A Falling, Folded String

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1 Problem

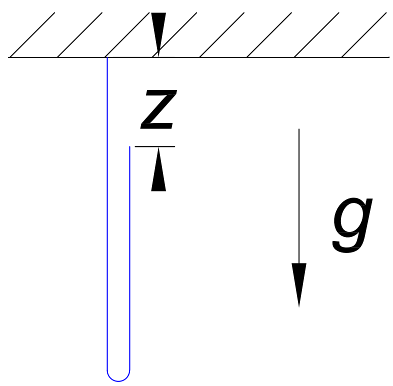
Discuss the motion of a folded (inextensible) string (or cable or chain), one end of which is fixed, after the other end is released from rest.[1]

2 Solution

2.1 Free Fall

We might suppose that there is no tension in the "free" portion of the string, which portion simply accelerates downward at rate g.[2]

For a configuration as sketched below, if the string has length l and is initially folded such that $z = 0$ at time $t = 0$ when one end is released, then subsequently $\ddot{z} = g$ until after time $t = \sqrt{2l/g}$ the string is entirely vertical, and again at rest.



[1]This classic problem was brought to the author's attention by Johann Otto.

[2]This view seems to be advocated by Routh (1896) in Ex. 4, p. 82 of [15].

Examples of falling chains were often considered at Cambridge U. [13, 14, 15, 16, 18, 20, 21], as recounted in a footnote on p. 80 of [15]: *Problems on infinitesimal impulses were solved in the lecture room of the late Mr Hopkins as long ago as 1850. A problem of this kind was set in the Smith's Prize examination in 1853 by Prof Challis, and a solution given in Tait and Steele's Dynamics*. For the latter, see pp. 250-251 of [7].

The earliest example of a variable-mass string problem may be due to Buquoy (1815) [5, 69, 94], to which the only published reference in the 19$^{\text{th}}$ century was by Poisson (1819) [6]. Also of note in the 19$^{\text{th}}$ century are papers by Cayley [8, 11], and those related to the laying of the transAtlantic cable [9, 10]. The latter activity includes the interesting phenomena of vertical "arching" as a slack string is pulled from a nominally horizontal configuration at rest [79, 88], which has led to the fascinating "chain fountain" [83, 84, 85, 89, 91, 93].

Variable-mass problems involving water jets were first considered by Torricelli (1644) [1] and first analyzed by Bernoulli (1738) [2, 3], and variable-mass rocket problems were perhaps first discussed by Moore (1813) [4]. A paper by de Mondesir (1887) [12] on variable-mass systems involving chains led to some debate, later discussed in [17].

This problem may have first appeared as Ex. II, p. 302 of [13] by Love (1897), where the above solution was advocated. Love also argued that the tension at the lowest point[3] of the portion of the string at rest is $T = \mu g^2 t^2 = 2Wz/l$, where μ is the mass per unit length and $W = \mu l g$ is the weight of the string, whose free end is at $z = gt^2/2$: *An element, of mass $\mu\, gt\, dt$, passes from motion with velocity gt to rest in an interval dt, so that the momentum destroyed by the impulse $T\, dt$ is $\mu g^2 t^2\, dt$. Hence $T = \mu g^2 t^2$.*

However, when the upper end of the string falls by dz = v dt = g t dt, only half of this length comes to rest,[4] Hence, the tension according to this analysis is actually $T = \mu g^2 t^2/2 = Wz/l$. This result was quoted in Ex. 5, p. 149 of [20] by Lamb (1914).[5]

Lamb [20] ended his discussion with the challenge: *Examine the loss of mechanical energy.* As the string falls its potential energy decreases by $Wl/4$, and the final kinetic energy is zero, although the kinetic energy of the string is nonzero while it is in motion. There must be some conversion of potential energy into "nonmechanical" energy as the string falls.[6] This conversion could be continual, or could occur only at the moment when the tip of the falling string comes to rest after briefly attaining very high velocity. In the latter scenario the falling string is like the cracking of a whip.[7] Another possibility is that kinetic plus potential energy is conserved as the string falls, and the analysis presented above is incorrect.[8] The most general possibility is that the motion is neither simply free fall, nor is mechanical energy conserved.

2.2 Mechanical Energy is Conserved

An early advocate of conservation of mechanical energy in motion where strings/chains/whips change their shape was Kucharski (1941) [22].[9] The first application of energy conservation to the present problem may have been by Hamel (1948), Ex. 100, pp. 643-645 of [28],[10] The first use energy conservation in a paper in English on a falling, folded string may be that of

[3]Called the bight in Ex. 5, p. 149 of [20].

[4]The length L of string at rest is related by $L = (l + z)/2$, so $dL = dz/2$.

[5]In prob. 8.30, p. 241 of [43] where the tension at the point of support of the string was stated to be $(W/2)(1+3z/l)$, which is the sum of the tension $T = 2(W/2)(z/l)$ at the bight plus the weight $\mu g(l+z)/2 = (W/2)(1 + z/l)$ of the portion of the string at rest. See also sec. II of [48].

[6]On p. 260 of [13], Love stated: *It is important to observe that discontinuous motions such as are considered here in general involve dissipation of energy.*

[7]The case of a string/tape wrapped around a massless spool which rolls down an incline, unwinding the string, was discussed in [23] (1941). It seems reasonable that energy is conserved during the rolling, but this implies that the remaining string on the spool attains very high kinetic energy, which is dissipated with a loud crack as the end of the tape comes to rest on the incline [25]. Regarding whips, see, for example, [36, 41, 44, 62, 63].

[8]The closely related problems of a string/chain sliding off a frictionless table, and a string falling off the table from a heap at its edge (Cayley's problem [8]), were contrasted by Sommerfeld (1943) in examples I.7 and I.8 of [24], where energy might be conserved in the first problem, but is not in the second. See also the Appendix below.

[9]One can consider the Lagrangian $\mathcal{L}(z) = T(z) - V(z)$ for the entire string, which does not depend explicitly on time, with the implication that energy is conserved. The resulting equation of motion for coordinate z is the same as eq. (2).

However, energy is only approximately conserved, so the Lagrangian method leads only to an approximate analysis, rather than an "exact" one.

[10]Hamel's argument also appeared in a few other German texts, as reported in [52].

[35] (1955).

The kinetic and potential energies are, taking the potential energy V, and the total energy $E = T + V$, to be zero at time $t = 0$ when one end of the string is released from rest at $z = 0$,

$$T = \frac{\mu(l-z)\dot{z}^2}{4}, \qquad V = -\frac{\mu g z(2l-z)}{4}, \qquad (l-z)\dot{z}^2 = gz(2l-z). \tag{1}$$

This leads to,

$$\ddot{z} = g + \frac{\dot{z}^2}{2(l-z)} = g\left(1 + \frac{z(2l-z)}{(l-z)^2}\right) > g, \tag{2}$$

and,

$$\begin{aligned} t &= \int_0^t dt = \int_0^z \frac{dz}{\dot{z}} = \int_0^z dz\sqrt{\frac{l-z}{gz(2l-z)}} = \sqrt{\frac{l}{g}}\int_0^{z/l} dx\sqrt{\frac{1-x}{x(2-x)}} \\ &= \sqrt{\frac{l}{g}}\int_0^{\sin^{-1}\sqrt{z/l}} d\theta\frac{\cos^2\theta}{\sqrt{1+\cos^2\theta}}, \end{aligned} \tag{3}$$

with the changes of variable $x = z/l = \sin^2\theta$. This is an elliptic integral, and in particular the entire fall time ($z = l$, $\theta = \pi/2$), turns out to be $0.85\sqrt{2l/g}$ [48], somewhat less than the time $\sqrt{2l/g}$ for free fall as in sec. 2.1 above.

For the string to accelerate downwards at a rate greater than g there must be a tension T_{bot} at the bottom of the string, where elements of the string are continually coming to rest, so one might have thought the string would under compression, rather than tension, there. This possibly surprising phenomenon is a general feature of the dynamics of strings, as reviewed in [77]. Here, we can use the equation of motion for the moving portion of the string to deduce T_{bot},

$$\mu\frac{l-z}{2}\ddot{z} = T_{\text{bot}} + \mu\frac{l-z}{2}g, \qquad T_{\text{bot}} = \frac{\mu\dot{z}^2}{4} = \frac{\mu g z(2l-z)}{4(l-z)}. \tag{4}$$

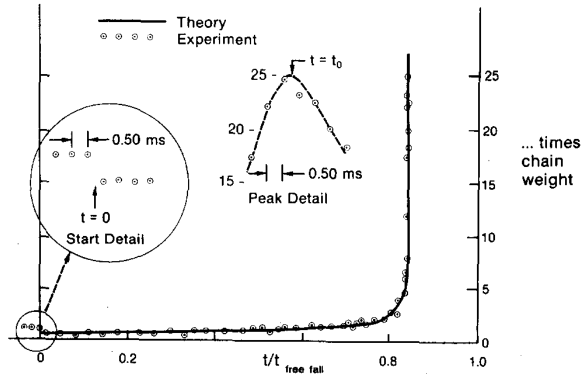
A quantity more accessible to experiment is the tension,

$$T_{\text{top}} = T_{\text{bot}} + \frac{\mu g(l+z)}{2} = \frac{\mu g(l^2 + 2lz - 2z^2)}{2(l-z)} = \frac{W}{2}\frac{1 + 2z/l - 2z^2/l^2}{1 - z/l}, \tag{5}$$

which starts from $\mu g l/2 = W/2$ and then diverges as the string falls.

Experiments on falling, folded strings have been reported in [35, 48, 54, 70, 75, 93]. The figure on the next page (from [48]) shows the observed tension T_{top} to be in excellent agreement with the above analysis, although of course the tension is never actually infinite.

Significant nonconservation of energy just before the tip of the string comes to rest prevents the velocity, acceleration and tension from becoming infinite, but conservation of energy appears to hold rather well during most of the motion of the falling, folded string.



Other discussions of this example are in [56, 60, 64, 65, 71, 72, 76, 87, 92]. General comments on variable-mass problems are also given in [24, 33, 34, 37, 39, 40, 43, 50, 58, 61, 66, 67, 81].

A Appendix: Cayley's Problem

In 1857, Cayley [8] considered the example where: *a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table.* A closely related example concerns the case where the portion of the chain on the (frictionless) table lies in a straight line perpendicular to its edge.

Here we review solutions based on the (naïve) assumption that the chain falls purely vertically, making a sharp bend by 90° at the edge of the table.

A.1 Chain "Heaped Up Close to the Edge of the Table"

When the end of the chain is distance z below the plane of the tabletop, the momentum of the (vertical portion of) the chain is $p_z = m(z/l)\dot{z}$, subject to the force of gravity, $F_z = m(z/l)g$. Hence, ignoring any force between the chain on the table and that below, the equation of motion can be written as,

$$\frac{l}{m}\frac{dp_z}{dt} = \frac{d(z\dot{z})}{dt} = \frac{l}{m}F_z = zg, \qquad z\dot{z}\frac{d(z\dot{z})}{dt} = z^2\dot{z}g, \qquad \frac{(z\dot{z})^2}{2} = \frac{(z^3 - z_0^3)g}{3}, \tag{6}$$

where z_0 is the length of the chain hanging over the edge at $t = 0$, when the system is at rest.

For the case that z_0 is very small, the equation of motion is approximately,

$$\dot{z}^2 = \frac{2gz}{3}, \qquad \frac{dz}{\sqrt{z}} \approx \sqrt{\frac{2g}{3}}\,dt, \qquad 2\sqrt{z} \approx \sqrt{\frac{2g}{3}}\,t, \qquad z \approx \frac{gt^2}{6}, \qquad \ddot{z} \approx \frac{g}{3}. \tag{7}$$

The kinetic energy of the chain is $T = m(z/l)\dot{z}^2/2$, and its gravitational potential energy is $V = -m(z/l)gz/2$. If mechanical energy were conserved, we would have $z\dot{z}^2 - gz^2 = -gz_0^2$, and for small z_0, $\dot{z}^2 = gz$, *i.e.*, $\ddot{z} = g/2$.[11]

[11] Likewise, if one invoked Lagrange's method for $\mathcal{L} = T - V$, one would find the equation of motion for small z_0 to be $\ddot{z} = g/2$.

Instead, we have,

$$E = \frac{mz}{l}\left(\frac{\dot{z}^2}{2} - \frac{gz}{2}\right) = -\frac{mz\dot{z}^2}{4l} = -\frac{3m\dot{z}^4}{8gl}, \qquad \dot{E} = -\frac{m\dot{z}^3}{2l} = -\frac{1}{2}\left(\frac{m}{l}\frac{dz}{dt}\right)\dot{z}^2 = -\frac{1}{2}\frac{dm_v}{dt}\dot{z}^2, \tag{8}$$

where $m_v = mz/l$, such that dm_v/dt is the rate at which mass changes from being at rest on the table to falling with velocity $\dot{z}$. The corresponding abrupt increase in kinetic energy at the edge of the table comes at the expense of the mechanical energy of the rest of the falling chain.[12]

See also [24, 33, 64, 80].

A.2 Chain in a Straight Line on the Table

In this case the entire chain has velocity $\dot{z}$ and acceleration $\ddot{z}$ (although in different directions for the horizontal and vertical portions of the chain). Again assuming that the portion of the chain off the table is purely vertical, the equation of motion is,[13]

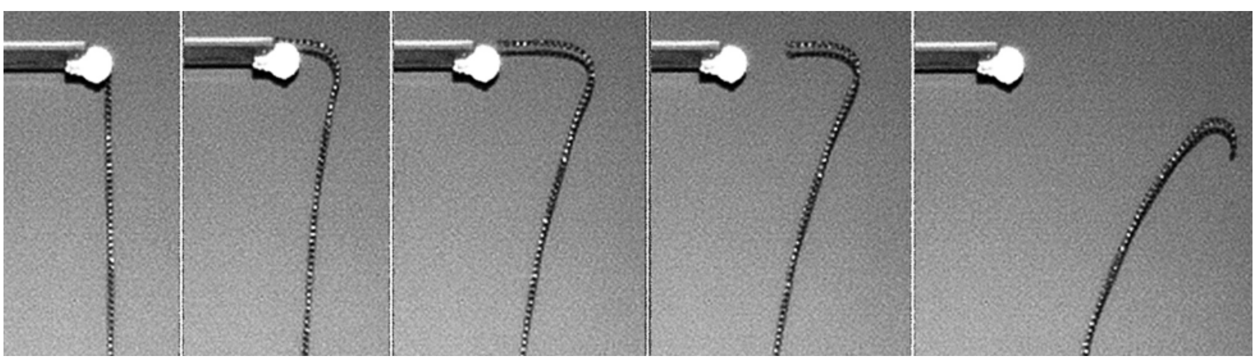
$$m\ddot{z} = \frac{mz}{l}g, \qquad z = \sqrt{\frac{g}{l}}\left(A\, e^{t\sqrt{g/l}} + B\, e^{-t\sqrt{g/l}}\right) = z_0 \cosh\left(t\sqrt{g/l}\right), \tag{9}$$

where length z_0 hangs off the table at time $t = 0$, when the system is at rest.

There are no abrupt movements of elements of the chain, so mechanical energy can be conserved,

$$E = \frac{m\dot{z}^2}{2} - \frac{mz}{l}\frac{gz}{2} = \frac{mgz_0^2}{2l}\left[\sinh^2\left(t\sqrt{g/l}\right) - \cosh^2\left(t\sqrt{g/l}\right)\right] = -\frac{mgz_0^2}{2l}. \tag{10}$$

However, in neither case does the chain fall purely vertically in practice. The photos below, from [86], are for an experiment where the portion of the chain on the "table" lies in a straight line.

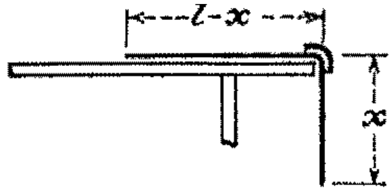


This phenomenon was implicit in the discussion by den Hartog (1948), p. 192 of [27], where a guide was specified to force the chain for fall purely vertically, as shown in his Fig. 164 below.[14]

[12] Other perspectives on this behavior are given in [38, 59].

[13] Here, conservation of mechanical energy is a good approximation, and leads to eq. (9). Likewise, a Lagrangian analysis leads to this equation of motion.

[14] However, den Hartog incorrectly evaluated the force on this guide, as remarked in footnote 14 of [42]. The need for a guide to keep the chain vertical was also mentioned in Ex. 12, p. 171 of [33] (1953).



For other discussions which assume that the chain falls vertically, see [19, 24, 39, 65, 93], while discussions of horizontal motion of the chain when off the table include [42, 46, 51]. A video showing the complex motion of the end of a moving chain as it unwraps from various objects is at [82].

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