The Transverse Momentum of an Electron in a Wave

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1 Problem

When a charged particle (of mass m and charge e) interacts with a linearly polarized plane wave with electric field $E_x = E_0 \cos(kz - \omega t)$, the particle's motion consists of a transverse oscillation. Hence, the particle has transverse momentum, while the wave carries only longitudinal momentum. How is Newton's 3rd law satisfied in this situation?

Hint: Consider the interaction field momentum.

This problem may be analyzed in the frame in which the particle is at rest on average. The longitudinal oscillation is negligible if $eE_0/m\omega c \ll 1$, as may be assumed.

2 Solution

The general sense of the answer has been given by Poynting [1], who noted that an electromagnetic field can be said to contain a flux of energy (energy per unit area per unit time) given by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \tag{1}$$

in Gaussian units, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field (taken to be in vacuum throughout this paper) and c is the speed of light.

Thomson [2, 3, 4] and Poincaré [5] noted that this flow of energy can also be associated with a momentum density given by

$$\mathbf{p}_{\rm field} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{(\mathbf{E}_{\rm wave} + \mathbf{E}_{\rm charge}) \times (\mathbf{B}_{\rm wave} + \mathbf{B}_{\rm charge})}{4\pi c}. \tag{2}$$

Hence, in the problem of a free electron in a plane electromagnetic wave we are led to seek an electromagnetic field momentum that is equal and opposite to the mechanical momentum of the electron. However, this field momentum should not include either of the self-momenta $(\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm wave})/4\pi c$ or $(\mathbf{E}_{\rm charge} \times \mathbf{B}_{\rm charge})/4\pi c$. The former is independent of the electron, while the latter can be considered as a part of the mechanical momentum of the electron according to the concept of "renormalization."

We desire to show that the **interaction** field momentum,

$$\mathbf{P}_{\rm int} = \int \mathbf{p}_{\rm int}\, d{\rm Vol} = \int d{\rm Vol} \frac{\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm charge} + \mathbf{E}_{\rm charge} \times \mathbf{B}_{\rm wave}}{4\pi c}, \tag{3}$$

is equal and opposite to the mechanical momentum of the electron.

We consider a plane electromagnetic wave that propagates in the $+z$ direction of a rectangular coordinate system. For linear polarization along $\mathbf{x}$,

$$\mathbf{E}_{\rm wave} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{x}}, \qquad \mathbf{B}_{\rm wave} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{y}}, \tag{4}$$

where $\omega = kc$ is the angular frequency of the wave, $k = 2\pi/\lambda$ is the wave number and $\hat{\mathbf{x}}$ is a unit vector in the x direction, *etc.*

2.1 Transverse Momentum of the Electron in a Weak Wave

A free electron of mass m oscillates in this field such that its average position is at the origin. This simple statement hides the subtlety that our frame of reference is the average rest frame of the electron when inside the wave, and is not the lab frame of an electron that is initially at rest, but which is overtaken by a wave. If the velocity of the oscillating electron is small, we can ignore the $\mathbf{v}/c \times \mathbf{B}$ force, and take the motion to be entirely in the plane $z = 0$. Then, (also ignoring radiation damping) the equation of motion of the electron is

$$m\ddot{\mathbf{x}} = e\mathbf{E}_{\rm wave}(0, t) = e\hat{\mathbf{x}}E_0 \cos\omega t. \tag{5}$$

Using eq. (4) we find the position of the electron to be

$$\mathbf{x} = -\frac{e}{m\omega^2}\hat{\mathbf{x}}E_0 \cos\omega t. \tag{6}$$

and the mechanical momentum of the electron is

$$\mathbf{P}_{\rm mech} = m\dot{\mathbf{x}} = \frac{e}{\omega}\hat{\mathbf{x}}E_0 \sin\omega t. \tag{7}$$

It is important to note that $\mathbf{P}_{\rm mech}$ is proportional to the first power of the wave field strength.

2.2 Longitudinal Motion of the Electron

The root-mean-square (rms) velocity of the electron is

$$v_{\rm rms} = \sqrt{\langle \dot{x}^2 \rangle} = \frac{eE_{\rm rms}}{m\omega c}c. \tag{8}$$

The condition that the $\mathbf{v}/c \times \mathbf{B}$ force be small is then,

$$\eta \equiv \frac{eE_{\rm rms}}{m\omega c} \ll 1, \tag{9}$$

where the dimensionless measure of field strength, η, is a Lorentz invariant. Similarly, the rms departure of the electron from the origin is

$$x_{\rm rms} = \frac{eE_{\rm rms}}{m\omega^2} = \frac{\eta\lambda}{2\pi}. \tag{10}$$

Thus, condition (9) also insures that the extent of the motion of the electron is small compared to a wavelength, and so we may use the dipole approximation when considering the fields of the oscillating electron.

In the weak-field approximation, we can now use eq. (7) for the velocity to evaluate the second term of the Lorentz force:

$$e\frac{\mathbf{v}}{c} \times \mathbf{B} = \frac{e^2 E_x^2}{2m\omega c}\hat{\mathbf{z}} \sin 2\omega t. \tag{11}$$

This term vanishes for circular polarization, in which case the motion is wholly in the transverse plane. However, for linear polarization the $\mathbf{v}/c \times \mathbf{B}$ force leads to oscillations along the z axis at frequency 2ω, as first analyzed in general by Landau [6]. For polarization along the x-axis, the x-z motion has the form of a "figure 8," which for weak fields ($\eta \ll 1$) is described by

$$x = -\frac{eE_x}{m\omega}\cos\omega t, \qquad z = -\frac{e^2E_x^2}{8m^2\omega^3 c}\sin 2\omega t. \tag{12}$$

If the electron had been at rest before the arrival of the plane wave, then inside the wave it would move with an average drift velocity given by

$$v_z = \frac{\eta^2/2}{1+\eta^2/2}c, \tag{13}$$

along the direction of the wave vector, as first deduced by McMillan [7]. In the present paper, we work in the frame in which he electron has no average velocity along the z axis. Therefore, prior to its encounter with the plane wave the electron had been moving in the negative z direction with speed given by eq. (13).

2.3 The Interference Term $\mathbf{P}_{\mathrm{wave,static}}$

The oscillating charge has oscillating fields, and the strength of those oscillating fields is proportional to the strength of the incident wave field. Hence, if we insert the oscillating fields of the charge into eq. (3), the interaction momentum will be quadratic in the wave field strength. This momentum cannot possibly balance the mechanical momentum (7).

For the interaction momentum (3) to yield a result proportional to the wave field strength, we need to insert a field associated with the charge that is independent of the wave field. Thus, we are led to consider the static field of the charge.

Indeed, the fields associated with the electron can be regarded as the superposition of those of an electron at rest at the origin plus those of a dipole consisting of the actual oscillating electron and a positron at rest at the origin. Thus, we can write the electric field of the electron as $\mathbf{E}_{\mathrm{static}} + \mathbf{E}_{\mathrm{osc}}$, and the magnetic field as $\mathbf{B}_{\mathrm{osc}}$.

The interaction field momentum density can now be written

$$\mathbf{p}_{\mathrm{int}} = \mathbf{p}_{\mathrm{wave,static}} + \mathbf{p}_{\mathrm{wave,osc}}, \tag{14}$$

where

$$\mathbf{p}_{\mathrm{wave,static}} = \frac{\mathbf{E}_{\mathrm{static}} \times \mathbf{B}_{\mathrm{wave}}}{4\pi c}. \tag{15}$$

and

$$\mathbf{p}_{\mathrm{wave,osc}} = \frac{\mathbf{E}_{\mathrm{wave}} \times \mathbf{B}_{\mathrm{osc}} + \mathbf{E}_{\mathrm{osc}} \times \mathbf{B}_{\mathrm{wave}}}{4\pi c}. \tag{16}$$

We recall from eqs. (7) and (12) that the transverse mechanical momentum of the oscillating electron has pure frequency ω. Since the wave and the oscillating part of the electron's field each have frequency ω, the term $\mathbf{p}_{\mathrm{wave,osc}}$ contains harmonic functions of ω^2, which can be resolved into a static term plus ones in frequency 2ω. Hence, we have a second reason why we should not expect this term to cancel the mechanical momentum. Rather, we look to the term $\mathbf{p}_{\rm wave,static}$, since this has pure frequency ω. The term $\mathbf{p}_{\rm wave,osc}$ cancels the longitudinal momentum associated with the "figure-8" motion, and also includes a "hidden momentum" related to the fact that the average rest frame of an electron inside the wave is not the rest frame of the electron in the absence of the wave, as sketched in secs. 3-4. See also [8].

The static field of the electron at the origin is, in rectangular coordinates,

$$\mathbf{E}_{\rm static} = \frac{e}{r^3}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}), \qquad (17)$$

where r is the distance from the origin to the point of observation. Combing this with eq. (4) we have

$$\mathbf{p}_{\rm wave,static} = \frac{e}{4\pi c r^3}\{-z\hat{\mathbf{x}} + x\hat{\mathbf{z}}\}E_0\cos(kz - \omega t). \qquad (18)$$

When we integrate this over all space to find the total field momentum, the term in $\hat{\mathbf{z}}$ vanishes as it is odd in x. Likewise, after expanding $\cos(kz - \omega t)$, the terms proportional to $z\cos kz$ vanish on integration. The remaining term is thus

$$\begin{aligned} \mathbf{P}_{\rm wave,static} &= \int d\mathrm{Vol}\, \mathbf{p}_{\rm wave,static} \\ &= -\frac{e}{4\pi c}\hat{\mathbf{x}}E_0\sin\omega t\int_V \frac{z\sin kz}{r^3} \\ &= -\frac{e}{\omega}\hat{\mathbf{x}}E_0\sin\omega t = -\mathbf{P}_{\rm mech}, \end{aligned} \qquad (19)$$

after an elementary volume integration (that involves integration by parts twice).

It is noteworthy that the integration is independent of any hypothesis as to the size of a classical electron. Indeed, the integrand of eq. (19) can be expressed as $\cos\theta\sin(kr\cos\theta)/r^2$ via the substitution $z = r\cos\theta$. Hence, the integral over a spherical shell is independent of r for $kr \ll 1$, and significant contributions to the integral occur for radii up to one wavelength of the electromagnetic wave. This contrasts with the self-momentum density of the electron which is formally divergent; if the integration is cut off at a minimum radius (the classical electron radius), the dominant contribution occurs within twice that radius.

3 The Momentum $\mathbf{P}_{\rm wave,osc}$

Several subtleties in the argument appear when we consider the other interference term $\mathbf{P}_{\rm wave,osc}$ in the momentum density (14).

3.1 Circular Polarization

After a somewhat lengthy calculation [8] we find that for a circularly polarized wave, the only the z component of $\mathbf{P}_{\rm wave,osc}$ is nonzero,

$$P_{{\rm wave,osc},z} = \int_V p_{{\rm wave,osc},z} = -\frac{4}{3}\eta^2 mc. \qquad (20)$$

Recall that we have performed the analysis in a frame in which the electron has no longitudinal momentum. However, as remarked in sec. 2.3, prior to its encounter with the wave the electron had velocity $v_z = -\eta^2 c/2$ (assuming $\eta^2 \ll 1$), and therefore had initial mechanical momentum $P_{\mathrm{mech},z} = -\eta^2 mc/2$. So, we would expect that this initial mechanical momentum had been converted to field momentum, if momentum is to be conserved.

We continue to be puzzled as to why the result (20) is 8/3 times larger than that required to satisfy momentum conservation.

3.2 Linear Polarization

We find the longitudinal component of the interference field momentum of a free electron in a linearly polarized wave to be [8]

$$P_{\mathrm{wave,osc},z} = -\frac{4}{3}\eta^2 mc + \frac{2}{3}\eta mc \cos 2\omega t. \tag{21}$$

The constant term is the same as that found in eq. (20) for circular polarization, and represents the initial mechanical momentum of the electron that became stored in the electromagnetic field once the electron became immersed in the wave.

As for the second term of eq. (21), recall from eq. (12) that for linear polarization the electron oscillates along the z axis at frequency 2ω. Hence. the z component of the mechanical momentum of the electron is

$$P_{\mathrm{mech},z} = m\dot{z} = -\frac{\eta^2 mc}{2}\cos 2\omega t. \tag{22}$$

The term in $P_{\mathrm{wave,osc},z}$ at frequency 2ω is $-4/3$ times the longitudinal component of the mechanical momentum associated with the "figure 8" motion of the electron. Thus, we have not been completely successful in accounting for momentum conservation when the small, oscillatory longitudinal momentum is considered.

The factors of 4/3 and 8/3 found here may be related to the well-known factor of 4/3 that arise in analyses of the electromagnetic energy and momentum of the self fields of an electron [11, 12]. A further appearance of a factor of 8/3 in the present example occurs when we consider the field energy of the interference terms.

4 The Interference Field Energy

It is also interesting to examine the electromagnetic field energy of an electron in a plane wave. As for the momentum density (2), we can write,

$$u_{\mathrm{total}} = \frac{(\mathbf{E}_{\mathrm{wave}} + \mathbf{E}_{\mathrm{static}} + \mathbf{E}_{\mathrm{osc}})^2 + (\mathbf{B}_{\mathrm{wave}} + \mathbf{B}_{\mathrm{osc}})^2}{8\pi}, \tag{23}$$

for the field energy density. Again, we no not consider the divergent energies of the self fields, but only the interference terms,

$$u_{\mathrm{int}} = u_{\mathrm{wave,static}} + u_{\mathrm{wave,osc}}, \tag{24}$$

where

$$u_{\mathrm{wave,static}} = \frac{\mathbf{E}_{\mathrm{wave}} \cdot \mathbf{E}_{\mathrm{static}}}{4\pi}. \tag{25}$$

and

$$u_{\text{wave,osc}} = \frac{\mathbf{E}_{\text{wave}} \cdot \mathbf{E}_{\text{osc}} + \mathbf{B}_{\text{wave}} \cdot \mathbf{B}_{\text{osc}}}{4\pi}. \tag{26}$$

In general, the interference field energy density is oscillating. Here, we look for terms that are nonzero after averaging over time. We see at once that

$$\langle u_{\text{wave,static}} \rangle = 0, \tag{27}$$

since all terms have time dependence of $\cos \omega t$ or $\sin \omega t$. In contrast, $\langle u_{\text{wave,osc}} \rangle$ will be nonzero as its terms are products of sines and cosines, and we find [8] that

$$U_{\text{int}} = \int_V \langle u_{\text{wave,osc}} \rangle = -\frac{4}{3}\eta^2 mc^2, \tag{28}$$

for waves of either linear or circular polarization. As with the case of the interference field momentum, this interference field energy is distributed over a volume of order a cubic wavelength around the electron. Being an interference term, its sign can be negative.

We can interpret the quantity

$$\frac{U_{\text{int}}}{c^2} = -\frac{4}{3}\eta^2 m \tag{29}$$

as compensation for the relativistic mass increase of the oscillating electron, which scales as v_{rms}^2/c^2 and hence as η^2 (for small η, recall eq. (8)). Indeed, a general result for the motion of an electron in a plane wave of arbitrary strength η is that its rms relativistic mass, often called its effective mass, is [9]

$$m_{\text{eff}} = m\sqrt{1+\eta^2}. \tag{30}$$

For small η, the increase in mass is

$$\Delta m \approx \frac{1}{2}\eta^2 m. \tag{31}$$

Thus, the decrease in field energy due to the interference terms between the electromagnetic fields of the wave and electron is $-8/3$ times the mass increase it should compensate.

4.1 The Radiation Reaction

Our analysis of the energy balance of an electron in a plane wave is not quite complete. We have neglected the energy radiated by the electron. Since the rate of radiation is constant (once the electron is inside the plane wave), the total radiated energy grows linearly with time, and eventually becomes large. The interference energy (28) is constant in time, and hence cannot account for the radiated energy.

In the present example, it appears that the radiated energy is not compensated by a decrease in the near zone electromagnetic energy. Rather, the mechanical energy of the charge must be decreasing, due to the effect of the radiation reaction force first identified by Lorentz [13]. This contrasts with the case of a uniformly accelerating charge, for which the mechanical radiation reaction vanishes while the radiated energy increases at the expense of the near zone field energy, as first discussed by Schott [14].

References

[1] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://physics.princeton.edu/~mcdonald/examples/EM/poynting_ptrsl_175_343_84.pdf

[2] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_pm_31_149_91.pdf

[3] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_recent_researches_sec_1-16.pdf

[4] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://physics.princeton.edu/~mcdonald/examples/thomson.pdf

[5] H. Poincaré, *Théorie de Lorentz et le Principe de la Réaction*, Arch. Neérl. **5**, 252 (1900),
http://physics.princeton.edu/~mcdonald/examples/EM/poincare_an_5_252_00.pdf

[6] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975), prob. 2, § 47 and prob. 2, § 49; p. 112 of the 1941 Russian edition,
http://physics.princeton.edu/~mcdonald/examples/EM/landau_teoria_polya_41.pdf

[7] E.M. McMillan, *The Origin of Cosmic Rays*, Phys. Rev. **79**, 498 (1950),
http://physics.princeton.edu/~mcdonald/examples/accel/mcmillan_pr_79_498_50.pdf

[8] K.T. McDonald and K. Shmakov, *The Classical "Dressing" of a Free Electron in a Plane Electromagnetic Wave*; http://xxx.lanl.gov/abs/physics/0003058,
http://physics.princeton.edu/~mcdonald/accel/dressing.pdf

[9] T.W.B. Kibble, *Frequency Shift in High-Intensity Compton Scattering*, Phys. Rev. **138**, B740, (1965), http://physics.princeton.edu/~mcdonald/examples/QED/kibble_pr_138_b740_65.pdf
Radiative Corrections to Thomson Scattering from Laser Beams, Phys. Lett. **20**, 627 (1966), http://physics.princeton.edu/~mcdonald/examples/QED/kibble_pl_20_627_66.pdf
Refraction of Electron Beams by Intense Electromagnetic Waves, Phys. Rev. Lett. **16**, 1054 (1966), http://physics.princeton.edu/~mcdonald/examples/QED/kibble_prl_16_1054_66.pdf
Mutual Refraction of Electrons and Photons, Phys. Rev. **150**, 1060 (1966),
http://physics.princeton.edu/~mcdonald/examples/QED/kibble_pr_150_1060_66.pdf
Some Applications of Coherent States, in *Cargèse Lectures in Physics*, Vol. 2, ed. by M. Lévy (Gordon and Breach, New York, 1968), pp. 299-345,
http://physics.princeton.edu/~mcdonald/examples/QED/kibble_cl_2_299_68.pdf

[10] V. Hnizdo, *Hidden momentum and the electromagnetic mass of a charge and current carrying body*, Am. J. Phys. **65**, 55 (1997), and references therein,
http://physics.princeton.edu/~mcdonald/examples/EM/hnizdo_ajp_65_55_97.pdf

[11] J. Schwinger, *Electromagnetic Mass Revisited*, Found. Phys. **13**, 373 (1983),
http://physics.princeton.edu/~mcdonald/examples/EM/schwinger_fp_13_373_83.pdf

[12] P. Moylan, *An Elementary Account of the Factor of 4/3 in the Electromagnetic Mass*, Am. J. Phys. **63**, 818 (1995),
http://physics.princeton.edu/~mcdonald/examples/EM/moylan_ajp_63_818_95.pdf

[13] H.A. Lorentz, *La Théorie Électromagnétique de Maxwell et son Application aux Corps Mouvants*, Arch. Neérl. **25**, 363 (1892), reprinted in *Collected Papers* (Martinus Nijhoff, The Hague, 1936), Vol. II, pp. 64-343,
http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_theorie_electromagnetique_92.pdf
Weiterbildung der Maxwellschen Theorie. Elektronen Theorie, Enzykl. Math. Wiss. **5**, part II, no. 14, 151-279 (1903), especially secs. 20-21;
Contributions to the Theory of Electrons, Proc. Roy. Acad. Amsterdam **5**, 608 (1903), reprinted in *Collected Papers*, Vol. III, pp. 132-154,
http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_praa_5_608_08.pdf
The Theory of Electrons (B.G. Teubner, Leipzig, 1906; reprinted by Dover Publications, New York, 1952), secs. 26 and 37 and note 18,
http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_theory_of_electrons_09.pdf

[14] G.A. Schott, *On the Motion of the Lorentz Electron*, Phil. Mag. **29**, 49 (1915),
http://physics.princeton.edu/~mcdonald/examples/EM/schott_pm_29_49_15.pdf