Tuval's Electromagnetic Spaceship

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1 Problem

In two recent e-prints, Tuval and Yahalom proposed "electromagnetic spaceships" based on the net force exerted on a pair of time-dependent current loops [1], or a time dependent current loop and a permanent magnet [2].

Show that rocket propulsion (net gain of mechanical momentum of an isolated system) is possible in the first example, but not in the second.

2 Solution

2.1 Development of the Concept of Electromagnetic Field Momentum

2.1.1 Ampère

Ampère argued [3] that the static forces on a pair of steady electrical currents are equal and opposite. He had no concept of electromagnetic (field) momentum.[1]

2.1.2 Maxwell

Maxwell (sec. 57 of [5]) interpreted Faraday's electrotonic state as the vector potential $\mathbf{A}$, and gave an interpretation of this as "electromagnetic momentum", in the sense that

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{Maxwell})} = \int \frac{\rho \mathbf{A}^{(\mathrm{C})}}{c} \, d\mathrm{Vol} \tag{1}$$

is electromagnetic momentum associated with an electric charge density ρ in the (Coulomb-gauge) vector potential (in Gaussian units, with c being the speed of light in vacuum). Maxwell also argued (sec. 792 of [6]) that the radiation pressure P of light[2] is equal to its energy density u,

$$P = u = \frac{D^2}{4\pi} = \frac{H^2}{4\pi} \tag{2}$$

for an electromagnetic wave with fields $\mathbf{D}$ and $\mathbf{H}$ in vacuum, but he did not explicitly associate this pressure with momentum in the electromagnetic field.

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[1] Ampère also had no concept of electromagnetic fields, which were first discussed by Faraday, sec. 2147 of [4].

[2] Apparently, Kepler considered the pointing of comets' tails away from the Sun as evidence for radiation pressure of light [7].

2.1.3 J.J. Thomson

In 1891, Thomson noted [8] that a sheet of electric displacement $\mathbf{D}$ (parallel to the surface) which moves perpendicular to its surface with velocity $\mathbf{v}$ must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} = (1/c)\,\partial \mathbf{D}/\partial t$.[3] Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [8],

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{Thomson})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}\,. \tag{3}$$

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [12], adding the remark (p. 9) that the momentum density (3) is closely related to the Poynting vector [13, 14],[4,5]

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \tag{4}$$

The form (3) was also used by Poincaré in 1900 [20], following Lorentz' convention [21] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector be $(c/4\pi)\,\mathbf{D} \times \mathbf{H}$. In 1903 Abraham [22] argued for

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{Abraham})} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}\,, \tag{5}$$

and in 1908 Minkowski [23] advocated the form[6,7]

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{Minkowski})} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}\,. \tag{6}$$

Thomson did not relate the momentum density (3) to the radiation pressure of light, eq. (2), until 1904 (p. 355 of [25]) when he noted that $P = F/A = c\,p_{\mathrm{EM}} = D^2/4\pi = H^2/4\pi$ for fields moving with speed c in vacuum, for which $D = H$. He also gave an argument (p. 348 of [25]) that the forms (1) and (3) for field momentum are equivalent once the sources of the fields are taken into account.[8]

[3] Variants of this argument were given by Heaviside in 1891, sec. 45 of [9], and much later in sec. 18-4 of [10], where it is noted that Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c)\,\partial \mathbf{B}/\partial t$, combined with the Maxwell equation for $\mathbf{H}$ implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted in sec. 265 of [11].

[4] The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [16], based on Euler's continuity equation [17] for mass flow, $\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = -\partial \rho/\partial t$.

[5] Thomson argued, in effect, that the field momentum density (3) is related by $\mathbf{p}_{\mathrm{EM}} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [8, 12]. See also eq. (19), p. 79 of [9], and p. 6 of [15]. It turns out that the energy flow velocity defined by $\mathbf{v} = \mathbf{S}/u$ can exceed c (see, for example, sec. 2.1.4 of [18] and sec. 4.3 of [19].

[6] Minkowski, like Poynting [13], Heaviside [14] and Abraham [22], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [23].

[7] For some remarks on the "perpetual" Abraham-Minkowski debate see [24].

[8] Possibly, Thomson delayed publishing the relation of radiation pressure to his expression (3) until he could demonstrate its equivalence to Maxwell's form (1). For other demonstrations of this equivalence, see [26].

The first discussion that a static electromagnetic field might contain momentum was given by Thomson on p. 347 of [25], where he deduced that the field momentum of a magnetic dipole $\mathbf{m}$, a small solenoid coil with vector potential $\mathbf{A}^{(\mathrm{C})} = \frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$, in the electric field $\mathbf{E}$ of a distance point charge q, is

$$\begin{aligned}\mathbf{P}_{\mathrm{EM}} &= \frac{\mathbf{E}\times\mathbf{H}_{\mathrm{in}}\mathrm{Vol}_{\mathrm{coil}}}{4\pi c} = \frac{\mathbf{E}\times\mathbf{m}}{c} \\ &= \int\frac{\rho\mathbf{A}^{(C)}}{c}\,d\mathrm{Vol} = \frac{\mathbf{m}\times q\hat{\mathbf{r}}}{cr^2} = \frac{\mathbf{E}\times\mathbf{m}}{c}\,. \end{aligned} \tag{7}$$

Thomsom remarked, in effect, that the first line of eq. (7) suggests the field momentum is associated with the magnetic dipole,[9] while the second line suggests it is associated with the electric charge. He then noted that if the magnetic dipole were a small permanent magnet (in the field of an electric charge), and this magnet were demagnetized by "tapping", the magnet would acquire the initial momentum (7) according to the first line, while the electric charge should aquire this momentum according to second line.

He did not conclude that these contradictory statements imply the total momentum of the system must be zero (when it is "at rest") [28], such that there exists a "hidden" mechanical momentum in the system equal and opposite to the field momentum.[10] Then, if the field momentum vanishes the "hidden" mechanical momentum does also, and the total momentum of the system remains zero.[11]

That this "hidden" momentum is of order $1/c^2$,[12] and so is a "relativistic" effect, was beyond the scope of discussions in 1904.[13]

2.1.4 Page and Adams

According to the Lorentz force law, the forces of two moving charges on each other are not equal and opposite in general. An explicit verification at order $1/c^2$ that the net force is equal and opposite to the time rate of change of the field momentum of the two charges was given by Page and Adams [36].[14]

[9]The field momentum can be computed by a third method, apparently first noted by Furry [27] (see also [26]), $\mathbf{P}_{\mathrm{EM}} = \int V^{(C)}\mathbf{J}\,d\mathrm{Vol}/c^2$, where $V^{(C)}$ is the scalar potential in the Coulomb gauge and $\mathbf{J}$ is the electric current density. This form is not particularly efficient in deducing eq. (7), but it reinforces the impression that the field momentum is associated with the magnetic dipole.

[10]The provocative term "hidden" momentum was introduced by Shockley [29].

[11]For Thomson's particular example, the magnetic moment drops to zero while the static electric field of the charge is unchanged. In this case, the "overt" mechanical momentum of the dipole changes according to $d\mathbf{p}_{m,\mathrm{overt}}/dt = -\dot{\mathbf{m}}\times\mathbf{E}/c$ (see, for example, sec. IV of [30], the last line on p. 53 of [31]) and sec. 2.5 of [32]), so the final "overt" momentum of the dipole is $\mathbf{p}_{m,\mathrm{overt}} = \mathbf{m}\times\mathbf{E}/c$ which equals the initial "hidden" mechanical momentum of the magnetic dipole in the electric field [27]. Meanwhile, the falling magnetic moments leads to an induced electric field at the charge q, such that the force on the charge is $\mathbf{F} = q\mathbf{E}_{\mathrm{ind}} = -q\partial\mathbf{A}/\partial ct = -q\,\dot{\mathbf{m}}\times\mathbf{r}/cr^3 = \dot{\mathbf{m}}\times\mathbf{E}/c = -d\mathbf{p}_{m,\mathrm{overt}}/dt$. The final ("overt") momentum of the charge is $\mathbf{p}_q = -\mathbf{m}\times\mathbf{E}/c = -\mathbf{p}_{m,\mathrm{overt}}$, so the final, total momentum of the system is also zero.

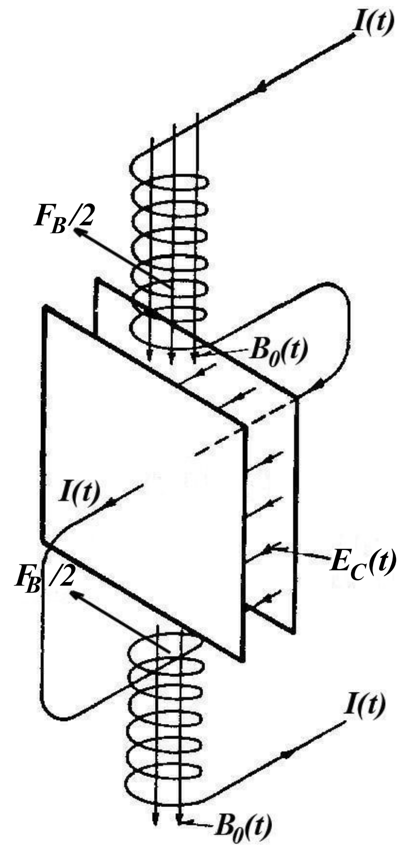
[12]A loop of area A that carries current I has magnetic moment $m = IA/c$, so the field momentum (7) is an effect of order $1/c^2$.

[13]For comments on the character of this "hidden" momentum, see [35].

[14]This demonstration is also readily given using the Darwin Lagrangian for interacting charges, which is accurate to order $1/c^2$ [37].

2.1.5 Slepian

In the late 1940's J. Slepian, a senior engineer at Westinghouse, posed a series of delightful pedagogic puzzles in the popular journal *Electrical Engineering*. One of these concerned how a capacitor in a cylindrical magnetic field might or might not be used to provide a form of rocket propulsion [38].



The current in Slepian's example is sinusoidal at a low enough frequency that radiation is negligible, so that system can be regarded as quasistatic. In this case, the electromagnetic field momentum is always equal and opposite to the "hidden" mechanical momentum, according to a general result of sec. 4.1.4 of [33]. Consequently, the Lorentz force on the system associated with the **E** and **B** field induced by the oscillating **B** and **E** fields are always equal and opposite to the "hidden" momentum forces associated with the oscillatory "hidden" momentum, and the total momentum of the system remains constant (no rocket propulsion).[15]

2.1.6 Cullwick

In the 1950's, Cullwick [41, 42] considered an example somewhat related to that of Tuval and Yahalom, namely an electric charge on the axis of a toroidal magnet. Cullwick noted that this example is paradoxical because no force is exerted on the moving charge when the current is constant in the toroid,but the moving charge exerts a nonzero force on the

[15]For additional discussion, see [40].

toroid.[16]

In the quasistatic limit, Cullwick's paradox is resolved by noting that the unbalanced force is equal and opposite to the time rate of change of the field momentum [46].

2.2 Two Time-Dependent Electrical Circuits

The force of circuit 1 that carries current $I_1(t)$, approximated as uniform around the circuit, on another circuit 2 that carries current $I_2(t)$ is given at time t by the generalized Biot-Savart law[17] (in Gaussian units) as

$$\mathbf{F}_{21}(t) = \oint \frac{I_2(t)\, d\mathbf{l}_2}{c} \times \left(\oint \frac{I_1(t - r/c)\, d\mathbf{l}_1 \times \hat{\mathbf{r}}}{cr^2} + \oint \frac{I_1^{(1)}(t - r/c)\, d\mathbf{l}_1 \times \hat{\mathbf{r}}}{c^2 r} \right), \tag{8}$$

where $I^{(n)}(t) = d^n I(t)/dt^2$, $\mathbf{r}$ is the vector from element $d\mathbf{l}_2$ to element $d\mathbf{l}_1$, and we suppose the circuits are massive enough that they do not change shape (or move as a whole) during the time interval of interest.

We follow Tuval and Yahalom [1] in expanding the (retarded) current as

$$I(t - r/c) = \sum_{n=0}^{\infty} \frac{I^{(n)}(t)}{n!} \left(\frac{-r}{c} \right)^n, \tag{9}$$

such that

$$\begin{aligned}
\mathbf{F}_{21}(t) &= \oint \frac{I_2(t)\, d\mathbf{l}_2}{c} \times \left(\oint \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} \left(\frac{-r}{c} \right)^n \frac{d\mathbf{l}_1 \times \hat{\mathbf{r}}}{cr^2} + \oint \sum_{n=0}^{\infty} \frac{I_1^{(n+1)}(t)}{n!} \left(\frac{-r}{c} \right)^n \frac{d\mathbf{l}_1 \times \hat{\mathbf{r}}}{c^2 r} \right) \\
&= \oint \frac{I_2(t)\, d\mathbf{l}_2}{c} \times \left(\oint \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} \left(\frac{-r}{c} \right)^n \frac{d\mathbf{l}_1 \times \hat{\mathbf{r}}}{cr^2} - \oint \sum_{n=1}^{\infty} \frac{I_1^{(n)}(t)}{(n-1)!} \left(\frac{-r}{c} \right)^n \frac{d\mathbf{l}_1 \times \hat{\mathbf{r}}}{cr^2} \right) \\
&= I_1(t) I_2(t) \oint \oint \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \hat{\mathbf{r}})}{c^2 r^2} - \sum_{n=2}^{\infty} \frac{n-1}{n!} I_1^{(n)}(t) I_2(t) \oint \oint \left(\frac{-r}{c} \right)^n \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \hat{\mathbf{r}})}{c^2 r^2}.
\end{aligned} \tag{10}$$

The first term of the last line of eq. (10) is the instantaneous static force, so the time-dependent force departs from this only if current 1 is nonlinear with time.

A famous result (due to Ampère) is that $\boldsymbol{\nabla} r^{m+1} = (m+1) r^m\, \hat{\mathbf{r}}$, so that

$$\begin{aligned}
\oint \oint r^m\, d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \hat{\mathbf{r}}) &= \oint \oint r^m (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}} - \oint d\mathbf{l}_1 \oint d\mathbf{l}_2 \cdot r^m\, \hat{\mathbf{r}} \\
&= \oint \oint r^m (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}} - \oint d\mathbf{l}_1 \oint \frac{d\mathbf{l}_2 \cdot \boldsymbol{\nabla} r^{m+1}}{m+1} \\
&= \oint \oint r^m (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}.
\end{aligned} \tag{11}$$

[16]This paradox was revived in [43, 44, 45], without reference to Cullwick.

[17]See, for example, [47], and eq. (5) of [48].

Thus, the force on circuit 2 can be written as

$$\mathbf{F}_{21}(t) = I_1(t)I_2(t)\oint\oint\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2} - \sum_{n=2}^{\infty}\frac{n-1}{n!}I_1^{(n)}(t)I_2(t)\oint\oint\left(\frac{-r}{c}\right)^n\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2}. \quad (12)$$

If we interchange indices 1 and 2, $\mathbf{r}\to-\mathbf{r}$ and the force on circuit 1 is

$$\mathbf{F}_{12}(t) = -I_1(t)I_2(t)\oint\oint\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2} + \sum_{n=2}^{\infty}\frac{n-1}{n!}I_2^{(n)}(t)I_1(t)\oint\oint\left(\frac{-r}{c}\right)^n\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2}. \quad (13)$$

The total force on the two circuits is[18]

$$\mathbf{F}(t) = \mathbf{F}_{12}(t)+\mathbf{F}_{21}(t) = \sum_{n=2}^{\infty}\frac{n-1}{n!}[I_1(t)I_2^{(n)}(t) - I_2(t)I_1^{(n)}(t)]\oint\oint\left(\frac{-r}{c}\right)^n\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2}. \quad (14)$$

If this force is to provide net momentum to the system (rocket propulsion), the time integral of the total force must be nonzero. Supposing the currents have time dependence only during the interval (t_1, t_2), the total mechanical momentum gained by the system is

$$\mathbf{P} = \int_{t_1}^{t_2}\mathbf{F}(t)\,dt = \sum_{n=2}^{\infty}\frac{n-1}{n!}\int_{t_1}^{t_2}[I_1(t)I_2^{(n)}(t) - I_2(t)I_1^{(n)}(t)]\,dt\oint\oint\left(\frac{-r}{c}\right)^n\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2}. \quad (15)$$

If we integrate by parts n times, noting that the time derivatives vanish at t_1 and t_2, we find that

$$\int_{t_1}^{t_2}[I_1(t)I_2^{(n)}(t) - I_2(t)I_1^{(n)}(t)]\,dt = (-1)^n\int_{t_1}^{t_2}[I_1^{(n)}(t)I_2(t) - I_2^{(n)}(t)I_1(t)]\,dt, \quad (16)$$

which implies that this time integral vanishes for all even n. Hence, the net momentum gain is

$$\mathbf{P} = \sum_{n=3,\mathrm{odd}}^{\infty}\frac{n-1}{n!}\int_{t_1}^{t_2}[I_1(t)I_2^{(n)}(t) - I_2(t)I_1^{(n)}(t)]\,dt\oint\oint\left(\frac{-r}{c}\right)^n\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{c^2r^2}. \quad (17)$$

Note that if one of the currents is constant in time (as would be the case of the equivalent current associated with a permanent magnet), then the momentum gain is zero, although the force $\mathbf{F}(t)$ is nonzero. This is, for constant I_1, the time integrals become

$$\int_{t_1}^{t_2}[I_1(t)I_2^{(n)}(t) - I_2(t)I_1^{(n)}(t)]\,dt = I_1\int_{t_1}^{t_2}I_2^{(n)}(t)\,dt = I_1[I_2^{(n-1)}(t_2) - I_2^{(n-1)}(t_1)] = 0, \quad (18)$$

since the time derivatives of I_2 vanish at times t_1 and t_2. **Hence, there can be no momentum gain for the case of a time-dependent circuit plus a permanent magnet**, contrary to the claim in [2].

[18] If the currents vary sinusoidally as $e^{-i\omega t}$, where $\omega = kc = 2\pi c/\lambda$, then the total force is proportional to $|I_1|\,|I_2|\,/c^2$.

There can be a gain of mechanical momentum in the case of two time-dependent circuits, each of whose time dependence has nonzero odd-order time derivatives, of order 3 or higher. Yet, we expect by conservation of momentum that a system initial at rest cannot spontaneously take on momentum. Indeed, we expect that the gain of mechanical momentum of the system is offset by an equal and opposite electromagnetic field momentum. In particular, we note that a system of two circuits can be regarded as a pair of magnetic dipoles, which form a magnetic quadrupole with a nonzero, time-dependent quadrupole moment. As noted in sec. 71 of [49], a quadrupole moment emits radiation if its 3rd time derivative is nonzero. Hence, we learn that if the 3rd-order integral is nonzero in the expression (17) for the gain in mechanical momentum by the system, then this momentum is compensated by the emission of momentum in the form of quadrupole radiation (and if the nth-order integral is nonzero, then there exists nth-order multipole radiation of momentum).

Of course, as the system emits momentum in its multipole radiation, it also emits energy, such that only a finite amount of energy and momentum can be radiated. Hence, the gain of mechanical momentum by the system is finite, just as in rocket propulsion based on chemical reactions.

In sum, the system of time-dependent circuits can serve as an "electromagnetic spaceship", with (weak) propulsion via the reaction to the radiation of momentum due to the time-varying quadrupole (or higher) moment of the system).

2.3 The Spaceship as an Antenna Array

We have seen that if there is net momentum gain by a pair of time-dependent circuits (after their time variation has ceased), this is associated with the radiation of momentum. Hence, an alternative analysis is to compute the radiated momentum, and note that the mechanical momentum gain is equal and opposite to this.

In general, computation of radiation by antenna systems are best performed numerically, but analytic calculations are possible for some simple systems. In particular, if the two circuits are small compared to the wavelength of the radiation, their electromagnetic fields can be modeled as those of "point" dipoles, as first argued by Hertz [50].

The radiation field of a single, time-dependent "point" dipole has sufficient symmetry that while energy is radiated, no momentum is radiated. Hence, the case of a single time-dependent circuit/dipole plus a permanent magnet [2] (which latter does not radiate) emits no momentum, and there can be no net mechanical force on this system.[19,20]

A system of two time-dependent circuits/dipoles can emit momentum, and can therefore accrue a net mechanical momentum in reaction to this radiation. For this, the two circuits should be arranged so that the radiation pattern is asymmetric.

[19] The electromagnetic spaceship of Slepian, sec. 2.1.5, operated a low frequency, is a variant of a small dipole antenna, and so does not emit momentum even if radiation is taken in to account. Hence, Slepian was correct in arguing that this device does not serve as a spaceship, even in principle.

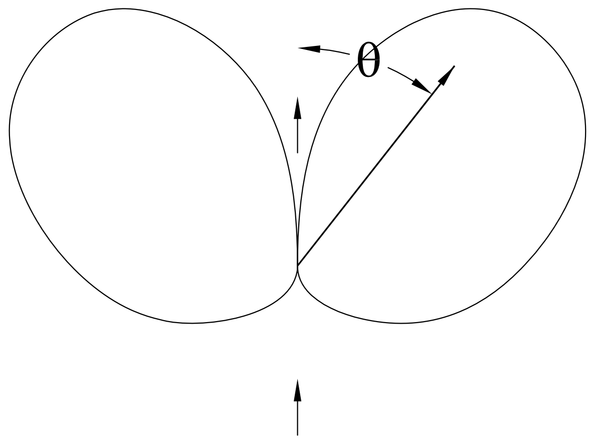
[20] If the second circuit were just a conductig loop, with no current except that induced by the first circuit, this "loop Yagi" antenna [51] would radiate momentum, and could serve as an electromagnetic spaceship.

A variant on an antenna-based spaceship is to keep the drive antenna/laser on the Earth, and have only a passive reflector on the spaceship. This configuration becomes very efficient as the spaceship approaches the speed of light [52].

A simple example of this has been discussed in [53], in which the dipoles have oscillatory currents with angular frequency ω, and are arrayed $\lambda/4 = \pi c/2\omega$ apart, and driven with their currents 90° out of phase. Then the pattern of radiated energy has the form

$$\frac{d\langle U\rangle}{d\Omega\, dt} = \frac{|[\ddot{\mathbf{m}}] \times \hat{\mathbf{r}}|^2}{8\pi c^3} = \frac{\omega^4 m_0^2 \sin^2\theta}{4\pi c^3}\left(1 + \sin\frac{\pi}{2}\cos\theta\right), \tag{19}$$

where m_0 is the magnitude of the magnetic dipole moment of the two circuits/antennas. This angular distribution is sketched in the figure on the next page. More energy is radiated into the forward hemisphere ($z > 0$) than the backward.



The total radiated energy is obtained by integration of eq. (19) over solid angle,

$$\frac{d\langle U\rangle}{dt} = \int \frac{d\langle U\rangle}{d\Omega\, dt}\, d\Omega = \frac{2\omega^4 m_0^2}{3c^3}. \tag{20}$$

Associated with the radial flow of energy $\langle U\rangle$ in the far zone is a radial flow of momentum,[21] $\langle \mathbf{P}\rangle = \langle U\rangle\, \hat{\mathbf{r}}/c$. Hence, the angular distribution of time-average momentum radiated by the antenna follows from eq. (19) as

$$\frac{d\langle \mathbf{P}\rangle}{d\Omega\, dt} = \frac{\omega^4 m_0^2 \sin^2\theta}{4\pi c^4}\left(1 + \sin\frac{\pi}{2}\cos\theta\right)\hat{\mathbf{r}}. \tag{21}$$

On integrating this over solid angle to find the total momentum radiated, only the z component is nonzero,

$$\begin{aligned}
\frac{d\langle \mathbf{P}\rangle_z}{dt} &= 2\pi\frac{\omega^4 m_0^2}{4\pi c^4}\int_{-1}^{1}\sin^2\theta\left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right]\cos\theta\; d\cos\theta \\
&= \frac{4\omega^4 m_0^2}{\pi^2 c^4}\int_0^{\pi/2} x\left(1 - \frac{4x^2}{\pi^2}\right)\sin x\; dx \qquad \text{where } x = \frac{\pi}{2}\cos\theta \\
&= \frac{8\omega^4 m_0^2}{\pi^2 c^4}\left(\frac{12}{\pi^2} - 1\right) = \frac{12}{\pi^2 c}\frac{d\langle U\rangle}{dt}\left(\frac{12}{\pi^2} - 1\right) \approx \frac{0.26}{c}\frac{d\langle U\rangle}{dt}.
\end{aligned} \tag{22}$$

[21] This is the classical version of the quantum relation for photons that $U = \hbar\omega$ and $\mathbf{P} = \hbar\,\mathbf{k} = \hbar\omega\,\hat{\mathbf{k}}/c = U\,\hat{\mathbf{k}}/c$.

The radiation reaction force on the antenna is $F_z = -dP_z/dt$.[22] For a broadcast antenna radiating 10^5 Watts, the reaction force would be only $\approx 10^{-4}$ N.

An array of many dipole circuits/antennas along a common axis, and properly phased, can have a narrower radiation pattern than that found above, such that $d\langle \mathbf{P}\rangle_z/dt \approx (1/c)d\langle U\rangle/dt$. A laser beam also has this maximal relation between radiated momentum and energy.

Electromagnetic spaceships have rather low thrust in practice, and their minimum ratio of power to thrust is $(dU/dt)/(dP/dt) = c = 3 \times 10^8$ W/N. Low-thrust devices find application as correction elements on satellites, but an electromagnetic/photon "rocket" is not particularly favorable compared to Hall-effect thrusters [54], for which the power-to-thrust ratio is of order 10^4 W/N.

For a review of efforts on "photon rockets", see [55].

Acknowledgment

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References

[1] M. Tuval and A. Yahalom, *Newton's Third Law in the Framework of Special Relativity* (Jan. 26, 2013), http://arxiv.org/abs/1302.2537v1
V2 and V3 of this e-note are somewhat different.

[2] M. Tuval and A. Yahalom, *Relativistic Engine Based on a Permanent Magnet* (June 30, 2015), http://arxiv.org/abs/1507.02897v1
V2 of this e-note is very different.

[3] A.M. Ampère, *Théorie mathématique des Phénomènes électro-dynamiques uniquement déduite de l'Expérience*, 2nd ed. (Paris, 1883),
http://physics.princeton.edu/~mcdonald/examples/EM/ampere_theorie_26_83.pdf

[4] M. Faraday, *Experimental Researches in Electricity.—Twentieth Series*, Phil. Trans. Roy. Soc. London **136**, 21 (1846),
http://physics.princeton.edu/~mcdonald/examples/EM/faraday_ptrsl_136_21_46.pdf

[5] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865),
http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_ptrsl_155_459_65.pdf

[22] Since $\omega/c = k = 2\pi/\lambda$, and the magnetic moment can be expressed as $m_0 = I_0 A/c$, we have that

$$F_z = -\frac{d\langle \mathbf{P}\rangle_z}{dt} = -\frac{144\pi^2 I_0^2 A^2}{\lambda^4 c^2}\left(\frac{12}{\pi^2} - 1\right) \propto \frac{I_0^2}{c^2}\,, \tag{23}$$

which agrees with the dimensions of the force found in eq. (14).

[6] J.C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon Press, Oxford, 1873), http://puhep1.princeton.edu/~mcdonald/examples/EM/maxwell_treatise_v2_73.pdf

[7] E.F. Nichols and G.F. Hull, *The Pressure Due to Radiation*, Phys. Rev. **17**, 26 (1903), http://physics.princeton.edu/~mcdonald/examples/EM/nichols_pr_17_26_03.pdf

[8] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891), http://physics.princeton.edu/~mcdonald/examples/EM/thomson_pm_31_149_91.pdf

[9] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Electrician Publishing Co., 1891), http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_electromagnetic_theory_1.pdf

[10] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 2 (Addison-Wesley, 1963), http://www.feynmanlectures.caltech.edu/II_17.html#Ch17-S4 http://www.feynmanlectures.caltech.edu/II_18.html#Ch18-S2

[11] J.J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism* (Cambridge U. Press, 1895), http://physics.princeton.edu/~mcdonald/examples/EM/thomson_EM_2nd_ed_97.pdf

[12] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893), http://physics.princeton.edu/~mcdonald/examples/EM/thomson_recent_researches_sec_1-16.pdf

[13] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884), http://physics.princeton.edu/~mcdonald/examples/EM/poynting_ptrsl_175_343_84.pdf

[14] O. Heaviside, *Electrical Papers*, Vol. 1 (Macmillan, 1894), http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_electrical_papers_1.pdf
See p. 438 for the Poynting vector. Heaviside wrote the momentum density in the Minkowski form (6) on p. 108 of [9].

[15] H. Bateman, *The Mathematical Analysis of Electrical and Optical Wave Motion*, (Cambridge U. Press, 1915), http://physics.princeton.edu/~mcdonald/examples/EM/bateman_wave_motion_15.pdf

[16] N. Umow, *Ableitung der Bewegungsgleichungen der Energie in continuirlichen Körpern*, Zeit. Math. Phys. **19**, 418 (1874), http://physics.princeton.edu/~mcdonald/examples/EM/umow_zmp_19_97,418_74.pdf http://physics.princeton.edu/~mcdonald/examples/EM/umov_theorem.pdf

[17] L. Euler, *Principes généraux du mouvement des fluides*, Acad. Roy. Sci. Belles-Lett. Berlin (1755), http://physics.princeton.edu/~mcdonald/examples/fluids/euler_fluids_55_english.pdf

[18] K.T. McDonald, *Momentum in a DC Circuit* (May 26, 2006), http://physics.princeton.edu/~mcdonald/examples/loop.pdf

[19] K.T. McDonald, *Flow of Energy and Momentum in the Near Zone of a Hertzian Dipole* (Apr. 11, 2007), http://physics.princeton.edu/~mcdonald/examples/hertzian_momentum.pdf

[20] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://physics.princeton.edu/~mcdonald/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*,
http://physics.princeton.edu/~mcdonald/examples/EM/poincare_an_5_252_00_english.pdf

[21] H.A. Lorentz, *La Théorie Électromagnétique Maxwell et Son Application aux Corps Mouvants*, (Brill, Leiden, 1892),
http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_theorie_electromagnetique_92.pdf

[22] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),
http://physics.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[23] H. Minkowski, *Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körper*, Nachr. Ges. Wiss. Göttingen **1**, 53 (1908),
http://physics.princeton.edu/~mcdonald/examples/EM/minkowski_ngwg_53_08.pdf
http://physics.princeton.edu/~mcdonald/examples/EM/minkowski_ngwg_53_08_english.pdf

[24] K.T. McDonald, *Abraham, Minkowski and "Hidden" Mechanical Momentum* (June 6, 2012), http://physics.princeton.edu/~mcdonald/examples/abraham.pdf

[25] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_pm_8_331_04.pdf

[26] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), http://physics.princeton.edu/~mcdonald/examples/pem_forms.pdf

[27] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://physics.princeton.edu/~mcdonald/examples/EM/furry_ajp_37_621_69.pdf

[28] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968),
http://physics.princeton.edu/~mcdonald/examples/EM/coleman_pr_171_1370_68.pdf

[29] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967),
http://physics.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf

[30] L. Vaidman, *Torque and force on a magnetic dipole*, Am. J. Phys. **58**, 978 (1990),
http://physics.princeton.edu/~mcdonald/examples/EM/vaidman_ajp_58_978_90.pdf

[31] M.G. Calkin, *Lagrangian and Hamiltonion Mechanics* (World Scientific, 1996),
http://puhep1.princeton.edu/~mcdonald/examples/EM/calkin_96_p49.pdf

[32] K.T. McDonald, *Forces on Magnetic Dipoles* (Oct. 26, 2014),
http://physics.princeton.edu/~mcdonald/examples/neutron.pdf

[33] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf

[34] P. Penfield, Jr. and H.A. Haus, *Electrodynamics of Moving Media* (MIT, 1967), p. 215, http://physics.princeton.edu/~mcdonald/examples/EM/penfield_haus_chap7.pdf

[35] K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012), http://physics.princeton.edu/~mcdonald/examples/penfield.pdf

[36] L. Page and N.I. Adams, Jr., *Action and Reaction Between Moving Charges*, Am. J. Phys. **13**, 141 (1945), http://physics.princeton.edu/~mcdonald/examples/EM/page_ajp_13_141_45.pdf

[37] C.G. Darwin, *The Dynamical Motions of Charged Particles*, Phil. Mag. **39**, 537 (1920), http://physics.princeton.edu/~mcdonald/examples/EM/darwin_pm_39_537_20.pdf

[38] J. Slepian, *Electromagnetic Space-Ship*, Electrical Engineering **68**, 145 (1949), http://physics.princeton.edu/~mcdonald/examples/EM/slepian_ee_68_145_49

[39] J. Slepian, *Answer to Previous Essay*, Electrical Engineering **68**, 245 (1949), http://physics.princeton.edu/~mcdonald/examples/EM/slepian_ee_68_245_49

[40] K.T. McDonald, *Slepian's Electromagnetic Spaceship* (June 18, 2006), http://physics.princeton.edu/~mcdonald/examples/slepian.pdf

[41] E.G. Cullwick, *Electromagnetic Momentum and Newton's Third Law*, Nature **170**, 425 (1952), http://physics.princeton.edu/~mcdonald/examples/EM/cullwick_nature_170_425_52.pdf

[42] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, Green & Co., London, 1959), pp. 232-238, http://physics.princeton.edu/~mcdonald/examples/EM/cullwick_ER_59.pdf

[43] G.T. Trammel, *Aharonov-Bohm Paradox*, Phys. Rev. **134**, B1183 (1964), http://physics.princeton.edu/~mcdonald/examples/EM/trammel_pr_134_B1183_64.pdf

[44] T.T. Taylor, *Electrodynamic Paradox and the Center-of-Mass Principle*, Phys. Rev. **137**, B467 (1965), http://physics.princeton.edu/~mcdonald/examples/EM/taylor_pr_137_B467_65.pdf

[45] J.M. Aguirregabiria, A Hernández and M Rivas, *Linear momentum density in quasistatic electromagnetic systems*, Eur. J. Phys. **25**, 255 (2004), http://physics.princeton.edu/~mcdonald/examples/EM/aguirregabiria_ejp_25_555_04.pdf

[46] K.T. McDonald, *Cullwick's Paradox: Charged Particle on the Axis of a Toroidal Magnet* (June 4, 2006), http://physics.princeton.edu/~mcdonald/examples/cullwick.pdf

[47] D.J. Griffiths and M.A. Heald, *Time-Dependent Generalizations of the Biot-Savart and Coulomb Laws*, Am. J. Phys. **59**, 111 (1991), http://physics.princeton.edu/~mcdonald/examples/EM/griffiths_ajp_59_111_91.pdf

[48] K.T. McDonald, *The Relation Between Expressions for Time-Dependent Electromagnetic Fields Given by Jefimenko and by Panofsky and Phillips* (Dec. 5, 1996), http://physics.princeton.edu/~mcdonald/examples/jefimenko.pdf

[49] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, Oxford, 1975).

[50] H. Hertz, *The Forces of Electrical Oscillations Treated According to Maxwell's Theory*, Ann. d. Phys. **36**, 1 (1889),
http://physics.princeton.edu/~mcdonald/examples/EM/hertz_ap_36_1_89.pdf
reprinted in chap. 9 of H. Hertz, *Electric Waves* (Macmillan, 1900),
http://physics.princeton.edu/~mcdonald/examples/EM/hertz_electric_waves.pdf
A translation by O. Lodge appeared in Nature **39**, 402 (1889),
ttp://physics.princeton.edu/~mcdonald/examples/EM/hertz.pdf

[51] A commercial loop Yagi antenna is at
http://www.mimomax.com/assets/Spec-Sheets/MiMOMax-All-Loop-Yagi-Antennas-v2.pdf

[52] G. Marx, *Interstellar Vehicle Propelled by Terrestrial Laser Beam*, Nature **211**, 22 (1966), http://physics.princeton.edu/~mcdonald/examples/EM/marx_nature_211_22_66.pdf

[53] K.T. McDonald, *The Force on an Antenna Array* (May 1, 1979),
http://physics.princeton.edu/~mcdonald/examples/antenna_force.pdf

[54] K.T. McDonald, *Electron Trajectories in a Hall Thruster* (Feb. 27, 2008),
http://physics.princeton.edu/~mcdonald/examples/thruster.pdf

[55] M.M. Michaelis and A. Forbes, *Laser propulsion: a review*, S. Afr. J. Sci. **102**, 289 (2006), http://physics.princeton.edu/~mcdonald/examples/EM/michaelis_sajs_102_289_06.pdf