Uncoiling on a Horizontal Surface

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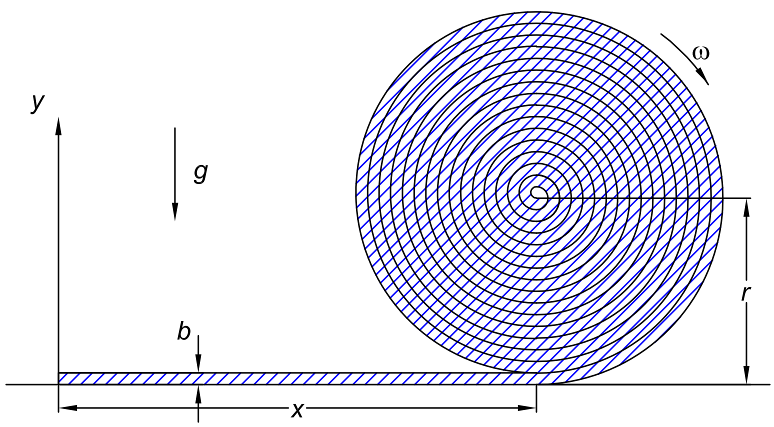
1 Problem

Discuss the motion of a coiled sheet or tape as it unrolls on a horizontal surface, supposing that there is no slipping, no springiness to the sheet, and no dissipation of energy during the uncoiling.



2 Solution

If the coil of initial mass M and radius R starts from rest (given a tiny push to start the motion), its center of mass falls as the coil unwinds, such that when the tape lies flat on the table its gravitational potential energy has decreased by MgR, where g is the acceleration due to gravity, and the tape ends up at rest on the horizontal support surface. This appears to be a violation of conservation of energy, as noted in [1].

The resolution to this type of paradox was given in [2], where it was argued that the velocity of the coil goes to infinity as its radius r goes to zero, and the kinetic energy MgR of the coil just before it completely unwinds is dissipated in a final, inelastic collision with the horizontal surface, accompanied in practice by a loud noise. See also [3, 4, 5].[1]

2.1 Energy Analysis

For the case of no slipping, the system has only one degree of freedom, so we seek a single equation of motion. Prior to the final unwinding for the coil, we approximate energy as conserved, which we use as the basis for an equation of motion.

We suppose the coil is made of a material of thickness b, with total mass M, initial outer radius R, and (for simplicity) inner radius 0. At some time t the line of contact of the coil,

[1]For the case of uncoiling during vertical fall (yo-yo), see, for example, [6].

then with radius $r < R$, with the horizontal surface is at x, with the left end of the uncoiled material at $x = 0$, as sketched below.

The mass of the coil is then $m = Mr^2/R^2$, and the moment of inertia of the coil about its axis is $I = mr^2/2 = Mr^4/2R^2$.

The no-slip condition can be expressed as,

$$\pi R^2 = bx + \pi r^2, \qquad \frac{dx}{dt} \equiv \dot{x} = -\frac{2\pi r\,\dot{r}}{b}\,, \tag{1}$$

and the angular velocity ω of the rotating coil is related by,[2]

$$\omega = \frac{\dot{x}}{r} = -\frac{2\pi\dot{r}}{b}\,. \tag{2}$$

The kinetic energy the system is,[3]

$$\begin{aligned} T &= \frac{m(\dot{r}^2 + \dot{x}^2)}{2} + \frac{I\omega^2}{2} = \frac{Mr^2\dot{r}^2}{2R^2}\left(1 + \frac{4\pi^2 r^2}{b^2}\right) + \frac{\pi^2 Mr^4\dot{r}^2}{b^2R^2} = \frac{Mr^2\dot{r}^2}{2R^2} + \frac{3\pi^2 Mr^4\dot{r}^2}{b^2R^2} \\ &= \frac{Mr^2\dot{r}^2}{2b^2R^2}(6\pi^2 r^2 + b^2). \end{aligned} \tag{3}$$

The gravitational potential energy of the system with respect to the horizontal support surface is,

$$V = mgr = \frac{Mgr^3}{R^2}\,, \tag{4}$$

ignoring the small potential energy $Mb^2xg/2R^2$ of the tape that lies on the horizontal surface. The constant energy (prior to the final "big bang") is

$$E = T_0 + V_0 = \frac{M\dot{r}_0^2}{2b^2}(6\pi^2 R^2 + b^2) + MgR = T + V = \frac{Mr^2\dot{r}^2}{2b^2R^2}(6\pi^2 r^2 + b^2) + \frac{Mgr^3}{R^2}\,. \tag{5}$$

The first term on the right side of eq. (5) is small compared to the second for $r > b$, such that,

$$\dot{r}^2 = \dot{r}_0^2 + \frac{2g\,b^2R^3}{6\pi^2 r^4 + b^2r^2}\left(1 - \frac{r^3}{R^3}\right) \approx \dot{r}_0^2 + \frac{g\,b^2R^3}{3\pi^2 r^4}\left(1 - \frac{r^3}{R^3}\right), \tag{6}$$

which indicates that $\dot{r}$ (and $\dot{x}$) diverge as r goes to zero, as mentioned above.

We can differentiate eq. (5) with respect to time to obtain an equation of motion,

$$(6\pi^2 r^2 + b^2)r\ddot{r} + (12\pi^2 r^2 + b^2)\dot{r}^2 + 3b^2gr = 0. \tag{7}$$

Even in a zero-g environment, uncoiling is possible, provided the tape sticks to the horizontal surface once it makes contact.

Furthermore,

$$r\ddot{r} = -\frac{3b^2gr}{6\pi^2 r^2 + b^2} - \frac{12\pi^2 r^2 + b^2}{6\pi^2 r^2 + b^2}\dot{r}^2 \approx -\frac{b^2 g}{2\pi^2 r} - 2\dot{r}_0^2 - \frac{2g\,b^2R^3}{3\pi^2 r^4}\left(1 - \frac{r^3}{R^3}\right) < 0. \tag{8}$$

[2]Formally, these relations are holonomic constraints.

[3]Care is required in evaluating the kinetic energy of a system that includes a variable mass, as noted in [7], where a variant of the present example is discussed.

2.2 Forces and Torques

While the energy analysis seems satisfactory in leading to an equation of motion (in the approximation of conservation of mechanical energy), it is difficult to reconcile the behavior of the unwinding coil with a Newtonian analysis based on forces an torques. As the center of the coil accelerates to the right, one associates this with a horizontal force F to the right at the line of contact of the coil (and with a possible horizontal force $F_{\rm int}$ between the portion of the tape on the horizontal surface and the portion in the coil just above the line of contact). Meanwhile, the angular momentum of the coil is increasing in a clockwise sense, which requires a clockwise torque, which seems to require the horizontal force to the left.

There must also be a vertical (normal) force $N_{\rm tot}$, at the interface between the coiled and horizontal portions of the tape, due to the effect of horizontal support surface on the coil at the line of contact of the coil with the horizontal surface, as well as a possible shear force between the portion of the tape on the horizontal surface and the portion in the coil. We can compute the vertical (constraint) force on the coil via the vertical-momentum equation, also using eqs. (6) and (8),

$$\begin{aligned} N_{\rm tot} - mg &= \frac{dP_y}{dt} = \frac{d}{dt}(m\dot{r}) = \frac{d}{dt}\left(\frac{Mr^2\dot{r}}{R^2}\right) = \frac{Mr}{R^2}(2\dot{r}^2 + r\ddot{r}) \\ &\approx -\frac{Mr\dot{r}_0^2}{R^2} - \frac{Mgb^2}{2\pi^2 R^2} = -\frac{Mr\dot{r}_0^2}{R^2} - mg\frac{b^2}{2\pi^2 r^2}\,, \qquad (9) \\ N_{\rm tot} &\approx -\frac{Mr\dot{r}_0^2}{R^2} + mg\left(1 - \frac{b^2}{2\pi^2 r^2}\right) \approx -\frac{Mr\dot{r}_0^2}{R^2} + mg. \qquad (10) \end{aligned}$$

Note that if $g = 0$, then the normal force N must be negative, and is then a shear force inside the tape just above the line of contact, rather than being a contact force with the horizontal surface. In this case, N is not a constraint force in the sense of Lagrange, but is an internal force between horizontal and coiled portions of the tape.

When considering the horizontal constraint force F at the line of contact, we must note that there is also a horizontal, internal force $F_{\rm int}$.

The horizontal momentum-equation involves the sum of these two forces,

$$\begin{aligned} F_{\rm tot} = F + F_{\rm int} = \frac{dP_x}{dt} &= \frac{d}{dt}(m\dot{x}) = -\frac{d}{dt}\left(\frac{2\pi Mr^3\dot{r}}{bR^2}\right) = -\frac{2\pi Mr^2}{bR^2}(3\dot{r}^2 + r\ddot{r}) \\ &\approx \frac{Mgb}{3\pi r^2 R^2}(5r^3 - 2R^3), \qquad (11) \end{aligned}$$

again using eqs. (6) and (8) in the approximation that $r \gg b$. This force is initially in the $+x$ direction, but changes sign when $r \approx \sqrt[3]{2/5)}R \approx 0.74R$.

Qualitatively, as the coil first begins to unwind, its center of mass moves to larger x, so if there were no horizontal force $F_{\rm tot}$, the coil would slide to smaller x to keep the x-coordinate of the c.m. fixed. The horizontal force $F_{|rmtot}$ opposes this tendency, and hence is in the positive x-direction initially. However, at later times, as the angular velocity of the unwinding coil increases, the linear momentum of the coil increases, and the coil would tend to slide towards more positive x-values, if there were no horizontal force at the line of contact. Thus, at later times, the horizontal force points in the negative x-direction.

We can also consider the torque equation for angular momentum L_O of the entire tape about the origin (at the left end of the tape at rest on the horizontal surface),[4]

$$\boldsymbol{\tau}_O = \frac{d\mathbf{L}_O}{dt}\,, \tag{12}$$

where $\boldsymbol{\tau}_O = \sum_i \mathbf{x}_i \times \mathbf{F}_{\mathrm{ext},i}$ is the torque about the origin due to external forces (which here include constraint forces) applied at positions $\mathbf{x}_i$.

The angular momentum of the system is only due to the motion of the coil, and can be written as the sum of the angular momentum $\mathbf{L}_{\text{coil cm}} = -I\omega\,\hat{\mathbf{z}}$ of the coil about its own center of mass plus the angular momentum of the motion of the center of mass of the coil (at $x\,\hat{\mathbf{x}} + r\,\hat{\mathbf{y}}$) relative to the origin,

$$\begin{aligned}
\mathbf{L}_{\text{of cm of coil,O}} &= x\,\hat{\mathbf{x}} + r\,\hat{\mathbf{y}} \times M\frac{r^2}{R^2}(\dot{x}\,\hat{\mathbf{x}} + \dot{r}\,\hat{\mathbf{y}}) = M\frac{r^2}{R^2}(\dot{r}x - \dot{x}r)\,\hat{\mathbf{z}} \\
&= M\frac{r^2}{R^2}\left(\dot{r}\frac{\pi(R^2 - r^2)}{b} + \frac{2\pi r^2\dot{r}}{b}\right)\hat{\mathbf{z}} = M\frac{\pi r^2\dot{r}}{bR^2}(R^2 + r^2)\,\hat{\mathbf{z}}, \qquad (13) \\
\mathbf{L}_O &= \mathbf{L}_{\text{coil cm}} + \mathbf{L}_{\text{of cm of coil,O}} = M\frac{r^4}{2R^2}\frac{2\pi\dot{r}}{b}\,\hat{\mathbf{z}} + M\frac{\pi r^2\dot{r}}{bR^2}(R^2 + r^2)\,\hat{\mathbf{z}} \\
&= M\frac{\pi r^2\dot{r}}{bR^2}(R^2 + 2r^2)\,\hat{\mathbf{z}}, \qquad (14) \\
\frac{d\mathbf{L}_O}{dt} &= M\frac{\pi(r^2\ddot{r} + 2r\dot{r}^2)}{bR^2}(R^2 + 2r^2)\,\hat{\mathbf{z}} + M\frac{\pi r^2\dot{r}}{bR^2}4r\dot{r}\,\hat{\mathbf{z}} \\
&= \frac{M\pi r}{bR^2}\left[r\ddot{r}(R^2 + 2r^2) + 2\dot{r}^2(R^2 + 4r^2)\right]\hat{\mathbf{z}}. \qquad (15)
\end{aligned}$$

The (external) forces that contribute to the torque $\boldsymbol{\tau}_O$ are $-mg\,\hat{\mathbf{y}}$ on the center of mass of the coil at (x, r), $F\,\hat{\mathbf{x}}$ and $N\,\hat{\mathbf{y}}$ on the line of contact at $(x, 0)$ of the coil with the horizontal surface, and the forces $\pm(M - m)g\,\hat{\mathbf{y}}$ on the tape at rest on the horizontal surface. The latter two forces act at $(x/2, 0)$ and $(x/2, b/2)$, and lead to no net torque about the origin. Then, the torque is,

$$\begin{aligned}
\boldsymbol{\tau}_O &= (x\,\hat{\mathbf{x}} + r\,\hat{\mathbf{y}}) \times -mg\,\hat{\mathbf{y}} + (x\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}}) \times (F\,\hat{\mathbf{x}} + N\,\hat{\mathbf{y}}) \\
&= (N - mg)x\,\hat{\mathbf{z}} = \frac{M\pi r}{bR^2}(2\dot{r}^2 + r\ddot{r})(R^2 - r^2)\,\hat{\mathbf{z}}. \qquad (16)
\end{aligned}$$

Equating (15) and (16), we have,

$$3r\ddot{r} + 10\dot{r}^2 = 0. \tag{17}$$

It is noteworthy that the force/torque method has led to an equation of motion (17) that does not include gravity, and differs from eq. (7) which followed from conservation of energy. However, the term in g in eq. (7) is proportional to b^2, which is of second-order of smallness. So, unless our model of the coil is accurate to order b^2, which is doubtful, the preceeding analysis could be unreliable. Since gravity plays some role in the uncoiling (which could

[4]It is problematic to consider a torque equation for a variable-mass system such as the coil.

still occur in zero gravity), the energy analysis is likely more accurate than the force/torque analysis.

Appendix A considers the case that the "coil" is only a single turn of tape on a massless spool. Here, the geometry is more precisely defined, and both the energy analysis and the force/torque analysis lead to the same equation of motion.

2.3 A Lagrangian Analysis

We can analyze the system as a whole via Lagrange's method, with Lagrangian $\mathcal{L} = T - V$ using the kinetic energy of eq. (3), the potential energy of eq. (4) and the coordinate as r. Then, the equation of motion is again eq. (7) as found above.

2.3.1 Constraint Forces

We can deduce the constraint forces via a Lagrangian method in which more than the minimum number of coordinates are used, as apparently first proposed by Routh [19, 20] for holonomic constraints,[5] as a special case of a method for problems with nonholonomic constraints given by Ferrers [22]. See also [23].

In this method, the minimum number n of independent coordinates is augmented with m additional coordinates, so that the total set of coordinates is q_i, $i = 1, \ldots, n + m$, and for which the m constraint equations $f_j(q_i) = 0$, $j = 1, \ldots, m$, are known, but not explicitly enforced initially. Then, we consider the $n + m$ modified Lagrange equations,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_i}\,, \tag{18}$$

where the λ_j are so-called Lagrange multipliers (which have the physical significance of being the constraint force F_j if the dimensions of the constraint equation $f_j = 0$ are chosen to be length).

2.4.1.1 Vertical Constraint Force

When considering the vertical (normal) constraint force N at the line of contact of the coil with the horizontal surface, it is appropriate to discuss the constraint that the vertical coordinate of the horizontal surface is, say, $y_s = 0$. In the language of the method of Lagrange multipliers, we can say that a third constraint is,

$$f_3 = y_s = 0. \tag{19}$$

We now consider the entire system of the coil plus its unwound portion to be described by two coordinates, r and y_s.

The constraints (1) and (2) still hold.

Then, the kinetic energy of the system is,

$$T = \frac{m[\dot{x}^2 + (\dot{r} + \dot{y}_s)^2]}{2} + \frac{I\omega^2}{2} = \frac{Mr^2}{2R^2}\left(\frac{4\pi^2 r^2 \dot{r}^2}{b^2} + \dot{r}^2 + 2\dot{r}\dot{y}_s + \dot{y}_s^2\right) + \frac{3\pi^2 M r^4 \dot{r}^2}{b^2 R^2}\,. \tag{20}$$

[5] The term holonomic was introduced by Hertz on p. 91 of [21].

The gravitational potential is now,

$$V = M\left(1 - \frac{r^2}{R^2}\right) g y_s + M\frac{r^2}{R^2} g(y_s + r). \tag{21}$$

We now deduce the equation of motion for coordinate y_s via eq. (18), including a single Lagrange multiplier λ_3 associated with f_3 of eq. (19).[6] First,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_s} - \frac{\partial T}{\partial y_s} = \frac{M}{R^2}[r^2(\ddot{r} + \ddot{y}_s) + 2r\dot{r}(\dot{r} + \dot{y}_s)], \qquad \frac{\partial V}{\partial y_s} = Mg, \tag{22}$$

and the equation of motion is,

$$\frac{M}{R^2}[r^2(\ddot{r} + \ddot{y}_s) + 2r\dot{r}(\dot{r} + \dot{y}_s)] + Mg = \lambda_3. \tag{23}$$

We can now enforce the constraint (19), *i.e.*, that $\dot{y}_s = 0 = \ddot{y}_s$, in which case eqs. (6) and (8) hold. Then, the horizontal constraint force for the entire system is given by, for $r \gg b$,

$$\lambda_3 = \frac{Mr}{R^2}\left(2\dot{r}^2 + r\ddot{r}\right) + Mg \approx Mg - \frac{Mgb^2}{2\pi^2 R^2} = Mg - mg\frac{b^2}{2\pi^2 r^2}\,. \tag{24}$$

However, we are interested in the normal force N on the coil, whereas eq. (24) includes the normal force on the mass $M - m$ that lies flat on the horizontal surface. That is,

$$N = \lambda_3 - (M - m)g \approx mg\left(1 - \frac{b^2}{2\pi^2 r^2}\right) \approx mg\,. \tag{25}$$

Thus, the constraint force N is the part of the normal force $N_{\rm tot}$ previously found in eq. (10) that depends on g.

2.4.1.2 Horizontal Constraint Force

A Lagrangian analysis can provide information about the horizontal constraint force F of the horizontal support surface on the coil, if we relax the no-slip constraint (1), we which rewrite, with dimensions of length so that the corresponding Lagrange multiplier will have dimensions of force, as,

$$f_1(r, x) = x + \frac{\pi}{b}(r^2 - R^2) = 0, \tag{26}$$

and consider the entire system of the coil plus its unwound portion to be described by two coordinates, r and x.

Constraint (2) still holds, in the form that $\omega = -2\pi\dot{r}/b$. The unwound portion of the tape, of mass $M(1-r^2/R^2)$, now slides on the horizontal surface with velocity $\dot{x} - r\,\omega = \dot{x} + 2\pi r\dot{r}/b$.

Then, the kinetic energy of the system is,

$$\begin{aligned} T &= M\left(1 - \frac{r^2}{R^2}\right)\frac{(\dot{x} + 2\pi r\dot{r}/b)^2}{2} + \frac{m(\dot{r}^2 + \dot{x}^2)}{2} + \frac{I\omega^2}{2} \\ &= M\frac{\dot{x}^2 + r^2\dot{r}^2}{2} + \frac{2\pi M r\dot{r}\dot{x}}{b}\left(1 - \frac{r^2}{R^2}\right) + \frac{\pi^2 M r^2\dot{r}^2}{b^2}\left(2 - \frac{r^2}{R^2}\right). \end{aligned} \tag{27}$$

[6] Alternatively, we could deduce the equation for coordinate r, as only one of the two equations will be needed.

The potential V is again given by eq. (4).

We now deduce the equation of motion for coordinate x via eq. (18), including a single Lagrange multiplier λ_1 associated with f_1 of eq. (26).[7]

For coordinate x,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = M\ddot{x} + \frac{2\pi M r\ddot{r}}{b}\left(1 - \frac{r^2}{R^2}\right) + \frac{2\pi M \dot{r}^2}{b}\left(1 - \frac{3r^2}{R^2}\right), \qquad \frac{\partial V}{\partial x} = 0, \tag{28}$$

and the equation of motion is,

$$M\ddot{x} + \frac{2\pi M r\ddot{r}}{b}\left(1 - \frac{r^2}{R^2}\right) + \frac{2\pi M \dot{r}^2}{b}\left(1 - \frac{3r^2}{R^2}\right) = \lambda_1. \tag{29}$$

We can now enforce the constraint (26), *i.e.*, $\ddot{x} = -2\pi(r\ddot{r} + \dot{r}^2)/b$, in which case eqs. (6) and (8) hold. Then, the horizontal constraint force F is given by, for $r \gg b$,

$$\begin{aligned} F = \lambda_1 &= -\frac{2\pi M r^3 \ddot{r}}{bR^2} - \frac{6\pi M r^2 \dot{r}^2}{bR^2} \\ &\approx -\frac{2\pi M r^2}{bR^2}\left[\frac{b^2 g}{2\pi^2 r} + \frac{2g\, b^2 R^3}{3\pi^2 r^4}\left(1 - \frac{r^3}{R^3}\right)\right] - \frac{6\pi M r^2}{bR^2}\frac{g\, b^2 R^3}{3\pi^2 r^4}\left(1 - \frac{r^3}{R^3}\right) \\ &\approx \frac{Mgb}{3\pi r^2 R^2}\left(7r^3 - 10R^3\right), \end{aligned} \tag{30}$$

which differs from both eqs. (11) for $F_{\rm tot}$.

2.4 Lagrangian Analysis of the Coil Only

If we regard the system of interest as only the coil, then the problem is one of variable mass.

A Lagrangian approach to variable-mass problems has been given in [9, 10].[8]

This method considers the kinetic energy $T(q_k, \dot{q}_k, t)$ (but not the potential energy) of a system described by coordinates q_k, and supplements the generalized forces of Lagrange with additional terms, related to a so-called *control volume* whose velocity is $\mathbf{w}$, according to eq. (5.6) of [9] and eq. (1) of [10],

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{q}_k} - \frac{\partial T_w}{\partial q_k} + \int \frac{\partial \tilde{T}}{\partial \dot{q}_k}(\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} - \int \tilde{T}\frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{q}_k} \cdot d\mathbf{Area} = Q_k, \tag{31}$$

where T_w is the kinetic energy within the control volume, $\tilde{T}$ is the kinetic energy per unit volume, $\mathbf{v}$ is the velocity of the material at a point in the system, and the generalized forces Q_k are related to the external forces on the system by,[9]

$$Q_k = \sum_i \mathbf{F}_i^{\rm ext} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_i \mathbf{F}_i^{\rm ext} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \tag{32}$$

[7] Alternatively, we could deduce the equation for coordinate r.

[8] Use of this approach by the author appears in Appendix B of [14] to a leaky tank at rest (Torricelli's problem), in sec. 2.2 of [15] to a leaky bucket suspended from a spring, in Appendix A of [16] to a rolling water pipe, and in Appendix B of [17] to a leaky tank car.

[9] We recall that Lagrange's method distinguishes between external and constraint forces. In the present example, the forces at the line of contact of the coil with the horizontal surface are constraint forces, and so are not included in the computation of the generalized force.

supposing the system to consist of particles with mass m_i at positions $\mathbf{r}_i$ subject to external forces $\mathbf{F}_i^{\rm ext}$. The forms (32) followed from arguments by d'Alembert [13]. If the external forces are deducible from potentials, $\mathbf{F}_i = -\partial V_i/\partial \mathbf{r}_i$, then the first form of eq. (32) simplifies to

$$Q_k = -\frac{\partial V}{\partial q_k}\,, \tag{33}$$

where $V = \sum_i V_i$.

In the present example we take the variable-mass system to be the coil, which system can be characterized by a single coordinate, r = radius of the coil (as in the figure on p. 1).

The external force on the coil is $-mg\,\hat{\mathbf{y}} = Mgr^2\,\hat{\mathbf{y}}/R^2$, which can be related to a potential $V = mgr = Mgr^2/R^2$. Then, according to eq. (33), the generalized force is,[10]

$$Q_r = -\frac{\partial}{\partial r}\frac{Mgr^3}{R^2} \quad = \quad -\frac{3Mgr^2}{R^2} = -3mg. \tag{34}$$

However, if we use the first form of eq. (32), and note that the locations $\mathbf{r}_i$ of particles of the coil do not depend directly on coordinate r, we would have that $Q_r = 0$. If if we use the second form of eq. (32), the velocities of a particles in the coil with cylindrical coordinates (r'_i, θ_i, z_i) with respect to the axis of the coil have velocities $r'_i\dot\theta_i\,\hat{\boldsymbol{\theta}}_i = r'_i\dot\theta_i(-\cos\theta_i\,\hat{\mathbf{x}} + \sin\theta_i\,\hat{\mathbf{y}})$ where the y-axis is upwards, θ increases clockwise from the line of contact of the coil with the horizontal surface, and $\dot\theta_i = \dot\theta = -2\pi\dot r/b$, according to eq. (2). Then, in the sum of particles i the terms in $\cos\theta_i$ and $\sin\theta_i$ average to zero, and the generalized force is zero according to the second form of eq. (32). That is, use of either of the forms of eq. (32) for this example would imply that there is no dependence of the motion on the acceleration g due to gravity. So, we infer that we must use eq. (34) here.[11]

We take the surface of the control volume to be just outside the physical surface of the coil. The velocity of the control volume is then $\mathbf{w} = \dot{\mathbf{x}}$.

The kinetic energy within the control volume is, as in eq. (3),

$$T_w = \frac{Mr^2\dot r^2}{2R^2} + \frac{3\pi^2 Mr^4\dot r^2}{b^2R^2}\,. \tag{35}$$

There is no matter of the system on the surface of the control volume, except at its interface with the portion of the tape at rest on the horizontal surface. The velocity $\mathbf{v}$ of the matter is zero on this surface, as is the kinetic energy per unit volume $\tilde T$ there.

From the kinetic energy (35) we have,

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot r} - \frac{\partial T_w}{\partial r} = \frac{Mr}{R^2}(r\ddot r + 2\dot r^2) + \frac{6\pi^2 Mr^3}{b^2R^2}(4\dot r^2 + r\ddot r) - \frac{Mr\dot r^2}{R^2} - \frac{12\pi^2 Mr^3\dot r^2}{b^2R^2}\,. \tag{36}$$

Since $\tilde T = 0$ on the surface of the control volume, the integrals in $\tilde T$ do not contribute to the equation of motion.

[10] Note that the generalized force (34) does not equal the normal force (instantaneous weight, $\approx mg$) on the coil that was found in eq. (10).

[11] For another example in which use eq. (34) is preferred, see [14].

Altogether, the equation of motion for coordinate r, according to eq. (31), is,

$$\frac{Mr^2\ddot{r}}{b^2R^2}(6\pi^2r^2+b^2)+\frac{Mr\dot{r}^2}{b^2R^2}(12\pi^2r^2+b^2)=-\frac{3Mgr^2}{R^2}\,, \tag{37}$$

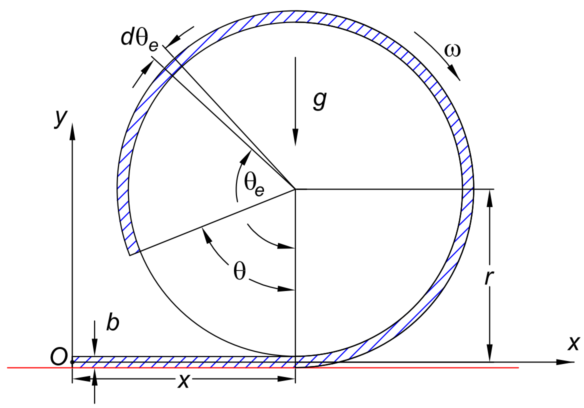
$$(6\pi^2r^2+b^2)r\ddot{r}+(12\pi^2r^2+b^2)\dot{r}^2+3b^2gr=0, \tag{38}$$

as found in eq. (7) above.[12]

A Appendix: Uncoiling of a Single Turn of Tape

The previous analysis made approximations related to the small thickness b of the tape, which may be correct only to order b/r. Since the equation of motion contains terms of order b^2, these approximations may not be sufficiently accurate.

Here, we consider a single turn of tape of mass M and thickness b, wrapped on a massless spool that rolls without slipping on a horizontal surface. The midline of the tape on the spool is at (constant) radius r. We take the origin of the x-y coordinate system to be at the midpoint of the left end of the tape (on the horizontal surface). The end of the tape still on the spool is at angle θ to the line from the center of the spool to the line of contact of the tape with the horizontal surface, as shown in the figure below.



The rolling constraint is that,

$$x=r\theta, \qquad \dot{x}=r\dot{\theta}, \tag{39}$$

where x is the horizontal coordinate of the line of contact of the spool with the horizontal surface. The coordinates of an element e of the tape of angular extent $d\theta_e$ on the spool at angle θ_e are,

$$x_e=r(\theta-\sin\theta_e), \qquad y_e=r(1-\cos\theta_e), \qquad \dot{x}_e=r\dot{\theta}(1-\cos\theta_e), \qquad \dot{y}_e=r\dot{\theta}\sin\theta_e, \tag{40}$$

noting that the angular velocity of all such elements is $\dot{\theta}_e=\dot{\theta}$. The kinetic energy of the tape is,

$$T \;=\; \int\frac{dm_e\,v_e^2}{2}=\int_\theta^{2\pi}M\frac{d\theta_e}{2\pi}\,\frac{\dot{x}_e^2+\dot{y}_e^2}{2}=\frac{Mr^2\dot{\theta}^2}{2\pi}\int_\theta^{2\pi}d\theta_e\,(1-\cos\theta_e)$$

[12]This problem was considered by a different "Lagrangian" method in sec. V.D of [18]. The Lagrangian of eq. (59) of [18] seems correct (and leads to eq. (7)), but eq. (60) displays a different equation of motion than either eqs. (7) or (17).

$$= \quad Mr^2\dot{\theta}^2\left(1-\frac{\theta-\sin\theta}{2\pi}\right). \tag{41}$$

and the potential energy with respect to the midline of the tape on the horizontal surface is,

$$V=\int dm_e\,gy_e=\int_\theta^{2\pi} M\frac{d\theta_e}{2\pi}gr(1-\cos\theta_e)=Mgr\left(1-\frac{\theta-\sin\theta}{2\pi}\right). \tag{42}$$

The constant energy is, for motion with initial angular velocity $\dot{\theta}_0$,

$$E=Mr^2\dot{\theta}_0^2+Mgr=T+V=M(r^2\dot{\theta}^2+gr)\left(1-\frac{\theta-\sin\theta}{2\pi}\right). \tag{43}$$

As $\theta\to 2\pi$, $\dot{\theta}\to\infty$, although when $\theta=2\pi$ the mechanical energy is zero, as noted in [1, 2].

Taking the time derivative of the energy (43), we find the equation of motion,

$$2Mr^2\dot{\theta}\ddot{\theta}\left(1-\frac{\theta-\sin\theta}{2\pi}\right)-M(r^2\dot{\theta}^2+gr)\dot{\theta}\frac{1-\cos\theta}{2\pi}=0, \tag{44}$$

$$2\ddot{\theta}\,(2\pi-\theta+\sin\theta)-\dot{\theta}^2(1-\cos\theta)=g\frac{1-\cos\theta}{r}\qquad(0<\theta<2\pi). \tag{45}$$

Note that the thickness b of the tape does not enter into the equation of motion (45), in contrast to the previous analyses.

A.1 Force and Torque Analysis

First, we consider the vertical-momentum equation for the entire system,

$$\begin{aligned} N-mg &= \frac{d}{dt}\int dm_e\,\dot{y}_e=\frac{d}{dt}\int_\theta^{2\pi}M\frac{d\theta_e}{2\pi}r\dot{\theta}\sin\theta_e=-\frac{d}{dt}M\frac{r\dot{\theta}}{2\pi}(1-\cos\theta) \\ &= -\frac{Mr}{2\pi}\left[\ddot{\theta}(1-\cos\theta)+\dot{\theta}^2\sin\theta\right], \end{aligned} \tag{46}$$

where N is the upward normal force at the line of contact, and $m=M(1-\theta/2\pi)$ is the mass of the tape on the spool. The net force (and torque) on the portion of the tape at rest is zero.

A.1.1 The Reference Point is the Origin

The torque about the origin, at the (fixed) left end of the tape, is,

$$\begin{aligned} \boldsymbol{\tau}_O &= x\,\hat{\mathbf{x}}\times N\,\hat{\mathbf{y}}+\int x_e\,\hat{\mathbf{x}}\times -dm_e g\,\hat{\mathbf{y}}=Nr\theta\,\hat{\mathbf{z}}-\int_\theta^{2\pi}r(\theta-\sin\theta_e)M\frac{d\theta_e}{2\pi}g\,\hat{\mathbf{z}} \\ &= (N-mg)r\theta\,\hat{\mathbf{z}}-\frac{Mgr}{2\pi}(1-\cos\theta)\,\hat{\mathbf{z}} \\ &= -\frac{Mr^2}{2\pi}\left[\theta\ddot{\theta}(1-\cos\theta)+\theta\dot{\theta}^2\sin\theta\right]\hat{\mathbf{z}}-\frac{Mgr}{2\pi}(1-\cos\theta)\,\hat{\mathbf{z}}. \end{aligned} \tag{47}$$

The angular momentum of the system about the origin is,

$$
\begin{aligned}
\mathbf{L}_O &= \int_{\text{coil}} (x_e\,\hat{\mathbf{x}} + y_e\,\hat{\mathbf{y}}) \times dm_e(\dot{x}_e\,\hat{\mathbf{x}} + \dot{y}_e\,\hat{\mathbf{y}}) = \int_\theta^{2\pi} M\frac{d\theta_e}{2\pi}(x_e\dot{y}_e - y_e\dot{x}_e)\,\hat{\mathbf{z}} \\
&= \frac{Mr^2\dot{\theta}}{2\pi}\int_\theta^{2\pi} d\theta_e\,\left[(\theta - \sin\theta_e)\sin\theta_e - (1-\cos\theta_e)^2\right]\,\hat{\mathbf{z}} \\
&= \frac{Mr^2\dot{\theta}}{2\pi}(-4\pi + \theta - 2\sin\theta + \theta\cos\theta)\,\hat{\mathbf{z}}. \qquad (48) \\
\frac{d\mathbf{L}_O}{dt} &= \frac{Mr^2\ddot{\theta}}{2\pi}(-4\pi + \theta - 2\sin\theta + \theta\cos\theta)\,\hat{\mathbf{z}} + \frac{Mr^2\dot{\theta}^2}{2\pi}(1 - \cos\theta - \theta\sin\theta)\,\hat{\mathbf{z}}. \qquad (49)
\end{aligned}
$$

The torque equation, $\boldsymbol{\tau}_O = d\mathbf{L}_O/dt$, then implies,

$$
2\ddot{\theta}\,(2\pi - \theta + \sin\theta) - \dot{\theta}^2(1-\cos\theta) = g\frac{1-\cos\theta}{r}\,, \qquad (50)
$$

which agrees with eq. (45) found via an energy analysis.

A.1.2 The Reference Point is on the Line of Contact

If we use $(x, 0) = (r\theta, 0)$ as the (accelerated) reference point, we must consider the torque associated with the "fictitious" coordinate force $-M\ddot{x}\,\hat{\mathbf{x}}$ on the center of mass of the entire tape, as well as that due to gravity.[13]

The coordinates of the center of mass of the entire tape are related by,

$$
\begin{aligned}
Mx_{\text{cm}} &= \int dm_e\,x_e = \frac{M\theta}{2\pi}\frac{x}{2} + \int_\theta^{2\pi}\frac{M\,d\theta_e}{2\pi}(x - r\sin\theta_e) \\
&= \frac{Mr\theta^2}{4\pi} + \frac{Mr}{2\pi}\int_\theta^{2\pi} d\theta_e(\theta - \sin\theta_e) = \frac{Mr\theta^2}{4\pi} + \frac{Mr}{2\pi}\left(2\pi\theta - \theta^2 + 1 - \cos\theta\right) \\
&= \frac{Mr}{2\pi}\left(2\pi\theta - \frac{\theta^2}{2} + 1 - \cos\theta\right), \qquad (51) \\
My_{\text{cm}} &= \int dm_e\,y_e = \frac{M\theta}{2\pi}0 + \int_\theta^{2\pi}\frac{M\,d\theta_e}{2\pi}(r - r\cos\theta_e) = \frac{Mr}{2\pi}(2\pi - \theta + \sin\theta). \qquad (52)
\end{aligned}
$$

The torque about $(x, 0)$ is, again noting that the net torque on the portion of the tape at rest is zero,

$$
\begin{aligned}
\boldsymbol{\tau}_x &= [(x_{\text{cm}} - x)\,\hat{\mathbf{x}} + y_{\text{cm}}\,\hat{\mathbf{y}}] \times -M\ddot{x}\,\hat{\mathbf{x}} + \int_{\text{coil}}[(x_e - x)\,\hat{\mathbf{x}} + y_e\,\hat{\mathbf{y}}] \times -dm_e\,g\,\hat{\mathbf{y}} \\
&= M\ddot{x}y_{\text{cm}}\,\hat{\mathbf{z}} - \int_\theta^{2\pi} Mg\frac{d\theta_e}{2\pi}(x_e - x)\,\hat{\mathbf{z}} \\
&= \frac{Mr^2\ddot{\theta}}{2\pi}(2\pi - \theta + \sin\theta)\,\hat{\mathbf{z}} + \frac{Mgr}{2\pi}\int_\theta^{2\pi} d\theta_e\,\sin\theta_e\,\hat{\mathbf{z}} \\
&= \frac{Mr^2\ddot{\theta}}{2\pi}(2\pi - \theta + \sin\theta)\,\hat{\mathbf{z}} - \frac{Mgr}{2\pi}(1 - \cos\theta)\,\hat{\mathbf{z}}. \qquad (53)
\end{aligned}
$$

[13]For another example in which the "fictitious torques" must be considered, see [24].

The angular momentum of the system about the moving point $(x, 0)$ is,

$$
\begin{aligned}
\mathbf{L}_x &= \int_{\text{coil}} [(x_e - x)\,\hat{\mathbf{x}} + y_e\,\hat{\mathbf{y}}] \times dm_e[(\dot{x}_e - \dot{x})\,\hat{\mathbf{x}} + \dot{y}_e\,\hat{\mathbf{y}}] \\
&= \int_\theta^{2\pi} M\frac{d\theta_e}{2\pi}[(x_e - x)\dot{y}_e - y_e(\dot{x}_e - \dot{x})]\,\hat{\mathbf{z}} \\
&= \frac{Mr^2\dot{\theta}}{2\pi}\int_\theta^{2\pi} d\theta_e\,\left[-\sin^2\theta_e + \cos\theta(1-\cos\theta_e)\right]\,\hat{\mathbf{z}} = \frac{Mr^2\dot{\theta}}{2\pi}\int_\theta^{2\pi} d\theta_e\,(-1+\cos\theta_e)\,\hat{\mathbf{z}} \\
&= \frac{Mr^2\dot{\theta}}{2\pi}(-2\pi + \theta - \sin\theta)\,\hat{\mathbf{z}}, \qquad (54) \\
\frac{d\mathbf{L}_x}{dt} &= \frac{Mr^2\ddot{\theta}}{2\pi}(-2\pi + \theta - \sin\theta)\,\hat{\mathbf{z}} + \frac{Mr^2\dot{\theta}^2}{2\pi}(1-\cos\theta)\,\hat{\mathbf{z}}. \qquad (55)
\end{aligned}
$$

The torque equation, $\boldsymbol{\tau}_x = d\mathbf{L}_x/dt$, then implies,

$$
2\ddot{\theta}\,(2\pi - \theta + \sin\theta) - \dot{\theta}^2(1-\cos\theta) = g\frac{1-\cos\theta}{r}\,, \qquad (56)
$$

which agrees with eq. (45) found via an energy analysis, and with eq. (50) via a torque analysis about the fixed origin.

A.2 Lagrangian Analysis

A.2.1 For the Entire Tape

The Lagrangian for the entire tape is $\mathcal{L} = T - V$ with the kinetic energy T and gravitational potential energy V given by eqs. (41)-(42). Since $\mathcal{L}$ does not depend explicitly on time, the energy $E = T + V$ is constant, and we expect the equation of motion from the Lagrangian analysis to be the same as eq. (45) from the energy analysis. Indeed,

$$
\begin{aligned}
\frac{2\pi M}{r^2}\mathcal{L} &= \left(\dot{\theta}^2 - \frac{g}{r}\right)(2\pi - \theta + \sin\theta)\,. \qquad (57) \\
0 &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 2\ddot{\theta}\,(2\pi - \theta + \sin\theta) + 2\dot{\theta}^2\,(-1+\cos\theta) - \left(\dot{\theta}^2 - \frac{g}{r}\right)(-1+\cos\theta) \\
&= 2\ddot{\theta}\,(2\pi - \theta + \sin\theta) - \dot{\theta}^2\,(1-\cos\theta) - \frac{g}{r}(1-\cos\theta), \qquad (58)
\end{aligned}
$$

as in eq. (45).

A.2.2 For the Variable-Mass Coil of Tape

We use the method of [9, 10], taking the control volume to be the coil of tape and the coordinate to be angle θ. According to eq. (1) of [10],

$$
\frac{d}{dt}\frac{\partial T_w}{\partial\dot{\theta}} - \frac{\partial T_w}{\partial\theta} + \int\frac{\partial\tilde{T}}{\partial\dot{\theta}}(\mathbf{v}-\mathbf{w})\cdot d\mathbf{Area} - \int\tilde{T}\frac{\partial(\mathbf{v}-\mathbf{w})}{\partial\dot{q}_k}\cdot d\mathbf{Area} = Q_\theta, \qquad (59)
$$

where T_w is the kinetic energy within the control volume, $\tilde{T}$ is the kinetic energy per unit volume, $\mathbf{v}$ is the velocity of the material at a point in the system, and since this system has a potential energy V, the generalized force is $Q_\theta = -\partial V/\partial\theta$.

The only material on the surface of the control volume is that at the interface between the coiled and horizontal portions of the tape, which material is at rest. Hence the kinetic energy $\tilde{T}$ per unit volume is zero on the surface of the control volume, so the integrals in eq. (59) vanish. Since the kinetic energy T_w in the control volume is T of eq. (41), eq. (59) reduces to,

$$\frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} = 0, \tag{60}$$

where $\mathcal{L} = T - V$ is the Lagrangian of the entire system. Therefore, use of eq. (59) also leads to the equation of motion eq. (45).[14]

B Appendix: Uncoiling on a Slope

We also consider the case that the coil rolls without slipping on a slope at angle α (which could be negative) with respect to the horizontal.

As sketched in the figure on the next page, we take the x-y coordinates parallel and perpendicular to the slope. The constraint equations (1)-(2) again apply.

---

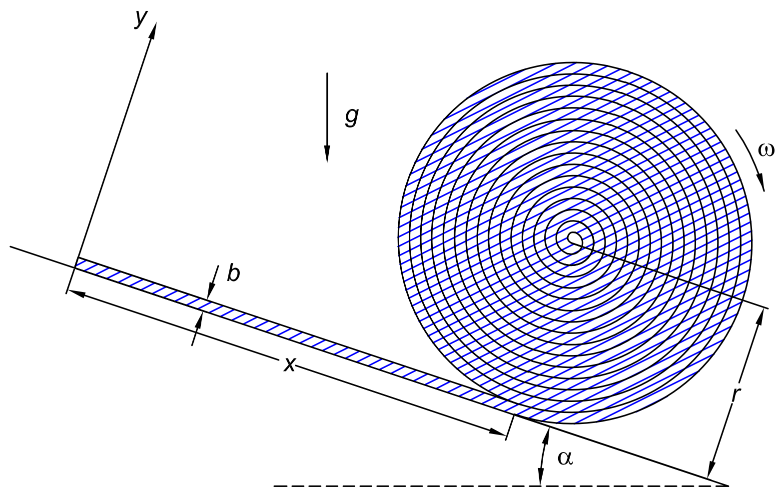
[14] If we consider the method of [18], in which partial derivatives of T do not act on the variable mass $m = M(1 - \theta/2\pi)$, we should rewrite the kinetic energy (41) as,

$$T = mr^2\dot\theta^2\left(1 + \frac{\sin\theta}{2\pi - \theta}\right). \tag{61}$$

Then, the equation of motion according to the method of [18] is,

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta} = \frac{d}{dt}\left[2mr^2\dot\theta\left(1 + \frac{\sin\theta}{2\pi-\theta}\right)\right] - mr^2\dot\theta^2\left(\frac{\cos\theta}{2\pi-\theta} + \frac{\sin\theta}{(2\pi-\theta)^2}\right)\\
&= \frac{d}{dt}\left[\frac{Mr^2\dot\theta}{\pi}(2\pi - \theta + \sin\theta)\right] - \frac{Mr^2\dot\theta^2}{2\pi}\left(\cos\theta + \frac{\sin\theta}{2\pi-\theta}\right)\\
&= \frac{Mr^2\ddot\theta}{\pi}(2\pi - \theta + \sin\theta) - \frac{Mr^2\dot\theta^2}{\pi}(1 - \cos\theta) - \frac{Mr^2\dot\theta^2}{2\pi}\left(\cos\theta + \frac{\sin\theta}{2\pi-\theta}\right)\\
&= \frac{Mr^2\ddot\theta}{\pi}(2\pi - \theta + \sin\theta) - \frac{Mr^2\dot\theta^2}{2\pi}\left(1 - \cos\theta + \frac{\sin\theta}{2\pi-\theta}\right)\\
&= Q_\theta = -\frac{\partial V}{\partial\theta} = \frac{Mgr}{2\pi}(1 - \cos\theta),\\
&2\ddot\theta(2\pi - \theta + \sin\theta) - \dot\theta^2\left(1 - \cos\theta + \frac{\sin\theta}{2\pi-\theta}\right) = \frac{g}{r}(1 - \cos\theta),
\end{aligned} \tag{62}$$

which differs from eq. (45) by the term $\dot\theta^2\sin\theta/(2\pi - \theta)$.



If the coil were released from rest, it would spontaneously roll downhill.[15]

The x-momentum equation is, similar to eq. (9),

$$N - mg\cos\alpha = \frac{dP_y}{dt} = \frac{d}{dt}(m\dot{r}) = \frac{d}{dt}\left(\frac{Mr^2\dot{r}}{R^2}\right) = \frac{Mr}{R^2}(2\dot{r}^2 + r\ddot{r}), \tag{63}$$

$$N = \frac{Mgr^2\cos\alpha}{R^2} + \frac{Mr}{R^2}(2\dot{r}^2 + r\ddot{r}). \tag{64}$$

The x-momentum equation is, similar to eq. (11), but now taking $F_{\rm tot}$ to point uphill,

$$mg\sin\alpha - F_{\rm tot} = \frac{dP_x}{dt} = \frac{d}{dt}(m\dot{x}) = -\frac{d}{dt}\left(\frac{2\pi Mr^3\dot{r}}{bR^2}\right) = -\frac{2\pi Mr^2}{bR^2}(3\dot{r}^2 + r\ddot{r}), \tag{65}$$

$$F_{\rm tot} = \frac{Mgr^2\sin\alpha}{R^2} + \frac{2\pi Mr^2}{bR^2}(3\dot{r}^2 + r\ddot{r}). \tag{66}$$

The angular-momentum $\mathbf{L}_O$ about the origin (at the upper end of the tape on the slope) is again related by eqs. (13)-(15), while the torque about the origin is now,

$$\begin{aligned}
\boldsymbol{\tau}_0 &= (x\,\hat{\mathbf{x}} + r\,\hat{\mathbf{y}}) \times mg(\sin\alpha\,\hat{\mathbf{x}} - \cos\alpha\,\hat{\mathbf{y}}) + (x\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}}) \times (F\,\hat{\mathbf{x}} + N\,\hat{\mathbf{y}}) \\
&= [(N - mg\cos\alpha)x - mgr\sin\alpha]\,\hat{\mathbf{z}} \\
&= \frac{M\pi r}{bR^2}(2\dot{r}^2 + r\ddot{r})(R^2 - r^2)\,\hat{\mathbf{z}} - \frac{Mr(R^2 - r^2)g\sin\alpha}{R^2}\,\hat{\mathbf{z}}.
\end{aligned} \tag{67}$$

Equating (15) and (67), we have,

$$r\ddot{r} + \frac{10}{3}\dot{r}^2 + \frac{gb\sin\alpha}{3\pi} = 0. \tag{68}$$

For an energy analysis, we note that the kinetic energy is again given by eq. (3), while the gravitational potential energy of the system with respect to origin is,

$$V = \frac{Mgr^3\cos\alpha}{R^2} - \frac{\pi Mg(R^4 - r^4)\sin\alpha}{2bR^2}\,. \tag{69}$$

[15]If the coil were given a large enough initial uphill velocity, it could roll uphill, and have its upper end at final position $x = -nl$ for $\alpha > 0$ or $x = l$ for $\alpha < 0$, where $l = \pi R^2/b$ is the length of the tape.

Unlike the case of uncoiling on a horizontal surface, the potential energy of the uncoiled tape (at rest on the slope) is significant, and for $\sin\alpha \gtrsim b/R$, the second term in eq. (69) dominates. The constant energy (prior to the final "big bang") is,

$$E = MgR\cos\alpha = T + V = \frac{Mr^2\dot{r}^2}{2R^2} + \frac{3\pi^2 Mr^4\dot{r}^2}{b^2R^2} + \frac{Mgr^3\cos\alpha}{R^2} - \frac{\pi Mg(R^4 - r^4)\sin\alpha}{2bR^2}. \quad (70)$$

The first term on the right side of eq. (70) is small compared to the second for $r \gg b$, such that,

$$\begin{aligned} \dot{r}^2 &= \frac{2g\,b^2R^3}{6\pi^2r^4 + b^2r^2}\left(1 - \frac{r^3}{R^3}\right)\cos\alpha + \frac{\pi gb(R^4 - r^4)\sin\alpha}{6\pi^2r^4 + b^2r^2} \\ &\approx \frac{g\,b^2R^3}{3\pi^2r^4}\left(1 - \frac{r^3}{R^3}\right) + \frac{\pi g(R^4 - r^4)\sin\alpha}{br^2}, \end{aligned} \quad (71)$$

which indicates that $\dot{r}$ (and $\dot{x}$) diverge as r goes to zero, as mentioned above.

We can differentiate eq. (70) with respect to time to obtain an equation of motion,

$$(6\pi^2r^2 + b^2)r\ddot{r} + (12\pi^2r^2 + b^2)\dot{r}^2 + 3b^2gr\cos\alpha + 4\pi gbr^2\sin\alpha = 0, \quad (72)$$

which for $r \gg b$ is approximately,

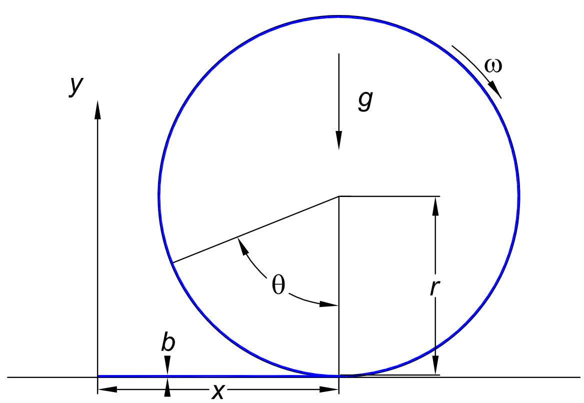
$$r\ddot{r} + 2\dot{r}^2 + \frac{2gb\sin\alpha}{3\pi} = 0, \quad (73)$$

which differs considerably from eq. (68). Furthermore,

$$\begin{aligned} r\ddot{r} &= -\frac{3b^2gr\cos\alpha}{6\pi^2r^2 + b^2} - \frac{12\pi^2r^2 + b^2}{6\pi^2r^2 + b^2}\dot{r}^2 - \frac{4\pi gbr^2\sin\alpha}{6\pi^2r^2 + b^2} \\ &\approx -\frac{b^2g\cos\alpha}{2\pi^2r} - \frac{2g\,b^2R^3}{3\pi^2r^4}\left(1 - \frac{r^3}{R^3}\right) - \frac{2\pi g(R^4 - r^4)\sin\alpha}{br^2} < 0. \end{aligned}$$

C Appendix: Uncoiling of a Thin Tape from a Spool

As another variant, whose equation of motion is integrable, we consider the case of a very thin tape (of mass M, length l and thickness b), with many turns initially wrapped around a massless spool (of radius $r \gg b$), as sketched below.[16]



[16]This variant was discussed in [2].

The no-slip condition is that the length x of tape at rest on the horizontal support surface and the angle θ through which the spool has rotated are related by,

$$x = r\theta. \tag{74}$$

Angle θ increases in a clockwise sense. Initially, x and θ are zero, and the coil is given a tiny push to start it moving to the right with initial angular velocity $\dot{\theta}_0$.

C.1 Energy Analysis

We analyze the motion in the approximation that mechanical energy is conserved.

We also adopt the approximation used in [2] that the length L of the tape is large compared to $2\pi r$, such that the center of mass of the tape on the spool is well approximated as being at the center of the spool.

When the coil has rotated through angle θ, the mass m of the tape remaining on the spool is,

$$m = M\left(1 - \frac{r\theta}{l}\right), \tag{75}$$

and the moment of inertia about its axis is mr^2. The center of the spool is at $(x, y) = (r\theta, r)$, the kinetic energy T is, noting that the moment of inertia I of the coil is $Mr^2(1 - r\theta/l)$,

$$T = T_{\rm cm} + T_{\rm rel\ to\ cm} = \frac{m\dot{x}^2}{2} + \frac{I\dot{\theta}^2}{2} = mr^2\dot{\theta}^2 = Mr^2\dot{\theta}^2\left(1 - \frac{r\theta}{l}\right), \tag{76}$$

the gravitational potential energy with respect to the horizontal surface is,

$$V = mgr = Mgr\left(1 - \frac{r\theta}{l}\right), \tag{77}$$

and the constant energy is,

$$E = Mr^2\dot{\theta}_0^2 + Mgr = T + V = Mr^2\dot{\theta}^2\left(1 - \frac{r\theta}{l}\right) + Mgr\left(1 - \frac{r\theta}{l}\right). \tag{78}$$

The equation of motion for θ is obtained by differentiating eq. (78) with respect to time,

$$2\ddot{\theta}\left(1 - \frac{r\theta}{l}\right) - \frac{r\dot{\theta}^2}{l} = \frac{g}{l}. \tag{79}$$

However, we learn more about the motion by rewriting eq. (78) as,

$$\dot{\theta}\sqrt{1 - r\theta/l} = \sqrt{\frac{g\theta}{l} + \dot{\theta}^2} \approx \sqrt{\frac{g}{r}}\sqrt{\frac{r\theta}{l}}, \qquad \dot{u}\sqrt{1-u} = \sqrt{\frac{g}{r}}\sqrt{u}, \qquad \dot{s}\sqrt{1-s^2} = \frac{1}{2}\sqrt{\frac{g}{r}}, \tag{80}$$

where the approximation is for small $\dot{\theta}_0$, and we define $u = r\theta/l$ to obtain the second form, and then $s = \sqrt{u} = \sqrt{r\theta/l}$ to obtain the third form. Note that the uncoiling occurs on the interval $0 \le s \le 1$.

The first form of eq. (80) indicates that $\dot\theta \to \infty$ as $\theta \to l/r = \theta_{\rm uncoiled}$, when the end of the tape strikes the horizontal surface with a "bang", as noted in [2].

The third form of eq. (80) can be integrated using Dwight 350.01 to give time t as a function of angle θ,

$$t = 2\sqrt{\frac{r}{g}}\int_0^{\sqrt{r\theta/l}} ds\,\sqrt{1-s^2} = \sqrt{\frac{r}{g}}\left(\sqrt{\frac{r\theta}{l}}\sqrt{1-\frac{r\theta}{l}} + \sin^{-1}\sqrt{\frac{r\theta}{l}}\right). \tag{81}$$

The time for the tape to become completely uncoiled, starting from rest, is,[17]

$$t_{\rm uncoiled} = \sqrt{\frac{r}{g}}\sin^{-1} 1 = \frac{\pi}{2}\sqrt{\frac{r}{g}}\,. \tag{82}$$

For later use, we note that from eq. (78) we have that,

$$r\dot\theta^2\left(1-\frac{r\theta}{l}\right) = \frac{gr\theta}{l}\,, \tag{83}$$

and then from eq. (79),

$$2\ddot\theta\left(1-\frac{r\theta}{l}\right)^2 = \frac{g}{l}\,. \tag{84}$$

C.2 Constraint Forces

C.2.1 Newtonian Approach

There must be a horizontal force $F_{\rm tot} = F + F_{\rm int}$ and a vertical (normal) force N at the line of contact of the coil with the horizontal surface, to maintain the constraint (74). As before, F is the horizontal force of the support surface on the coil, and $F_{\rm int}$ is the horizontal, internal force between the horizontal portion of the tape and the portion still in the coil.

We can compute the vertical constraint force on the coil via the vertical momentum equation,

$$N - mg = \frac{dP_y}{dt} = \frac{d}{dt}(m\dot r) = 0, \qquad N = mg = Mg\left(1-\frac{r\theta}{l}\right). \tag{85}$$

We can compute the total, horizontal force $F_{\rm tot}$ via the horizontal momentum equation, also using eqs. (83) and (84),

$$\begin{aligned} F_{\rm tot} &= \frac{dP_x}{dt} = \frac{d}{dt}(m\dot x) = \frac{d}{dt}\left[M\left(1-\frac{r\theta}{l}\right)r\dot\theta\right] = M\left(1-\frac{r\theta}{l}\right)r\ddot\theta - \frac{Mr^2\dot\theta^2}{l} \\ &= \frac{Mgr}{2(l-r\theta)} - \frac{Mgr^2}{l^2} \end{aligned} \tag{86}$$

This force is always in the $+x$ direction.

[17] The time (82) is 1/4 of the period of a simple pendulum of length r.

Or, we could compute the horizontal force $F_{\rm tot}$ via the angular-momentum equation, using the axis of the coil as the reference,

$$F_{\rm tot} = \frac{1}{r}\frac{dL}{dt} = \frac{1}{r}\frac{d}{dt}(I\omega) = \frac{d}{dt}\left[Mr\left(1 - \frac{r\theta}{l}\right)\dot\theta\right], \tag{87}$$

as in eq. (86).

C.2.2 Lagrangian Approach

As discussed in sec. 2.3 above, we can also compute the constraint forces via a Lagrangian method in which a constraint is relaxed, introducing another coordinate, and terms involving Lagrange multiplies in the equation of motion.

We first consider the vertical constraint force, associated with the constraint,

$$f_2 = y_s = 0, \tag{88}$$

where y_s is the vertical coordinate of the horizontal support surface. The constraint (74) still holds.

The kinetic energy of the system is now, recalling eq. (76),

$$T = \frac{M\dot y_s^2}{2} + Mr^2\dot\theta^2\left(1 - \frac{r\theta}{l}\right), \tag{89}$$

and the gravitational potential energy with respect to the horizontal surface is,

$$V = (M - m)gy_s + mg(r + y_s) = Mgy_s + Mgr\left(1 - \frac{r\theta}{l}\right), \tag{90}$$

The equation of motion for coordinate y_s includes the multiplier λ_2,

$$\lambda_2\frac{\partial f_2}{\partial y_s} = \lambda_2 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot y_s} - \frac{\partial\mathcal{L}}{\partial y_s} = M\ddot y_s + Mg. \tag{91}$$

We can now enforce the constraint (88), *i.e.*, that $\ddot y_s = 0$, to find that the vertical constraint (normal) force is,

$$\lambda_2 = Mg. \tag{92}$$

However, this normal force includes the force on the mass $Mr\theta/l$ that lies at rest on the horizontal surface, so the normal force N on the line of contact with the coil is,

$$N = \lambda_2 - \frac{Mgr\theta}{l} = Mg\left(1 - \frac{r\theta}{l}\right) = mg, \tag{93}$$

as in eq. (85).

C.3 Torque Analysis

For a Newtonian analysis, we note that the horizontal surface exerts a (constraint) force on its line of contact with the coil, which has horizontal component F (positive to the right) and vertical component N. We now consider the system to be just the spool and the tape thereon, which is a variable-mass system.

The horizontal equation of motion is,

$$F = \frac{dp_{\mathrm{coil},x}}{dt} = \frac{d(m\dot{x})}{dt} = \frac{d}{dt}\left[Mr\dot{\theta}\left(1 - \frac{r\theta}{l}\right)\right] = Mr\ddot{\theta}\left(1 - \frac{r\theta}{l}\right) - \frac{Mr^2\dot{\theta}^2}{l}. \tag{94}$$

For a second equation involving F (but not the unknown normal force N), we consider the torque τ and angular momentum $I\omega$ about the center of the spool,[18]

$$\tau = -rF = \frac{d(I\omega)}{dt} = \frac{d}{dt}\left[Mr^2\dot{\theta}\left(1 - \frac{r\theta}{l}\right)\right]. \tag{95}$$

After dividing by the constant r, this equation is the negative of eq. (94), which could only be consistent if the tape cannot unwind!

C.4 A Lagrangian Analysis of the Variable-Mass Coil

We apply the Lagrangian method of [9, 10]) to the system of the variable-mass coil, taking the control volume to be just outside the coil, and the single generalized coordinate to be angle θ.

Since the matter on the surface of the control volume (at the line of contact with the horizontal surface) is at rest, the kinetic energy $\tilde{T}$ per unit volume on that surface is zero, so the general equation (31) simplifies to,

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{\theta}} - \frac{\partial T_w}{\partial \theta} = Q_\theta. \tag{96}$$

The kinetic energy T_w inside the control volume is given by eq. (76), such that,

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{\theta}} - \frac{\partial T_w}{\partial \theta} = 2Mr^2\ddot{\theta}\left(1 - \frac{r\theta}{l}\right) - \frac{Mr^3\dot{\theta}^2}{L}. \tag{97}$$

The (vertical) external force (due to gravity) on the coil is associated with a potential, eq. (77), so the generalized force is,[19]

$$Q_\theta = -\frac{\partial V}{\partial \theta} = \frac{Mgr^2}{l}. \tag{98}$$

[18]This reference point is accelerated, and in general, torque analyses with an accelerated reference point must include a "fictitious", coordinate force, $-m\ddot{\mathbf{x}}_{\mathrm{cm}}$, that acts on the center of mass. But, when the reference point is the center of mass, this "fictitious" force exerts no torque.

[19]In the present approximation, the vertical coordinates of points on the coil do not depend on θ, so the generalized force would be zero according to the first form of eq. (32). While the velocities of these points do depend on $\dot{x} = r\dot{\theta}$, the vertical component of the velocity of point i varies as $\sin\theta_i$, which averages to zero around the entire coil (in our present approximation), so the generalized force would also be zero according to the second form of eq. (32). That is, the form (33) must be used for the generalized force in this example.

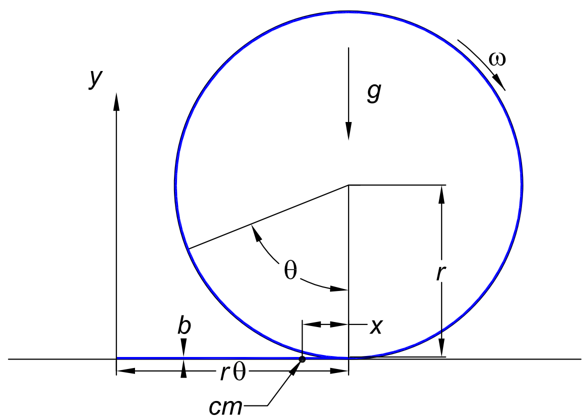
Hence, the equation of motion according to eq. (96) is,

$$2\ddot{\theta}\left(1-\frac{r\theta}{l}\right)-\frac{r\dot{\theta}^2}{l}=\frac{g}{l}\,, \tag{99}$$

as previously found in eq. (79).

D Appendix: Uncoiling without Friction

As yet another variant, we consider uncoiling of a thin tape of mass M and length l from a massless spool of radius r, but with no friction between the tape and the horizontal support surface, as sketched below. Then, there is no net horizontal force on the entire system, so the x-coordinate of the center of mass remains fixed at, say, $x = 0$, if the system starts from rest, and is given a small initial angular velocity $\dot{\theta}_0$ at $t = 0$.[20]



The rotation of the coil leads to tape being "ejected" in the $-x$ direction from the coil at its line of contact with the horizontal surface. This "rocket propulsion" results in the coil being accelerated in the $+x$ direction. For this to be possible, the tape must be under compression along the line of contact with the horizontal surface, which is an unusual state for a tape, which typically is considered to support tension but not compression.[21]

The length of the tape on the horizontal surface is $r\theta$, and its center has x-coordinate $x_{\rm horiz} = x - r\theta/2$, where x is the coordinate of the center of the coil (and of the line of contact). The mass of the horizontal tape is $Mr\theta/l$ and the mass of the coil is $M(1 - r\theta/l)$. The center of mass of the whole system is at $x = 0$, which implies that,

$$0=\frac{r\theta}{l}\left(x-\frac{r\theta}{2}\right)+\left(1-\frac{r\theta}{l}\right)x, \qquad x=\frac{r^2\theta^2}{2l}\,, \qquad \dot{x}=\frac{r^2\theta\dot{\theta}}{l}\,, \qquad \ddot{x}=\frac{r^2(\theta\ddot{\theta}+\dot{\theta}^2)}{l}\,. \tag{100}$$

When the coil has completely unwound, $r\theta = l$, and $x = l/2$ is the coordinate of the right end of the tape (whose center is, of course, then at $x = 0$) at rest on the horizontal surface.

[20] If the coil were given only a small initial velocity $\dot{x}_0$, then it would simply slide without rotating at the constant velocity $\dot{x}_0$.

[21] A possible scenario is that the ejected tape "bunches up" close to the initial line of contact, such that the center of the spool can remain fixed as the coil rotates. This "bunching" could be avoided it the tape has some stiffness against bending, but then some energy would be stored in the initial bending of the tape around the spool, and the energy analysis would need to be modified.

As in the previous analyses, it seems reasonable to consider the approximation that mechanical energy (kinetic plus gravitational potential energy) is conserved during the motion, and that we ignore elastic energy. The kinetic energy of the system is,

$$\begin{aligned} T &= T_{\rm horiz} + T_{\rm coil} = M\frac{r\theta}{l}\frac{\dot{x}_{\rm horiz}^2}{2} + M\left(1 - \frac{r\theta}{l}\right)\frac{\dot{x}^2}{2} + M\left(1 - \frac{r\theta}{l}\right) r^2 \frac{\dot{\theta}^2}{2} \\ &= M\frac{r\theta}{2l}\left(\dot{x} - \frac{r\dot{\theta}}{2}\right)^2 + M\left(1 - \frac{r\theta}{l}\right)\frac{r^4\theta^2\dot{\theta}^2}{2l^2} + M\left(1 - \frac{r\theta}{l}\right) r^2 \frac{\dot{\theta}^2}{2} \\ &= M\frac{r^3\theta\dot{\theta}^2}{8l^3}(2r\theta - l)^2 + M\frac{r^2\dot{\theta}^2}{2l^3}(l - r\theta)\left(l^2 + r^2\theta^2\right) = M\frac{r^2\dot{\theta}^2(4l - 3r\theta)}{8l}. \end{aligned} \tag{101}$$

The gravitational potential energy with respect to the horizontal surface is,

$$V = mgr = Mgr\left(1 - \frac{r\theta}{l}\right), \tag{102}$$

and the constant energy is,

$$E = \frac{Mr^2\dot{\theta}_0^2}{2} + Mgr = T + V = M\frac{r^2\dot{\theta}^2(4l - 3r\theta)}{8l} + Mgr\left(1 - \frac{r\theta}{l}\right), \tag{103}$$

$$\dot{\theta}^2(4l - 3r\theta) = 8g\theta + 4l\dot{\theta}_0^2. \tag{104}$$

The equation of motion for θ is obtained by differentiating eq. (104) with respect to time,

$$\ddot{\theta}(4l - 3r\theta)^2 = 16gl + 6rl\dot{\theta}_0^2. \tag{105}$$

According to eq. (104), the angular velocity $\dot{\theta}_f$ as $\theta \to \theta_f = l/r$ is given by $\dot{\theta}_f^2 = 8g/r$, while according to eq (100), $\dot{x} \to \dot{x}_f = r\dot{\theta}_f$, but the center of the horizontal tape has velocity $\dot{x}_{\rm horiz} \to \dot{x}_f - r\dot{\theta}_f/2 = r\dot{\theta}/2$. And, according to eq. (103) the system energy as $\theta \to \theta_f$ becomes concentrated in the kinetic energy of sliding horizontal tape, $T \to T_f = Mr\theta_f\dot{\theta}_f^2/8l = Mgr$. These results are somewhat inconsistent, and may indicate the inadequacy of our model for the last few turns, when the center of mass of the tape on the spool is not longer well approximated as being at the center of the spool.[22]

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[22]To clarify this, we also consider the case of a tape of mass M and thickness b, coiled as in the figure on p. 1, but with no friction at the horizontal support surface.

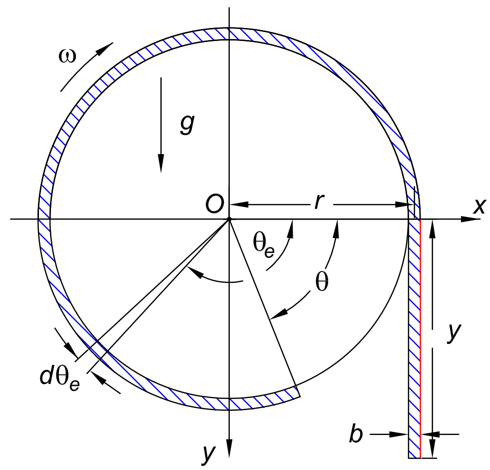
Here, we use the radius r of the coil as the coordinate, where $r = R$ initially. The mass of the coil of radius r is Mr^2/R^2, such that the mass of the tape on the horizontal surface is $M(1 - r^2/R^2)$ and its length is $l_{\rm horiz} = \pi(R^2 - r^2)/b$. The center of mass of the whole system is at $x = 0$, which implies that when the line of contact of the coil with the horizontal surface is at x, and the center of mass of the horizontal tape is at $x_{\rm horiz} = x - l_{\rm horiz}/2 = x - \pi(R^2 - r^2)/2b$,

$$0 = \left(1 - \frac{r^2}{R^2}\right)\left(x - \frac{l_{\rm horiz}}{2}\right) + \frac{r^2}{R^2}x, \qquad x = \frac{\pi(R^2 - r^2)^2}{2bR^2}, \qquad \dot{x} = -\frac{2\pi r\dot{r}(R^2 - r^2)}{bR^2}, \tag{106}$$

$$\dot{x}_{\rm horiz} = \dot{x} + \frac{\pi r\dot{r}}{b} = -\frac{\pi r\dot{r}(R^2 - 2r^2)}{bR^2}. \tag{107}$$

E Appendix: Uncoiling Vertically

In this Appendix we consider a tape of mass M and thickness b on a massless spool of radius $r - b/2$ with a fixed axis, such that the tape uncoils vertically, without friction. For simplicity, we suppose the tape has length $2\pi r$, *i.e.*it initially forms only a single turn on the spool.



As in eq. (2), the angular velocity of the unwinding coil is related by,

$$\dot{\theta} = \omega = \frac{\dot{x}}{r} = -\frac{2\pi \dot{r}(R^2 - r^2)}{bR^2}. \tag{108}$$

As in the previous analyses, it seems reasonable to consider the approximation that mechanical energy is conserved during the motion. The kinetic energy of the system is,

$$\begin{aligned} T &= T_{\rm horiz} + T_{\rm coil} = M\left(1 - \frac{r^2}{R^2}\right)\frac{\dot{x}_{\rm horiz}^2}{2} + M\frac{r^2}{R^2}\frac{\dot{x}^2}{2} + M\frac{r^2}{R^2}r^2\frac{\dot{\theta}^2}{2} \\ &= M\left(1 - \frac{r^2}{R^2}\right)\frac{\pi^2 r^2 \dot{r}^2 (R^2 - 2r^2)^2}{2b^2 R^4} + M\frac{r^2}{R^2}\frac{4\pi^2 r^2 \dot{r}^2 (R^2 - r^2)^2}{b^2 R^4} \\ &= M\left(1 - \frac{r^2}{R^2}\right)\frac{\pi^2 r^2 \dot{r}^2}{2b^2 R^4}(R^4 + 4r^2 R^2 - 4r^4). \end{aligned} \tag{109}$$

The gravitational potential energy with respect to the horizontal surface is, as in eq. (4),

$$V = mgr = \frac{Mgr^3}{R^2}, \tag{110}$$

and the constant energy is, for motion with initial angular velocity $\dot{\theta}_0$,

$$E = \frac{Mr^2\dot{\theta}_0^2}{2} + Mgr = T + V = M\left(1 - \frac{r^2}{R^2}\right)\frac{\pi^2 r^2 \dot{r}^2}{2b^2 R^4}(R^4 + 4r^2 R^2 - 4r^4) + \frac{Mgr^3}{R^2}, \tag{111}$$

$$\dot{r}^2(R^4 + 4r^2R^2 - 4r^4) = \frac{2gb^2R^4}{\pi^2 r} + \frac{b^2 R^6 \dot{\theta}_0^2}{\pi^2(R^2 - r^2)}. \tag{112}$$

As $r \to 0$, $\dot{r}$ diverges, but $r\dot{r} \to 0$, such that $\dot{x} \to 0$ and $\dot{x}_{\rm horiz} \to 0$, which indicates that the awkward limiting behavior of $\dot{x}$ and $\dot{x}_{\rm horiz}$ for a thin coil wrapped on a massless spool was an artifact of the poor approximation that the center of mass of the coil remained at the center of the spool at all times.

E.1 Energy Analysis

We take the origin to be on the axis of the spool, the x-axis to the horizontal and to the right, and the y-axis to be vertical and downwards. The end of the tape on the spool makes angle θ to the x-axis, as shown in the figure. The vertical section of the tape has length $y = r\theta$.

All points on the tape have velocity $r\dot{\theta}$, so the kinetic energy is,

$$T = \frac{Mr^2\dot{\theta}^2}{2} = T_{\rm coil} + T_{\rm vert} = M\left(1 - \frac{\theta}{2\pi}\right)\frac{r^2\dot{\theta}^2}{2} + M\frac{\theta}{2\pi}\frac{r^2\dot{\theta}^2}{2}\,. \tag{113}$$

The gravitational potential energy is,

$$\begin{aligned} V &= V_{\rm coil} + V_{\rm vert} = -\int_{\rm coil} dm_e\, g y_e - M\frac{\theta}{2\pi}g\frac{y}{2} = -\int_\theta^{2\pi} M\frac{d\theta_e}{2\pi}gr\sin\theta - \frac{Mgr\theta^2}{4\pi} \\ &= -\frac{Mgr}{2\pi}(1-\cos\theta) - \frac{Mgr\theta^2}{4\pi}\,. \end{aligned} \tag{114}$$

The total energy $E = T + V$ is constant, and zero if the system starts from rest. In that case, the tape has velocity $v + \sqrt{2\pi gr}$ as it leaves the spool. However, the tape leaves the spool in a finite time only if the initial angular velocity is nonzero.

Taking the time derivative of the constant energy, we obtain the equation of motion,

$$\ddot{\theta} = \frac{g}{2\pi r}(\theta - \sin\theta). \tag{115}$$

E.2 Torque Analysis

Taking the reference point on the fixed axis of the spool, the torque on the tape is

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{\tau}_{\rm coil} + \boldsymbol{\tau}_{\rm vert} = \int_{\rm coil} \mathbf{r}_e \times dm_e\, g\,\hat{\mathbf{y}} + r\,\hat{\mathbf{x}} \times M\frac{\theta}{2\pi}g\,\hat{\mathbf{y}} \\ &= \int_\theta^{2\pi} r(\hat{\mathbf{x}}\cos\theta_e + \hat{\mathbf{y}}\sin\theta_e) \times M\frac{d\theta_e}{2\pi}g\,\hat{\mathbf{y}} + \frac{Mgr\theta}{2\pi}\,\hat{\mathbf{z}} = \frac{Mgr(\theta - \sin\theta)}{2\pi}\,\hat{\mathbf{z}}. \end{aligned} \tag{116}$$

The angular momentum of the system is simply,

$$\mathbf{L} = Mr^2\dot{\theta}\,\hat{\mathbf{z}}, \qquad \frac{d\mathbf{L}}{dt} = Mr^2\ddot{\theta}\,\hat{\mathbf{z}}, \tag{117}$$

so the torque equation $\boldsymbol{\tau} = d\mathbf{L}/dt$ also leads to the equation of motion (115).

E.3 Lagrangian Analysis

E.3.1 For the Entire Tape

The Lagrangian for the entire tape is $\mathcal{L} = T - V$, using T and V from eqs. (113)-(114) and the coordinate as θ. Then, Lagrange's equation leads to the equation of motion (115).

E.3.2 For the Variable-Mass Coil

We make a Lagrangian analysis for the variable-mass coil using the method of [9, 10], taking the control volume to surround the coil. The velocity $\mathbf{w}$ of the control volume is 0, and the only mass on the surface of the control volume is at the interface between the coil and the vertical portion of the tape.

The kinetic energy of the mass within the control volume is,

$$T_w = T_{\rm coil} = M\left(1 - \frac{\theta}{2\pi}\right)\frac{r^2\dot{\theta}^2}{2}\,, \tag{118}$$

$$\frac{d}{dt}\frac{\partial T_w}{\partial\dot{\theta}} - \frac{\partial T_w}{\partial\theta} = Mr^2\ddot{\theta}\left(1 - \frac{\theta}{2\pi}\right) - \frac{Mr^2\dot{\theta}^2}{2\pi} + \frac{Mr^2\dot{\theta}^2}{4\pi} = Mr^2\ddot{\theta}\left(1 - \frac{\theta}{2\pi}\right) - \frac{Mr^2\dot{\theta}^2}{4\pi}\,. \tag{119}$$

The generalized force associated with the coil includes the generalized force of gravity on the coil, and the generalized force of the vertical tape on the coil. The former is $-\partial V_{\rm coil}/\partial\theta$ and the latter is $-\partial V_{\rm vert}/\partial\theta$. Altogether

$$Q_\theta = -\frac{\partial V}{\partial\theta} = \frac{Mgr(\theta - \sin\theta)}{2\pi}\,. \tag{120}$$

The matter on the surface of the control volume, at the interface of the coil with the vertical tape, has velocity $\mathbf{v} = r\dot{\theta}\,\hat{\mathbf{y}}$. The product of the kinetic energy $\tilde{T}$ per unit volume on the surface of the control volume and the area element $d\mathbf{Area}$ is,

$$\tilde{T}\,d\mathbf{Area} = \frac{1}{2}\frac{M}{2\pi r}r^2\dot{\theta}^2\,\hat{\mathbf{y}} = \frac{Mr\dot{\theta}^2}{4\pi}\,\hat{\mathbf{y}}, \qquad \frac{\partial\tilde{T}}{\partial\dot{\theta}}\,d\mathbf{Area} = \frac{Mr\dot{\theta}}{2\pi}\,\hat{\mathbf{y}}. \tag{121}$$

Also, at the surface of the control volume where there is matter,

$$\mathbf{v} - \mathbf{w} = r\dot{\theta}\,\hat{\mathbf{y}}, \qquad \frac{\partial(\mathbf{v} - \mathbf{w})}{\partial\dot{\theta}} = r\,\hat{\mathbf{y}}. \tag{122}$$

Then, the equation of motion according to eq. (31) is,

$$Mr^2\ddot{\theta}\left(1 - \frac{\theta}{2\pi}\right) - \frac{Mr^2\dot{\theta}^2}{4\pi} + \frac{Mr\dot{\theta}}{2\pi}\,\hat{\mathbf{y}}\cdot r\dot{\theta}\,\hat{\mathbf{y}} - \frac{Mr\dot{\theta}^2}{4\pi}\,\hat{\mathbf{y}}\cdot r\,\hat{\mathbf{y}} = \frac{Mgr(\theta - \sin\theta)}{2\pi}\,, \tag{123}$$

$$\ddot{\theta}\left(1 - \frac{\theta}{2\pi}\right) = \frac{g}{2\pi r}(\theta - \sin\theta). \tag{124}$$

It seems that the method of [9, 10] does not work well for the variable-mass coil. A dfficulty is that use of a kinetic energy of only part of the system, as in T_w, with a Lagrange-like computation, $(d/dt)(\partial T_w/\partial\dot{\theta}) - \partial T_w/\partial\theta$, will not yield the correct term in $\ddot{\theta}$ in the equation of motion (115). This comment applies also the the method proposed in [18].

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