were used by Thomson.

[20] Thomson did not notice at this time that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

[21] Due to Maxwell's awkward discussion in sec. 98 of [33], even the 2001 review article [70] missed that Maxwell's eq. (77) is the gauge transformation of the scalar potential.

References

[1] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686),
http://physics.princeton.edu/~mcdonald/examples/mechanics/newton_principia.pdf

[2] J. Bernoulli, *Extrait de la Réponse de M. Bernoulli à M. Herman*, Hist. Acad. Sci. Belles Lett. (Berlin, 1710), p. 521,
http://physics.princeton.edu/~mcdonald/examples/mechanics/bernoulli_hars_521_10.pdf

[3] D. Bernoulli, *Hydrodynamica* (Strassburg, Austria, 1738),
http://physics.princeton.edu/~mcdonald/examples/fluids/bernoulli_hydrodynamica_38.pdf
Hydrodynamics by Daniel Bernoulli and Hydraulics by Johann Bernoulli, trans. by T. Carmody and H. Kobus (Dover, 1968),
http://physics.princeton.edu/~mcdonald/examples/fluids/bernoulli_ch3_13.pdf

[4] D. Bernoulli, *Remarques sur le Principe de le Conservation des Forces Vives pris dans un Sens General*, Hist. Acad. Sci. Belles Lett. (Berlin, 1748), p. 356, ,
http://physics.princeton.edu/~mcdonald/examples/mechanics/bernoulli_harsbl_356_48.pdf

[5] J.-L. de Lagrange, *Sur l'Équation Séculaire de le Lune*, Mem. Acad. Roy. Sci. Paris **7**, (1773), http://physics.princeton.edu/~mcdonald/examples/mechanics/lagrange_arsp_7_73.pdf

[6] C.-A. de Coulomb, *Premier Mémoire sur l'Électricité et le Magnétisme*, Mém. Acad. Roy. Sci., 569 (1785), http://physics.princeton.edu/~mcdonald/examples/EM/coulomb_mars_88_569_85.pdf

[7] C.-A. de Coulomb, *Second Mémoire sur l'Électricité et le Magnétisme*, Mém. Acad. Roy. Sci., 578 (1785), http://physics.princeton.edu/~mcdonald/examples/EM/coulomb_mars_88_578_85.pdf

[8] P.-S. Laplace, *Traite de Mécanique Céleste*, Vol. 2 (Duprat, Paris, 1799),
http://physics.princeton.edu/~mcdonald/examples/mechanics/laplace_mc_v2.pdf

[9] S.D. Poisson, *Remarques sur une équation qui se présente dans la théorie des attractions des sphéroides*, Bull. Sci. Soc. Phil. Paris **3**, 383 (1813),
http://physics.princeton.edu/~mcdonald/examples/mechanics/poisson_nbssp_3_388_13.pdf

[10] J.C. Örsted, *Experimenta circa Effectum Conflictus Electrici in Acum Magneticam* (pamphlet, Berlin, 1829),
http://physics.princeton.edu/~mcdonald/examples/EM/oersted_experimenta.pdf
Experiments on the Effect of a Current of Electricity on the Magnetic Needle, Ann. Phil. **16**, 273 (1820),
http://physics.princeton.edu/~mcdonald/examples/EM/oersted_ap_16_273_20.pdf

[11] S.D. Poisson, *Sur la Théorie du Magnétisme*, Mém. Acad. Roy. Sci. **5**, 247, 486 (1826),
http://physics.princeton.edu/~mcdonald/examples/EM/poisson_24.pdf

[12] A.M. Ampère, *Théorie mathématique des Phénomènes électro-dynamiques uniquement déduite de l'Expérience*, 2nd ed. (Paris, 1883),
http://physics.princeton.edu/~mcdonald/examples/EM/ampere_theorie_26_83.pdf
English translation in [76].

[13] S.D. Poisson, *Sur la Théorie du magnétisme en mouvement*, Mém. Acad. Roy. Sci., **6**, 441 (1826), http://physics.princeton.edu/~mcdonald/examples/EM/poisson_26.pdf

[14] G. Ohm, *Die galvanishe Kette* (Berlin, 1827),
http://physics.princeton.edu/~mcdonald/examples/EM/ohm_galvanische_kette.pdf
The Galvanic Circuit Mathematically Investigated (Van Nostrand, 1891),
http://physics.princeton.edu/~mcdonald/examples/EM/ohm_galvanic_circuit.pdf

[15] G. Green, *Mathematical Papers* (Macmillan, 1871),
http://physics.princeton.edu/~mcdonald/examples/EM/green_papers.pdf

[16] H. Grassmann, *Neue Theorie der Elektrodynamik*, Ann. d. Phys. **64**, 1 (1845),
http://physics.princeton.edu/~mcdonald/examples/EM/grassmann_ap_64_1_45.pdf
http://physics.princeton.edu/~mcdonald/examples/EM/grassmann_ap_64_1_45_english.pdf

[17] F.E. Neumann, *Allgemeine Gesetze der inducirten elektrischen Ströme*, Abh. König. Akad. Wiss. Berlin **1**, 45 (1845),
http://physics.princeton.edu/~mcdonald/examples/EM/neumann_akawb_1_45.pdf

[18] W. Weber, *Elektrodynamishe Maassbestimmungen*, Abh. König. "achs. Gesell. Wissen. p. 209 (1846), http://physics.princeton.edu/~mcdonald/examples/EM/weber_aksgw_209_46.pdf
Translated as *Determinations of Electrodynamic Measure*,
http://physics.princeton.edu/~mcdonald/examples/EM/weber_aksgw_209_46_english.pdf

[19] W. Thomson, *On a Mechanical Representation of Electric, Magnetic, and Galvanic Forces*, Cam. Dublin Math. J. **2**, 61 (1847),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_cdmj_2_61_47.pdf

[20] H. Helmholtz, *Über die Erhaltung der Kraft* (Reimer, Berlin, 1847),
http://physics.princeton.edu/~mcdonald/examples/EM/helmholtz_kraft_47.pdf
On the Conservation of Force, in *Scientific Memoirs*, J. Tyndall and W. Francis, eds. (Taylor and Francis, 1853), p. 114,
http://physics.princeton.edu/~mcdonald/examples/EM/helmholtz_psb_47_english.pdf

[21] F.E. Neumann, *Über ein allgemeines Princip der mathematischen Theorie inducirter elektrischer Ströme*, Abh. König. Akad. Wiss. Berlin **1**, 47 (1847),
http://physics.princeton.edu/~mcdonald/examples/EM/neumann_akawb_1_47.pdf

[22] W. Weber, *Elektrodynamishe Maa∫sbestimmungen*, Ann. d. Phys. Chem. **73**, 193 (1848), http://physics.princeton.edu/~mcdonald/examples/EM/weber_apc_73_193_48.pdf
Translated as *On the Measurement of Electro-dynamic Forces*,
http://physics.princeton.edu/~mcdonald/examples/EM/weber_apc_73_193_48_english.pdf

[23] W. Thomson, *A Mathematical Theory of Magnetism*, Phil. Trans. Roy. Soc. London **141**, 269 (1851), http://physics.princeton.edu/~mcdonald/examples/EM/thomson_ptrsl_141_269_51.pdf

[24] W. Thomson, *On the Theory of Magnetic Induction in Crystalline and Non-crystalline Substances*, Phil. Mag. **1**, 177 (1851),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_pm_1_177_51.pdf

[25] W.J.M. Rankine, *On the General Law of the Transformation of Energy*, Phil. Mag. **5**, 106 (1853), http://physics.princeton.edu/~mcdonald/examples/statmech/rankine_pm_5_106_53.pdf

[26] R. Kohlsrausch and W. Weber *Elektrodynamische Maassbestimmungen inbesondere Zurückführung der Stromintensitäts-Messungen auf Mechansishes Maass*, Abh. König. Sächs. Gesell. Wiss. 219 (1856), http://physics.princeton.edu/~mcdonald/examples/EM/weber_aksgw_219_56.pdf

[27] W. Weber and R. Kohlsrausch, *Ueber die Elektricitätsmenge, welche bei galvanischen Strömen durch den Querschnitt der Kette fliesst*, Ann. d. Phys. **99**, 10 (1856), http://physics.princeton.edu/~mcdonald/examples/EM/weber_ap_99_10_56.pdf http://physics.princeton.edu/~mcdonald/examples/EM/weber_ap_99_10_56_english.pdf

[28] G. Kirchhoff, *Ueber die Bewegung der Elektricitat in Drähten*, Ann. d. Phys. Chem. **100**, 193 (1857), http://physics.princeton.edu/~mcdonald/examples/EM/kirchhoff_apc_100_193_57.pdf *On the Motion of Electricity in Wires*, Phil. Mag. **13**, 393 (1857), http://physics.princeton.edu/~mcdonald/examples/EM/kirchhoff_pm_13_393_57.pdf

[29] G. Kirchhoff, *Ueber die Bewegung der Elektricitat in Leitern*, Ann. d. Phys. Chem. **102**, 529 (1857), http://physics.princeton.edu/~mcdonald/examples/EM/kirchhoff_apc_102_529_57.pdf P. Graneau and A.K.T. Assis, *Kirchhoff on the Motion of Electricity in Conductors*, Apeiron **19**, 19 (1994), http://physics.princeton.edu/~mcdonald/examples/EM/kirchhoff_apc_102_529_57_english.pdf

[30] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, Crelles J. **55**, 25 (1858), http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz_jram_55_25_58.pdf

[31] J.C. Maxwell, *On Physical Lines of Force. Part II.—The Theory of Molecular Vortices applied to Electric Currents*, Phil. Mag. **21**, 281, 338 (1861), http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_pm_21_281_61.pdf

[32] B. Riemann, *Schwere, Electricität und Magnetismus*, ed. by K. Hattendorf (Rümpler, 1880), http://physics.princeton.edu/~mcdonald/examples/EM/riemann_vorlesungen_80.pdf

[33] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_ptrsl_155_459_65.pdf

[34] B. Riemann, *Ein Beitrag zur Elektrodynamik*, Ann. d. Phys. **207**, 237 (1867), http://physics.princeton.edu/~mcdonald/examples/EM/riemann_ap_207_237_67.pdf *A Contribution to Electrodynamics*, Phil. Mag. **34**, 368 (1867), http://physics.princeton.edu/~mcdonald/examples/EM/riemann_pm_34_368_67.pdf

[35] L. Lorenz, *Ueber die Identität der Schwingungen des Lichts mit den elektrischen Strömen*, Ann. d. Phys. **207**, 243 (1867), http://physics.princeton.edu/~mcdonald/examples/EM/lorenz_ap_207_243_67.pdf *On the Identity of the Vibration of Light with Electrical Currents*, Phil. Mag. **34**, 287 (1867), http://physics.princeton.edu/~mcdonald/examples/EM/lorenz_pm_34_287_67.pdf

[36] C. Neumann, *Resultate einer Untersuchungen über Die Principien der Elektrodynamik*, Nachr. Königl. Gesell. Wissen. Georg-Augusts-U. Göttingen, 223 (1868), http://physics.princeton.edu/~mcdonald/examples/EM/neumann_njgwgau_223_68.pdf

[37] C. Neumann, *Die Principien der Elektrodynamik*, Tübinger U. 50th Jubil. (1868), Math. Ann. **17**, 400 (1880), http://physics.princeton.edu/~mcdonald/examples/EM/neumann_ma_17_400_80.pdf

[38] H, Helmholtz, *Ueber die Bewenungsgleichungen der Elektricität für ruhende leitende Körper*, J. Reine Angew. Math. **72**, 57 (1870), http://physics.princeton.edu/~mcdonald/examples/EM/helmholtz_jram_72_57_70.pdf

[39] H, Helmholtz, *On the Theory of Electrodynamics*, Phil. Mag. **44**, 530 (1872), http://physics.princeton.edu/~mcdonald/examples/EM/helmholtz_pm_44_530_72.pdf

[40] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_treatise_v2_73.pdf

[41] J.C. Maxwell, *Matter and Motion* (Society for the Promotion of Christian Knowledge, 1876), http://physics.princeton.edu/~mcdonald/examples/mechanics/maxwell_matter_motion_76.pdf http://physics.princeton.edu/~mcdonald/examples/mechanics/maxwell_matter_motion.pdf

[42] H.A. Rowland, *On the Magnetic Effect of Electric Convection*, Am. J. Sci. **15**, 30 (1878), http://physics.princeton.edu/~mcdonald/examples/EM/rowland_ajs_15_30_78.pdf

[43] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2, 3rd ed. (Clarendon Press, 1892), http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_treatise_v2_92.pdf

[44] G.F. Becker, *"Potential" a Bernoullian Term*, Am. J. Sci. **45**, 97 (1893), http://physics.princeton.edu/~mcdonald/examples/mechanics/becker_ajs_45_97_93.pdf

[45] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, Nature **53**, 509 (1896), http://physics.princeton.edu/~mcdonald/examples/EM/gibbs_nature_53_509_96.pdf

[46] J. Larmor, *Aether and Matter* (Cambridge U. Press, 1900), http://physics.princeton.edu/~mcdonald/examples/EM/larmor_aether_matter_00.pdf

[47] C. Neumann, *Über die Maxwell–Hertz'sche Theorie* (Teubner, 1901), http://physics.princeton.edu/~mcdonald/examples/EM/neumann_maxwell_01.pdf

[48] H.M. Macdonald, *Electric Waves* (Cambridge U. Press, 1902), http://physics.princeton.edu/~mcdonald/examples/EM/macdonald_electric_waves_02.pdf

[49] H.A. Lorentz, *Weiterbildung der Maxwellschen Theorie. Elektronentheorie*, Enzykl. Math. Wiss. **5**, part II, 145 (1904), especially secs. 20-21, http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_emw_5_2_145_04.pdf

[50] G.A. Schott, *Electromagnetic Radiation and the Mechanical Reactions Arising from it* (Cambridge U. Press, 1912), http://physics.princeton.edu/~mcdonald/examples/EM/schott_radiation_12.pdf

[51] H. Weyl, *Gruppentheorie und Quantenmechanik* (Hirzel, Leipzig, 1928), pp. 8788; *Theory of Groups and Quantum Mechanics*, 2nd ed. (1930), pp. 100101, http://physics.princeton.edu/~mcdonald/examples/QM/weyl_groups_qm_30.pdf

[52] A. O'Rahilly, *Electromagnetic Theory* (Longmans Green, 1938), http://physics.princeton.edu/~mcdonald/examples/EM/orahilly_EM.pdf

[53] L. Page and N.I. Adams, Jr., *Action and Reaction Between Moving Charges*, Am. J. Phys. **13**, 141 (1945), http://physics.princeton.edu/~mcdonald/examples/EM/page_ajp_13_141_45.pdf

[54] L. Rosenfeld, *The Velocity of Light and the Evolution of Electrodynamics*, Nuovo Cim. Suppl. **4**, 1630 (1957), http://physics.princeton.edu/~mcdonald/examples/EM/rosenfeld_ncs_4_1630_57.pdf

[55] A.E. Woodruff, *Action at a Distance in Nineteenth Century Electrodynamics*, Isis **53**, 439 (1962), http://physics.princeton.edu/~mcdonald/examples/EM/woodruff_isis_53_439_62.pdf

[56] A.J. Pacey and S.J. Fisher, *Daniel Bernoulli and the Vis Viva of Compressed Air*, Brit. J. Hist. Sci. **3**, 388 (1967), http://physics.princeton.edu/~mcdonald/examples/mechanics/pacey_bjhs_3_388_67.pdf

[57] A.E. Woodruff, *The Contributions of Hermann von Helmholtz to Electrodynamics*, Isis **59**, 300 (1968), http://physics.princeton.edu/~mcdonald/examples/EM/woodruff_isis_59_300_68.pdf

[58] T. Hirosige, *Origins of Lorentz' Theory of Electrons and the Concept of the Electromagnetic Field*, Hist. Stud. Phys. Sci. **1**, 151 (1969), http://physics.princeton.edu/~mcdonald/examples/EM/hirosige_hsps_1_151_69.pdf

[59] C. Iltis, *Leibniz and the Vis Viva Controversy*, Isis **62**, 21 (1971), http://physics.princeton.edu/~mcdonald/examples/mechanics/iltis_isis_62_21_71.pdf

[60] S. d'Agostino, *Hertz's Researches on Electromagnetic Waves*, Hist. Stud. Phys. Sci. **6**, 261 (1975), http://physics.princeton.edu/~mcdonald/examples/EM/dagostino_hsps_6_261_75.pdf

[61] K.-H. Yang, *Gauge Transformations and Quantum Mechanics II. Physical Interpretation of Classical Gauge Transformations*, Ann. Phys. **101**, 97 (1976), http://physics.princeton.edu/~mcdonald/examples/QM/yang_ap_101_97_76.pdf

[62] M.N. Wise, *German concepts of force, energy and the electromagnetic ether: 1845-1880*, in G.N. Cantor and M.J.S. Hodge, *Conceptions of ether*, (Cambridge U. Press, 1981), pp. 269-307, http://physics.princeton.edu/~mcdonald/examples/EM/wise_force_81.pdf

[63] T. Archibald, *Carl Neumann versus Rudolf Clausius on the propagation of electrodynamic potentials*, Am. J. Phys. **168**, 104 (1986), http://physics.princeton.edu/~mcdonald/examples/EM/archibald_ajp_54_786_86.pdf

[64] K.-H. Yang and D.H. Kobe, *Superluminal, Advanced and Retarded Propagation of Electromagnetic Potentials in Quantum Mechanics*, Ann. Phys. **168**, 104 (1986), http://physics.princeton.edu/~mcdonald/examples/QM/yang_ap_168_104_86.pdf

[65] T. Archibald, *Energy and the Mathematization of Electrodynamics in Germany, 1845-1875*, Arch. Int. Hist. Sci. **39**, 276 (1989),
http://physics.princeton.edu/~mcdonald/examples/EM/archibald_aihs_39_276_89.pdf

[66] O. Darrigol, *The Electrodynamic Revolution in Germany as Documented by Early German Expositions of "Maxwell's Theory"*, Arch. Hist. Exact Sci. **45**, 189 (1993),
http://physics.princeton.edu/~mcdonald/examples/EM/darrigol_ahes_45_189_93.pdf

[67] A.K.T. Assis, *Weber's Electrodynamics* (Springer, 1994),
http://physics.princeton.edu/~mcdonald/examples/EM/assis_weber.pdf

[68] M.A. Bueno and A.K.T. Assis, *A new method for inductance calculations*, J. Phys. D **28**, 1802 (1995), http://physics.princeton.edu/~mcdonald/examples/EM/bueno_jpd_28_1802_95.pdf

[69] D. Speiser, *The Kepler Problem from Newton to Johann Bernoulli*, Arch. Hist. Exact Sci. **50**, 103 (1996),
http://physics.princeton.edu/~mcdonald/examples/mechanics/speiser_ahes_50_103_96.pdf

[70] J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), http://physics.princeton.edu/~mcdonald/examples/EM/jackson_rmp_73_663_01.pdf

[71] J.D. Jackson, *From Lorenz to Coulomb and other explicit gauge transformations*, Am. J. Phys. **70**, 917 (2002), http://physics.princeton.edu/~mcdonald/examples/EM/jackson_ajp_70_917_02.pdf

[72] E. Hecht, *An Historico-Critical Account of Potential Energy: Is* PE *Really Real?* Phys. Teach. **41**, 486 (2003),
http://physics.princeton.edu/~mcdonald/examples/mechanics/hecht_pt_41_486_03.pdf

[73] K.-H. Schlote, *Carl Neumanns Contributions to Electrodynamics*, Phys. Persp. **6**, 252 (2004), http://physics.princeton.edu/~mcdonald/examples/EM/schlote_pip_6_252_04.pdf

[74] K.-H. Yang, *The physics of gauge transformations*, Am. J. Phys. **73**, 742 (2005),
http://physics.princeton.edu/~mcdonald/examples/EM/yang_ajp_73_742_05.pdf

[75] K.T. McDonald, *Maxwell and Special Relativity* (May 26, 2014),
http://physics.princeton.edu/~mcdonald/examples/maxwell_rel.pdf

[76] A.K.T. Assis and J.P.M.C. Chahib, *Ampère's Electrodynamics* (Aperion, 2015),
http://physics.princeton.edu/~mcdonald/examples/EM/assis_ampere_15.pdf

[77] J. Coopersmith, *Energy, the Subtle Concept* (Oxford U. Press, 2015),
http://physics.princeton.edu/~mcdonald/examples/mechanics/coopersmith_energy_15.pdf

[78] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), http://physics.princeton.edu/~mcdonald/examples/gauge.pdf