Capacitance of a Thin Conducting Disk and of Conducting Spheroids

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(February 25, 2003)

1 Problem

Estimate the capacitance of a thin conducting disk in the form of a right circular cylinder of radius a and height $2b$, where $b \ll a$.

Compare this case with that of a disk in the form of a thin oblate spheroid of diameter $2a$ and height $2b$.

Note that an analytic expression can be given for the capacitance of an oblate or prolate spheroid of any ratio b/a.

2 Solution

2.1 Thin Conducting Disk (Right Circular Cylinder)

The problem of a conducting right circular cylinder has been considered by Maxwell [1] in one of his last papers, and by Smythe [2], who obtained somewhat different results.

Maxwell's method was to calculate the electrical energy U stored by a disk with charge Q according to

$$U = \frac{1}{2}\int_1\int_2 \frac{\sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)}{r_{12}} d\text{Area}_1\, d\text{Area}_2 = \frac{Q^2}{2C}\,, \tag{1}$$

where σ is the surface charge density and C is the capacitance, in Gaussian units. However, he appears to have neglected the contribution of the charge density on the narrow cylindrical surface at $r = a$ (while commenting on this in his final sentence).

Smythe considers the charge distribution on a cylinder (where $a \approx b$) to be made up of a set of rings whose charges are adjusted so that the potential inside the cylinder is uniform.

Here, we evaluate the potential at the center of the cylinder,

$$V(0) = \int \frac{\sigma(\mathbf{r})}{r} d\text{Area} = \frac{Q}{C}\,, \tag{2}$$

supposing (like Maxwell), that the charge density σ over (most of) the flat surface the disk has same form as for an infinitely thin disk [3], namely

$$\sigma(r) = \frac{Q'}{4\pi a\sqrt{a^2 - r^2}}\,. \tag{3}$$

If we let s measure the distance radially inward from the circumference of the disk, then the charge density (3) varies as $1/\sqrt{s}$ for small s. However, the charge density near the edge of a conductor whose surfaces intersect at an exterior angle of $3\pi/2$, as in the present problem, is known to vary as $1/\sqrt[3]{s}$, for s measured normal to the edge along either surface [4].

Our approximation is to suppose that the surface charge density obeys

$$\sigma_s = \frac{K}{\sqrt[3]{s}} \qquad (s \le b) \tag{4}$$

for distance $s < b$ on both the flat and cylindrical surfaces of the disk, and that it obeys eq. (3) for $s > b$, *i.e.*, for $r < a - b$. Continuity of the charge density at $r = a - b$ requires that

$$\frac{K}{\sqrt[3]{b}} = \frac{Q'}{4\pi a\sqrt{a^2 - (a-b)^2}} \approx \frac{Q'}{4\pi a\sqrt{2ab}}\,. \tag{5}$$

Hence,

$$K = \frac{Q'}{4\pi a\sqrt{2a}\,b^{1/6}}\,. \tag{6}$$

In this approximation, the total charge on the disk for $s < b$ is

$$Q_1 = 4\int_0^b \frac{K}{\sqrt[3]{s}} 2\pi a\, ds = 12\pi a b^{2/3} K = 3\sqrt{\frac{b}{2a}}Q' = \frac{3}{2}\sqrt{\frac{2b}{a}}Q'. \tag{7}$$

The amount of charge on the cylindrical surface of the disk is not of order b/a, but of order of the much larger quantity $\sqrt{b/a}$. Neglect of this charge result in the correction to the capacitance being of the wrong order of smallness in b/a, as seen below.

The total charge on the flat surfaces of the disk for $r < a - b$ is

$$Q_2 = 2\int_0^{a-b} \frac{Q'}{4\pi a\sqrt{a^2 - r^2}} 2\pi r\, dr = Q'\left(1 - \frac{\sqrt{a^2 - (a-b)^2}}{a^2}\right) \approx Q'\left(1 - \sqrt{\frac{2b}{a}}\right). \tag{8}$$

Of course, $Q_1 + Q_2 = Q$, so that

$$Q' \approx Q\left(1 - \frac{1}{2}\sqrt{\frac{2b}{a}}\right) = Q\left(1 - \sqrt{\frac{b}{2a}}\right). \tag{9}$$

Finally, we evaulate the potential V at the center of the disk. The contribution to the potential due to the charge with $s < b$ near the edge of the disk is, to the first approximation,

$$V_1 \approx \frac{Q_1}{a} \approx 3\sqrt{\frac{b}{2a}}\frac{Q}{a}\,. \tag{10}$$

The potential at the origin due to the charge at $r < a - b$ is, using Dwight 380.001 and the fact that $\sin^{-1}(1-\epsilon) \approx \pi/2 - \sqrt{2\epsilon}$,

$$\begin{aligned} V_2 &\approx 2\int_0^{a-b} \frac{Q'}{4\pi a\sqrt{a^2 - r^2}} \frac{2\pi r\, dr}{\sqrt{r^2 + b^2}} = \frac{Q'}{2a}\int_0^{(a-b)^2} \frac{dr^2}{\sqrt{-r^4 + (a^2 - b^2)r^2 + a^2 b^2}} \\ &= -\frac{Q'}{2a}\sin^{-1}\frac{a^2 - b^2 - 2r^2}{a^2 + b^2}\bigg|_0^{a-b} \approx \frac{Q'}{2a}\left[\frac{\pi}{2} + \sin^{-1}\left(1 - \frac{4b}{a}\right)\right] \\ &\approx \frac{Q}{2a}\left(1 - \sqrt{\frac{b}{2a}}\right)\left(\pi - 4\sqrt{\frac{b}{2a}}\right) \approx \frac{\pi Q}{2a} - \frac{Q}{2a}(4+\pi)\sqrt{\frac{b}{2a}}\,. \end{aligned} \tag{11}$$

The total potential at the origin, and hence the potential of the conductor, is

$$V = V_1 + V_2 \approx \frac{\pi Q}{2a} - \frac{Q}{2a}(\pi - 2)\sqrt{\frac{b}{2a}} = \frac{\pi Q}{2a}\left(1 - \frac{\pi - 2}{\pi}\sqrt{\frac{b}{2a}}\right). \tag{12}$$

The capacitance is therefore,[1]

$$C \approx \frac{2a}{\pi}\left(1 + \frac{\pi - 2}{\pi}\sqrt{\frac{b}{2a}}\right) \approx \frac{2a}{\pi}\left(1 + 0.26\sqrt{\frac{b}{a}}\right) \qquad (b \ll a). \tag{13}$$

The result of Smythe's numerical calculation [2] for $1/8 < b/a < 8$ is that

$$C_{\rm Smythe} \approx \frac{2a}{\pi}\left[1 + 0.87\left(\frac{b}{a}\right)^{0.76}\right] \qquad (1/8 < b/a < 8). \tag{14}$$

In contrast, Maxwell's calculation [1] for $b \ll a$ is

$$C_{\rm Maxwell} \approx \frac{2a}{\pi}\left(1 + \frac{1}{2\pi}\frac{b}{a}\ln\frac{a}{b}\right) \qquad (b \ll a). \tag{15}$$

If we ignore the charge density on the cylindrical surface, then $Q' = Q$ and $V = V_2$ where the upper limit of integration in eq. (11) is a rather than $a - b$. Then we would find

$$\begin{aligned} V &\approx 2\int_0^a \frac{Q}{4\pi a\sqrt{a^2 - r^2}}\frac{2\pi r\, dr}{\sqrt{r^2 + b^2}} = -\frac{Q}{2a}\sin^{-1}\frac{a^2 - b^2 - 2r^2}{a^2 + b^2}\Bigg|_0^a \\ &\approx \frac{Q}{2a}\left[\frac{\pi}{2} + \sin^{-1}\left(1 - \frac{2b^2}{a^2}\right)\right] \approx \frac{\pi Q}{2a}\left(1 - \frac{2}{\pi}\frac{b}{a}\right), \end{aligned} \tag{16}$$

and hence,

$$C \approx \frac{2a}{\pi}\left(1 + \frac{2}{\pi}\frac{b}{a}\right) \qquad (b \ll a). \tag{17}$$

Although the assumptions leading to this result are essentially the same as those of Maxwell, we do not find the logarithmic factor seen in eq. (15). And, we see from eqs. (13) and (17) that the inclusion of the charge density on the cylindrical surface changes the dependence of the correction term from b/a to $\sqrt{b/a}$. It is amusing that Symthe's result lies between these two cases (although his calculation does not necessarily apply for $b \ll a$).

[1] As a slight generalization, we might suppose that the $1/\sqrt[3]{s}$ dependence of the charge density on the flat surface of the disk extends inwards a distance kb rather than b. If so, the coefficient 0.26 in eq. (13) would be multiplied by $(3 - k^{2/3})/2k^{1/6}$. For $k > 3^{3/2} = 5.2$, the sign of the correction to the capacitance would reverse. However, it is intuitive that the increase of surface area as height b increases should be associated with an increase in capacity of the conductor.

2.2 Conducting Spheroids

We now consider the example of a spheriod (about the z axis), whose surface is given by

$$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1. \tag{18}$$

For $b < a$ the spheroid is oblate, while for $b > a$ it is prolate; of course, for $b = a$ it is a sphere.

We take advantage of the result that for any spheroid of equatorial radius a, the projection onto the plane $z = 0$ of the charge density on the upper ($z > 0$) surface of the spheroid is given by eq. (3), where Q' equals the total charge Q [3]. Hence, the potential at the origin can be calculated similarly to eqs. (11) or (16),

$$\begin{aligned} V &= 2\int_0^a \frac{Q}{4\pi a\sqrt{a^2 - r^2}} \frac{2\pi r\, dr}{\sqrt{r^2 + z^2}} = 2\int_0^a \frac{Q}{4\pi a\sqrt{a^2 - r^2}} \frac{2\pi r\, dr}{\sqrt{r^2 + b^2 - r^2 b^2/a^2}} \\ &= \frac{Q}{2a}\int_0^{a^2} \frac{dr^2}{\sqrt{-r^4(1 - b^2/a^2) + (a^2 - 2b^2)r^2 + a^2 b^2}}. \end{aligned} \tag{19}$$

The cases of $b < a$, $b = a$ and $b > a$ must be treated separately.

For a solution via spheroidal coordinates, see problems 467 and 469 of [5].

2.2.1 Oblate Spheroid ($b < a$)

Again using Dwight 380.001 (or Gradshteyn and Ryzhik 2.261), the potential at the origin follows from eq. (19) as

$$\begin{aligned} V &= -\frac{Q}{2a\sqrt{1 - b^2/a^2}} \sin^{-1} \left. \frac{a^2 - 2b^2 - 2r^2(1 - b^2/a^2)}{\sqrt{(a^2 - 2b^2)^2 + 4a^2 b^2 (1 - b^2/a^2)}} \right|_0^a \\ &= \frac{Q}{2\sqrt{a^2 - b^2}} \left(\frac{\pi}{2} + \sin^{-1}(1 - 2b^2/a^2)\right) = \frac{Q}{\sqrt{a^2 - b^2}} \sin^{-1} \frac{\sqrt{a^2 - b^2}}{a} = \frac{Q}{C}. \end{aligned} \tag{20}$$

For $b \ll a$ this becomes

$$V \approx \frac{\pi Q}{2a}\left(1 - \frac{2}{\pi}\frac{b}{a}\right), \tag{21}$$

so the capacitance of a thin oblate spheroid is

$$C \approx \frac{2a}{\pi}\left(1 + \frac{2}{\pi}\frac{b}{a}\right). \tag{22}$$

This is the same as our result (17) for a thin conducting disk (right circular cylinder) with neglect of the charge on the cylindrical surface of that disk.

As b approaches a, *i.e.*, as the spheroid approaches a sphere, both the numerator and the denominator of eq. (21) approach zero. In this limit, the arcsine can be written as

$$\sin^{-1}\left(2\frac{a^2 - b^2}{a^2} - 1\right) = -\sin^{-1}\left(1 - 2\frac{a^2 - b^2}{a^2}\right) \approx -\frac{\pi}{2} + \frac{2\sqrt{a^2 - b^2}}{a}, \tag{23}$$

so the potential goes to Q/a as for a sphere.

2.2.2 Sphere ($a = b$)

The capacitance of a sphere is, of course, simply its radius a. Here, we confirm that our method of calculating the potential at the origin leads to this well-known result.

When $b = a$, eq. (19) becomes

$$V = \frac{Q}{2a^2}\int_0^{a^2} \frac{dr^2}{\sqrt{a^2 - r^2}} = -\frac{Q}{a^2}\sqrt{a^2 - r^2}\bigg|_0^a = \frac{Q}{a}\,, \tag{24}$$

as expected.

2.2.3 Prolate Spheroid ($b > a$)

Equation (19) now integrates to[2]

$$\begin{aligned} V &= \frac{Q}{2a\sqrt{b^2/a^2 - 1}} \ln\left| 2\sqrt{\frac{b^2}{a^2} - 1}\sqrt{\left(\frac{b^2}{a^2} - 1\right) r^4 + (a^2 - 2b^2) r^2 + a^2 b^2} + 2\left(\frac{b^2}{a^2} - 1\right) r^2 + a^2 - 2b^2 \right|_0^a \\ &= \frac{Q}{2\sqrt{b^2 - a^2}} \ln \frac{a^2}{2b^2 - a^2 - 2b\sqrt{b^2 - a^2}} = \frac{Q}{\sqrt{b^2 - a^2}} \ln \frac{a}{b - \sqrt{b^2 - a^2}} \\ &= \frac{Q}{\sqrt{b^2 - a^2}} \ln \frac{b + \sqrt{b^2 - a^2}}{a} = \frac{Q}{2\sqrt{b^2 - a^2}} \ln \frac{b + \sqrt{b^2 - a^2}}{b - \sqrt{b^2 - a^2}} = \frac{Q}{C}\,. \end{aligned} \tag{26}$$

Again, as b approaches a both the numerator and denominator of eq. (26) approach zero. In this limit, the logarithm can be approximated by

$$\ln\left(1 + \frac{\sqrt{b^2 - a^2}}{a}\right) \approx \frac{\sqrt{b^2 - a^2}}{a}, \tag{27}$$

so the potential goes to Q/a as expected for a sphere.

For $b \gg a$ the prolate spheroid takes the form of a conducting needle of length b and radius a, and eq. (26) becomes, to a first approximation,

$$V \approx \frac{Q}{b} \ln \frac{2b}{a}\,, \qquad \Rightarrow \qquad C \approx \frac{b}{\ln(2b/a)}\,. \tag{28}$$

Maxwell [1] has disccussed higher approximations to the capacitance of a needle. See also [7].

As noted in [3], the charge distribution as projected onto the z axis is uniform for a conducting spheroid of any ratio b/a, so in particular, the charge distribution along the axis of our prolate needle is uniform. Jackson [8] has considered an amusing paradox concerning the uniform charge distribution of a needle. See also [9].

---

[2] The capacitance of a conducting prolate spheroid can also be calculated by noting that the field due to charge Q on the spheroid is the same as that due to charge Q spread uniformly along the line $|x| \le c \equiv \sqrt{b^2 - a^2}$ [6]. The voltage on the conductor is conveniently evaluated at $z = b$ to be

$$V = \int_{-c}^{c} \frac{Q}{2c}\frac{dz}{b - z} = -\frac{Q}{2c}\ln(b - z)\bigg|_{-c}^{c} = \frac{Q}{\sqrt{b^2 - a^2}} \ln \frac{b + \sqrt{b^2 - a^2}}{a}\,. \tag{25}$$

3 References

[1] J.C. Maxwell, *On the Electrical Capacity of a long narrow Cylinder, and of a Disk of sensible Thickness*, in *Scientific Papers* (Dover, New York, 1965), Vol. II, p. 672,
http://puhep1.princeton.edu/~mcdonald/examples/EM/maxwell_plms_9.pdf

[2] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, New York, 1968), sec. 5.39; *Charged Right Circular Cylinder*, J. Appl. Phys. **27**, 917 (1956); **33**, 2966 (1962),
http://puhep1.princeton.edu/~mcdonald/examples/EM/smythe_jap_27_917_56.pdf
http://puhep1.princeton.edu/~mcdonald/examples/EM/smythe_jap_33_2966_62.pdf

[3] K.T. McDonald, *Conducting Ellipsoid and Circular Disk* (Oct. 19, 2002),
http://puhep1.princeton.edu/~mcdonald/examples/ellipsoid.pdf

[4] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999), sec. 2.11.

[5] N.N. Lebedev, I.P. Skalskaya and Y.S. Uflyand, *Worked Problems in Applied Mathematics* (Dover, New York, 1979).

[6] R. Becker, *Electromagnetic Fields and Interactions* (Dover, New York, 1982), sec. 22.

[7] J.B. Keller, *Charge density on a slender axially symmetric conducting body*, Am. J. Phys. **71**, 282 (2003),
http://puhep1.princeton.edu/~mcdonald/examples/EM/keller_ajp_71_282_03.pdf

[8] J.D. Jackson, *Charge density on thin straight wire, revisted*, Am. J. Phys. **68**, 789 (2000),
http://puhep1.princeton.edu/~mcdonald/examples/EM/jackson_ajp_68_789_00.pdf *Charge density on a thin straight wire: The first visit*, Am. J. Phys. **70**, 409 (2002),
http://puhep1.princeton.edu/~mcdonald/examples/EM/jackson_ajp_70_409_02.pdf

[9] J.C. Maxwell, *On a Paradox in the Theory of Attractions*, in *Scientific Papers* (Dover, New York, 1965), Vol. II, p. 599,
http://puhep1.princeton.edu/~mcdonald/examples/EM/maxwell_pcps_3_77.pdf