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Notes on Synchrotron Radiation

Synchrotron radiation occurs when a charged particle, typically an electron, experiences an acceleration transverse to its velocity, as caused by a magnetic field. The following notes emphasize the case when the particle's velocity approaches the speed of light, the limit in which synchrotron radiation becomes prominent.

1 Total Rate

The total (classical) radiation rate is obtained from the Larmor formula:

$$P = \frac{dU}{dt} = \frac{2}{3}\frac{e^2 a^{\star 2}}{c^3} = \frac{2}{3}\frac{e^2\gamma^4 c}{\rho^2}, \tag{1}$$

where $a^\star$ is the acceleration in the electron's rest frame, and ρ is the radius of curvature. We have used $a_{\rm lab} = v^2/\rho \approx c^2/\rho$ for an electron with relativistic velocity. Also, $a^\star = \gamma^2 a_{\rm lab}$ with $\gamma = 1/\sqrt{1 - v^2/c^2}$, noting that the acceleration is transverse, so the transformation involves two powers of the time dilation.

2 The Angular Distribution

The angular distribution of radiated power (for time t measured at the charge, *i.e.*, for retarded time) is obtained via the invariant transformation of the Larmor angular distribution:

$$\frac{dU}{dt d\Omega} = \frac{e^2\gamma^4 c}{4\pi\rho^2}\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}, \tag{2}$$

where $\beta = v/c$ and angles (θ, ϕ) are measured with respect to the direction of the electron's motion and with the x-axis towards the center of the electron's orbit.

For highly relativistic motion, $\gamma \gg 1$, the radiation is peaked forward with characteristic angle $\theta \approx 1/\gamma$. Then $1 - \beta\cos\theta \approx (\theta^2 + 1/\gamma^2)/2$. In the plane of the orbit we have

$$\frac{dU}{dt d\Omega} \approx \frac{2e^2\gamma^{10} c}{\pi\rho^2}\frac{[1-(\gamma\theta)^2]^2}{[1+(\gamma\theta)^2]^5}. \tag{3}$$

3 The Critical Frequency

The 'cone' of synchrotron radiation passes over a fixed angle θ in (retarded) time

$$\Delta t' \approx \frac{\Delta\theta}{\omega_0} = \frac{1}{\gamma\omega_0}, \tag{4}$$

where $\omega_0 = v/\rho \approx c/\rho$ is the angular frequency of the electron's motion and $\Delta\theta \approx 1/\gamma$ is the width of the angular distribution.

For a distant observer the corresponding time interval Δt is related by the usual transformation for retarded time:

$$\Delta t = \Delta t'(1 - \vec{\beta} \cdot \hat{r}) \approx \Delta t'(1 - \beta) \approx \frac{\Delta t'}{2\gamma^2} = \frac{1}{2\gamma^3\omega_0}. \tag{5}$$

Then a Fourier analysis of the pulse of width Δt will have spectral width $\Delta\omega \approx 1/\Delta t = 2\gamma^2\omega_0$. The radiation is strong at high harmonics of the orbital frequency and can be regarded as a continuum spectrum.

Detailed analysis of the frequency spectrum leads one to define the 'critical frequency' as

$$\omega_C \equiv \frac{3}{2}\gamma^3\omega_0 = \frac{3}{2}\frac{\gamma^3 c}{\rho}. \tag{6}$$

[Note that Jackson defines the critical frequency slightly differently. Eq. (6) is the usual convention.]

4 The QED Critical Field Strength

From quantum mechanics we know that the radiation at classical frequency ω consists of photons of energy $\hbar\omega$. For strong enough acceleration a single photon at the critical frequency will carry away all of the particle's energy. In this case

$$\gamma mc^2 = \hbar\omega_C \approx \gamma^3\hbar\omega_0 = \frac{\gamma^3\hbar c}{\rho}, \tag{7}$$

where m is the particle's rest mass. In this regime quantum corrections to the classical analysis are very important.

The field that can produce the needed large acceleration (as measured in the particle's rest frame, where the electric field is $E^\star = \gamma B$) is called the QED critical field strength. In c.g.s. units the critical electric and magnetic fields are equal: $B_{\rm crit} = E_{\rm crit}$.

To find this we note that $F = ma$ for circular motion in a magnetic field B perpendicular to the plane of the particle's orbit can be written

$$\gamma m\frac{v^2}{\rho} \approx \gamma m\frac{c^2}{\rho} = \frac{evB}{c} \approx eB, \tag{8}$$

Combining eqs. (7) and (8) and recalling that laboratory magnetic field B transforms to γB in the particle's rest frame, we have

$$B_{\rm crit} = \gamma B = \frac{m^2c^3}{e\hbar} = 4.41 \times 10^{13} \text{ gauss}. \tag{9}$$

[The QED critical electric field is $E_{\rm crit} = m^2c^3/e\hbar = 1.32\times 10^{16}$ V/cm is such that the energy gain across a Compton wavelength is an electron rest mass: $eE_{\rm crit}(\hbar/mc) = mc^2$. At such field strengths the QED vacuum fluctuations into electron-positron pairs materialize at a high rate and the vacuum 'sparks'.]

The critical frequency, (6), of the synchrotron radiation can be usefully expressed in terms of the critical magnetic field. The result is more memorable when one considers the critical energy $u_C = \hbar\omega_C$:

$$u_C = \frac{3}{2}\gamma^3\frac{\hbar c}{\rho} = \frac{3}{2}\gamma^3\frac{e\hbar B}{\gamma mc} = \frac{3}{2}\gamma^2 mc^2\frac{e\hbar B}{m^2c^3} = \frac{3}{2}u\frac{\gamma B}{B_{\rm crit}}. \tag{10}$$

The invariant rate of radiation, (1), can also be re-expressed in terms of the QED critical field. It is perhaps quickest to note that in the particle's rest frame there is an electric field of strength $E^\star = \gamma B$, so the acceleration there is $a^\star = eE^\star/m = e\gamma B/m$, and hence

$$P = \frac{2}{3}\frac{e^2a^{\star 2}}{c^3} = \frac{2}{3}\frac{e^2}{c^3}\frac{e^2\gamma^2B^2}{m^2}\left[\frac{m^4c^6}{B_{\rm crit}^2e^2\hbar^2}\right] = \frac{2}{3}\frac{e^2}{\hbar c}\frac{mc}{\hbar}c\cdot mc^2\left(\frac{\gamma B}{B_{\rm crit}}\right)^2 = \frac{2}{3}\alpha\frac{c}{\lambda_C}mc^2\left(\frac{\gamma B}{B_{\rm crit}}\right)^2, \tag{11}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and $\lambda_C = \hbar/mc$ is the Compton wavelength of the particle. That is, when the field (in the particle's rest frame) equals the critical field the energy radiated in the QED characteristic time λ_C/c is α times the particle's energy.

5 The Frequency Spectrum

5.1 Approximate Calculation

A next approximation to the frequency spectrum can be obtained using an argument due to Feynman [1]. [Supposedly it was this problem that led him to invent his version of the fields of an accelerated charge.]

A charge e moves in a circle of radius ρ about the origin in the x-y plane:

$$\begin{aligned} x &= \rho\sin\omega_0 t, \\ y &= \rho\cos\omega_0 t. \end{aligned} \tag{12}$$

We observe the radiation at $(x, y) = (r_0, 0)$ where $r_0 \gg \rho$. Feynman tells us that the radiation field has y-component

$$E_y \approx -\frac{e}{c^2 r_0}\frac{d^2y(t')}{dt^2}, \tag{13}$$

where $t' = t - r(t')/c \approx t - r_0/c + (\rho/c)\sin\omega_0 t'$ is the retarded time. Then noting that

$$\frac{dt}{dt'} = 1 - \beta\cos\omega_0 t', \tag{14}$$

where $\beta = \rho\omega_0/c$ is the particle's velocity, we find

$$\frac{d^2y(t')}{dt^2} = \rho\omega_0^2\frac{\beta - \cos\omega_0 t'}{(1-\beta\cos\omega_0 t')^3}. \tag{15}$$

The radiation is big only for $\omega t' \approx 2n\pi$. We will take the Fourier analysis of only the pulse near $t' = 0$. For this we eliminate t' in favor of $T = t - r_0/c$. For $\beta \approx 1$ we find $t' \approx 2\gamma^2 T$, and

$$\cos\omega_0 t' - \beta \approx \frac{1 - 4\gamma^6\omega_0^2 T^2}{2\gamma^2}, \qquad 1 - \beta\cos\omega_0 t' \approx \frac{1 + 4\gamma^6\omega_0^2 T^2}{2\gamma^2}, \tag{16}$$

and hence

$$E_y(T) \approx \frac{1 - 4\gamma^6\omega_0^2 T^2}{1 + 12\gamma^6\omega_0^2 T^2}. \tag{17}$$

The Fourier transform of this varies as [2]

$$E_y(\omega) = e^{-\omega/2\sqrt{3}\gamma^3\omega_0}. \tag{18}$$

The power spectrum of the pulse, U_ω goes as

$$U_\omega \propto E_y^2(\omega) \propto e^{-\omega/\sqrt{3}\gamma^3\omega_0} \equiv e^{-\omega/\omega_C}, \tag{19}$$

where the critical frequency is

$$\omega_C = \sqrt{3}\gamma^3\omega_0. \tag{20}$$

A Fourier analysis of the pulse train rather than of only a single pulse, reduces the $\sqrt{3}$ in the critical frequency to 3/2 and reveals the roll-off at frequencies below $\omega_C/3$.

5.2 Detailed Results

The radiation of charged particles executing large numbers of cycles of circular motion was first studied by Schott [3], but became well known only with the works of Arzimovitch and Pomeranchuk [4] and Schwinger [5, 6]. This note follows sec. 5 of the review by Sands [7],

A detailed Fourier analysis of the radiation rate P leads to the following form for the power spectrum

$$U_\omega = \frac{P}{\omega_C} F(\omega/\omega_C), \tag{21}$$

where the function $F(x)$ is illustrated in Figs. 1 and 2.

At high frequencies the function F obeys

$$F(x) \approx 0.78\sqrt{x}e^{-x}, \qquad (x \gg 1), \tag{22}$$

which form was anticipated by the argument of Feynman. At low frequencies,

$$F(x) \approx 1.3x^{1/3}, \qquad (x \ll 1), \tag{23}$$

An "elementary" argument for the low-frequency limit is given in [8]. A useful approximation to the spectral function $F(x)$ is shown in Fig. 3.

In practice, considerations of synchrotron radiation involve numbers of photons radiated rather than a frequency spectrum. We desire the number $n(u)$ per energy interval du per second. Since $un(u)$ is the power spectrum, we have

$$un(u)du = U_\omega d\omega = \frac{P}{\omega_C} F(\omega/\omega_C)\frac{du}{\hbar} = \frac{P}{u_C} F(u/u_C)du, \tag{24}$$

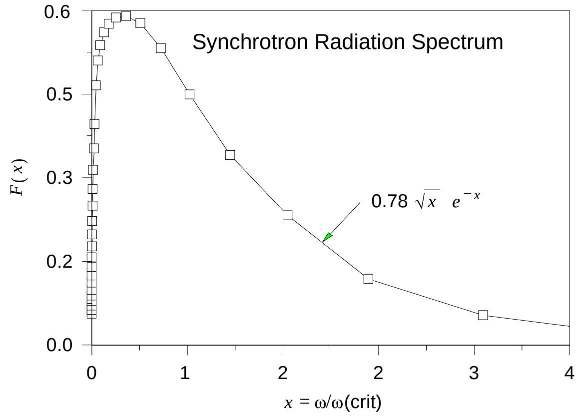

Figure 1: The synchrotron radiation spectral function $F(x)$.

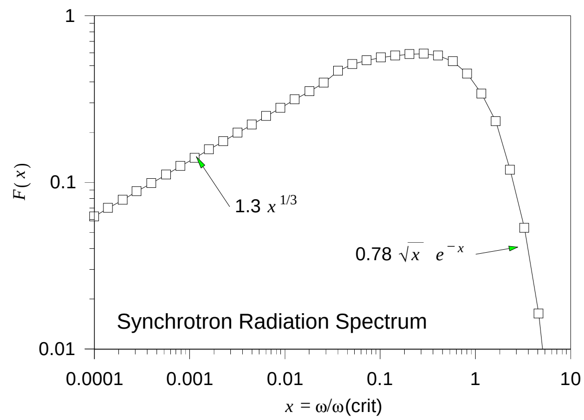

Figure 2: The synchrotron radiation spectral function $F(x)$.

and hence

$$n(u) = \frac{P}{u_C^2} \frac{F(u/u_c)}{u/u_C}. \tag{25}$$

An interesting variation describes the number of photons emitted into angle $\Delta\theta$ along the particle's orbit. The corresponding time interval is $\Delta t = \rho\Delta\theta/c$. Then the total number of photons emitted is $dN = \Delta t \int n(u)du$. Here we rewrite eqs. (1) and (6) as

$$P = \frac{2}{3}\frac{e^2\gamma^4 c}{\rho^2}, \qquad \text{and} \qquad u_C = \frac{3}{2}\frac{\gamma^3 c}{\rho}. \tag{26}$$

Combining these we find

$$dN = \frac{4}{9}\gamma\alpha\Delta\theta \int F(x)\frac{dx}{x} \approx \gamma\alpha\Delta\theta. \tag{27}$$

A detailed calculation indicates that the numerical coefficient is $5/2\sqrt{3} = 1.44$. The number of photons emitted per unit angle is independent of the strength of the field, *i.e.*, of the critical energy! Of course, the total energy emitted is roughly the number of photons times the critical energy, so this varies linearly with the critical energy.

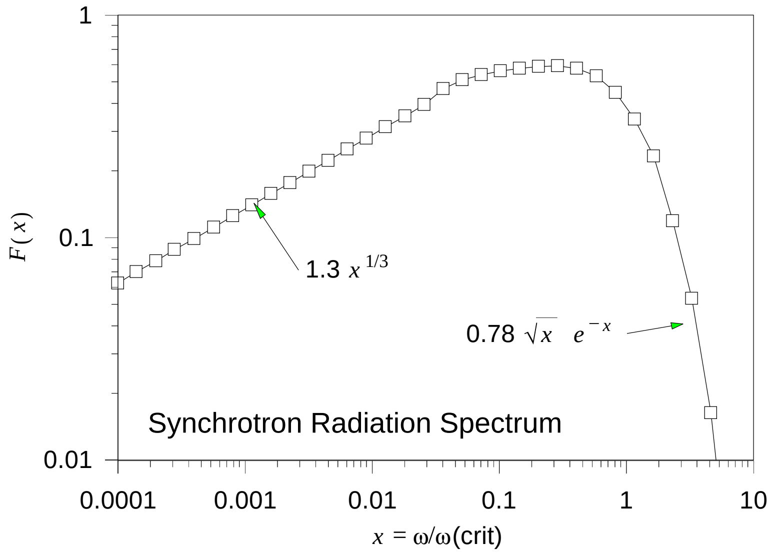

Figure 3: A useful approximation to the synchrotron radiation spectral function $F(x)$ is $1.1x^{0.3}e^{-x}$.

6 Synchrotron Radiation in a 'Short' Magnet

The detailed calculation of the synchrotron radiation spectrum is based on a Fourier analysis of cyclic motion. Yet one often speaks of synchrotron radiation of particles that undergo only a small angular deflection upon crossing a short region of magnetic field. Could the path length in the magnet over which the synchrotron radiation is produced could be so short that the above expression does not apply?

This effect was first discussed by Coïsson's [9, 10] who noted that the standard results for the spectrum of synchrotron radiation hold only if the particle is deflected by an angle larger than the characteristic width of the angular distribution of the radiation, namely $1/\gamma$.

Coïsson also noted that if the magnetic field strength changes significantly over a distance corresponding to a deflection angle less than $1/\gamma$, then the radiation spectrum is distorted from that due to an adiabatic approximation. In general, the spectrum in a rapidly varying field is wider than that due to a uniform field, with enhancements at both high and low frequencies. The enhancement at high frequencies has been verified by measurement of optical synchrotron radiation from protons at CERN [11, 12].

Synchrotron radiation in 'short' magnets is discussed further in papers by Bagrov *et al.* [13, 14], and by Hoffman and Méot [15]. A paper by Pomeranchuk [16] written in 1939 deserves notice. In this the earth is considered as a 'short' magnet that leads to large radiation from cosmic ray electrons. Indeed, Pomeranchuk notes that for electrons of energy greater than 10^{19} eV the earth's field appears to exceed the QED critical field strength and pair production is likely.

7 Interference of Synchrotron Radiation

Nikitin *et al.* [17] have observed an interesting interference effect in synchrotron radiation from the ends of two magnets separated by a straight section. Interference is prominent if the time delay between electrons and photons crossing the straight section is roughly one cycle. That is

$$\frac{\lambda}{c} \approx \Delta t = \frac{L}{\beta c} - \frac{L}{c} \approx \frac{L}{2\gamma^2 c}. \tag{28}$$

The interference can therefore be observed at wavelengths

$$\lambda \approx \frac{L}{\gamma^2}. \tag{29}$$

8 Coherence Effects

For wavelengths longer than the typical spacing between electrons in the beam bunch, coherence effects may be important.

The question remains as to what the coherence effect might be. In the first approximation there may be little effect, following the argument of sec. 20-4 (see also sec. 22-4) of Panofsky and Phillips [18] that in the presence of random density fluctuations the intensity of the radiation remains directly proportional to the number of charges.

On the other hand, in the continuum limit the resulting interference effects must be destructive. There is no radiation from a completely steady current loop!

9 An Example

Shown below are numerical calculations of synchrotron radiation of a pulse of 10^{10} electrons of 50 GeV energy $\gamma \approx 10^5$ at the Stanford Linear Accelerator Center. While most x-rays are produced in the strongest magnet, the earth's magnetic field gives the greatest contribution of optical photons!

Table 1: Parameters of magnets considered in the synchrotron radiation calculations.

Magnet	B (Gauss)	L (m)	E_{critical} (keV)
Alnico dump magnet	4500	1	750
Soft bends	500	2	83
Very soft bends	60	2	10
Earth's magnetic field	0.33	50	0.054
Final Focus quadrupoles	9500 G/26 mm	2	≈ 85

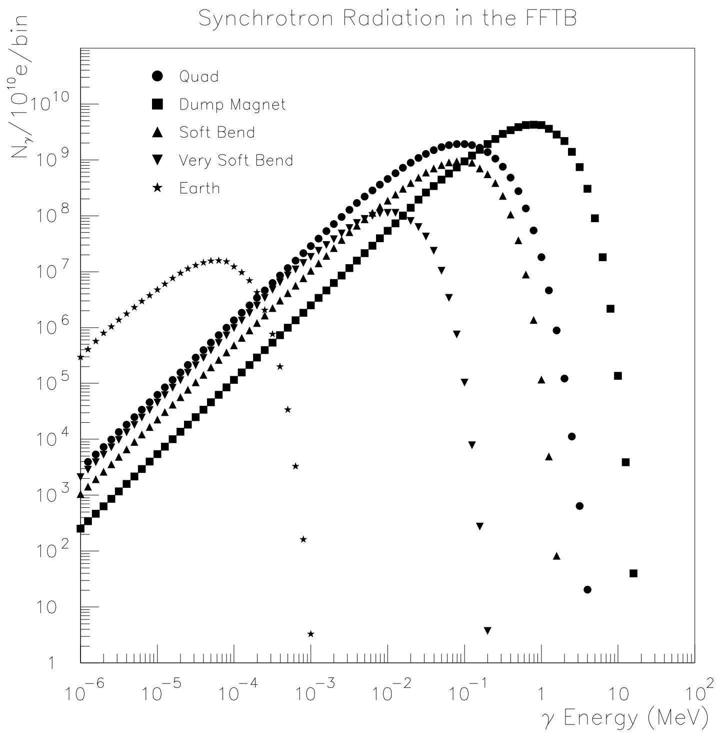

Figure 4: Synchrotron radiation spectra for the magnets in Table 1 and a pulse of 10^{10} 50-GeV electrons.

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