Pressure in Fluid Flow Past a Sphere

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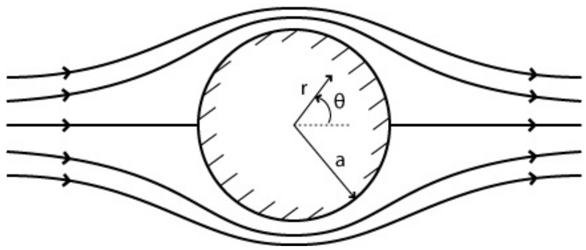
1 Problem

Deduce the pressure in an incompressible fluid of negligible viscosity that flows slowly past a small sphere of radius a at depth $H \gg a$ below the surface of the fluid, where v is the (horizontal) fluid velocity far from the sphere. This flow is irrotational.



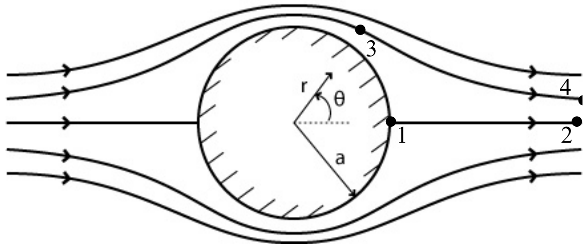
This problem was suggested by Johann Otto.

2 Solution

2.1 Solution in the Rest Frame of the Sphere

2.1.1 Pressure at the Stagnation Point

At points with large horizontal distance from the sphere the fluid streamlines are horizontal, say in the z-direction, and the fluid velocity is $v\,\hat{\mathbf{z}}$. The pressure at such a point is $P_0 + \rho g h$, where P_0 is the pressure at the surface of the fluid, ρ is the mass density of the fluid, and h is the depth of the point below the surface.[1] For example, the pressure at point 2 in the figure below, assumed to be far from the sphere, is $P_2 = P_0 + \rho g H$.



[1]This relation does not follow from Bernoulli's equation as the streamlines far from the sphere are horizontal not vertical, but rather obtains by considering the rest frame of the fluid (far from the sphere), and noting that the pressure at the bottom of a column of fluid at depth h is the weight per unit area, $\rho g h$, of the fluid in the column, plus the pressure P_0 at the surface. And, the pressure in the rest frame of the sphere is the same as that in the rest frame of the distant fluid, as force and area are invariant under a Galilean transformation. *We, of course, ignore relativistic corrections in this problem.*

We can now apply Bernoulli's law to the streamline from the stagnation point 1, where the fluid velocity is zero, to the distant point 2 at the same depth,

$$P_1 - \rho g H = P_2 + \frac{\rho v^2}{2} - \rho g H = P_0 + \frac{\rho v^2}{2}, \qquad P_1 = P_0 + \rho g H + \frac{\rho v^2}{2}. \tag{1}$$

2.1.2 Pressure at an Arbitrary Point

To find the pressure at an arbitrary point on a streamline (in the rest frame of the sphere), we can use Bernoulli's equation if we know the equation of the streamline and the velocity along it.

Strictly, the assumption of an incompressible fluid and flat bounding surfaces to the fluid are not compatible, but with the assumption that the sphere is far from all such bounding surfaces we make little error in supposing that the streamlines are the same as for bounding surfaces at infinity. This latter case was treated by Stokes [1], whom we follow in this section.

The low-velocity flow is irrotational, meaning that the flow velocity $\mathbf{u}$ obeys

$$\boldsymbol{\nabla} \times \mathbf{u} = 0, \tag{2}$$

in which case the velocity can be deduced from a scalar potential Φ. Following Stokes, we write

$$\mathbf{u} = \boldsymbol{\nabla}\Phi. \tag{3}$$

For an incompressible fluid (whose density is independent of time), we have that

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{4}$$

which implies that the scalar potential obeys Laplace's equation,[2]

$$\nabla^2 \Phi = 0. \tag{5}$$

In spherical coordinates (r, θ, ϕ (with angle θ measured with respect to the z-axis, which is parallel to the distant flow velocity $\mathbf{v}$), solutions to Laplace's equation (5) can be written as

$$\Phi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta), \tag{6}$$

where P_n is a Legendre polynomial, $P_0 = 1$ $P_1 = \cos\theta$, *etc.* For large r, we have that

$$\begin{aligned} \mathbf{u} &= v\,\hat{\mathbf{z}} = v\cos\theta\,\hat{\mathbf{r}} - v\sin\theta\,\hat{\boldsymbol{\theta}} \\ &= \boldsymbol{\nabla}\Phi = \frac{\partial\Phi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\,\hat{\boldsymbol{\theta}} \to \sum_{n=0}^{\infty} nA_n r^{n-1} P_n(\cos\theta)\,\hat{\mathbf{r}} + \sum_{n=0}^{\infty} A_n r^{n-1} \frac{dP_n(\cos\theta)}{d\theta}\,\hat{\boldsymbol{\theta}}, \end{aligned} \tag{7}$$

[2] According to the Helmholtz decomposition of vector fields [2], if $\boldsymbol{\nabla} \times \mathbf{u} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ everywhere in space, the field $\mathbf{u}$ must be constant. But, the velocity $\mathbf{u}$ is not defined inside the sphere, so there exists a nontrivial solution for the velocity outside it.

and hence the only nonvanishing coefficient A_n is $A_1 = v$. Furthermore, the radial velocity vanishes at the surface of the sphere,

$$u_r(r=a) \quad = \quad 0 = \frac{\partial \Phi(r=a)}{\partial r} = v\cos\theta - \sum_{n=0}^{\infty} \frac{(n+1)B_n}{a^{n+1}} P_n(\cos\theta), \tag{8}$$

and hence the only nonvanishing coefficient B_n is $B_1 = a^3 v/2$.

Thus, the scalar potential (outside the sphere) is

$$\Phi(r \geq a) = v\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \tag{9}$$

The velocity is

$$\mathbf{u}(r \geq a) = v\left(1 - \frac{a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} - v\left(1 + \frac{a^3}{2r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}. \tag{10}$$

The tangential velocity $u_\theta(r=a)$ is nonzero at the surface of the sphere (except at the stagnation points $(r = a, \theta = 0, \pi)$), which requires that the viscosity be zero, as assumed here.

The potential (9) can also be written in cylindrical coordinates (ϱ, ϕ, z) as

$$\Phi(\varrho, z) = vz\left(1 + \frac{a^3}{2(\varrho^2 + z^2)^{3/2}}\right) \tag{11}$$

such that by the fluid velocity is given (for $r > a$) by

$$\mathbf{u}(z, \varrho) = \frac{\partial \Phi}{\partial z}\,\hat{\mathbf{z}} + \frac{\partial \Phi}{\partial \varrho}\,\hat{\boldsymbol{\varrho}} = v\left(\hat{\mathbf{z}} - \frac{a^3(2z^2 - \varrho^2)}{2(z^2+\varrho^2)^{5/2}}\,\hat{\mathbf{z}} - \frac{3a^3\varrho z}{2(z^2+\varrho^2)^{5/2}}\,\hat{\boldsymbol{\varrho}}\right). \tag{12}$$

The flow is independent of azimuth ϕ (in the limit that any bounding surfaces are far from the sphere), so we follow Stokes in introducing a stream function $\Psi(\varrho, z)$ that is constant along a velocity streamline. For this, we note that the condition (4) for an incompressible fluid (often called conservation of mass) can be written in cylindrical coordinates as

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 = \frac{\partial u_z}{\partial z} + \frac{1}{\varrho}\frac{\partial \varrho u_\varrho}{\partial \varrho}, \qquad \textit{i.e.}, \qquad \frac{\partial \varrho u_z}{\partial z} = -\frac{\partial \varrho u_\varrho}{\partial \varrho}. \tag{13}$$

This implies that the vector field $\mathbf{w} = (w_\varrho, w_\phi, w_z) = (-\varrho u_z, 0, \varrho u_\varrho)$ obeys $\boldsymbol{\nabla}\times\mathbf{w} = 0$, *i.e.*, $\partial w_r/\partial z = \partial w_z/\partial r$, and hence can be derived from a scalar function Ψ, the stream function,[3]

$$\mathbf{w} = \boldsymbol{\nabla}\Psi, \qquad \frac{w_r}{\varrho} = u_z = \frac{1}{\varrho}\frac{\partial \Psi}{\partial \varrho}, \qquad -\frac{w_z}{\varrho} = u_\varrho = -\frac{1}{\varrho}\frac{\partial \Psi}{\partial z}. \tag{14}$$

Since $\mathbf{u}\cdot\mathbf{w} = \mathbf{u}\cdot\boldsymbol{\nabla}\Psi = -\varrho u_\varrho u_z + \varrho u_z u_\varrho = 0$, the fluid velocity $\mathbf{u}$ lies on surfaces of constant Ψ, and in particular the vector $\mathbf{u}(\varrho, z)$ lies along a line of constant $\Psi(\varrho, z)$ at a fixed azimuth ϕ; these lines are the fluid streamlines.

[3]The fluid-dynamics literature typically uses the notion of an exact differential in going from eq. (13) to (14).

Similarly, the condition (4) can be written (again assuming azimuthal symmetry) in spherical coordinates as

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 = \frac{1}{r^2}\frac{\partial r^2 u_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta\, u_\theta}{\partial\theta}, \qquad i.e., \qquad \frac{\partial r^2\sin\theta\, u_r}{\partial r} = -\frac{\partial r\sin\theta\, u_\theta}{\partial\theta}. \tag{15}$$

This implies that the vector field $\mathbf{w} = (w_r, w_\theta, w_\phi) = (-r\sin\theta\, u_\theta, r\sin\theta\, u_r, 0)$ obeys $\boldsymbol{\nabla}\times\mathbf{w} = 0$, *i.e.*, $\partial w_r/\partial\theta = \partial(r w_\theta)/\partial r$, and hence can be derived from a scalar function $\Psi(r,\theta)$, the stream function,

$$\mathbf{w} = \boldsymbol{\nabla}\Psi, \qquad \frac{w_\theta}{r\sin\theta} = u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \qquad -\frac{w_r}{r\sin\theta} = u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}. \tag{16}$$

Since $\mathbf{u}\cdot\mathbf{w} = \mathbf{u}\cdot\boldsymbol{\nabla}\Psi = -r\sin\theta\, u_r u_\theta + r\sin\theta\, u_\theta u_r = 0$, the fluid velocity $\mathbf{u}$ lies on surfaces of constant Ψ, and in particular the vector $\mathbf{u}(r,\theta)$ lies along a line of constant $\Psi(r,\theta)$ at a fixed azimuth ϕ; these lines are the fluid streamlines.

For the present example, we use eqs. (10) and (16) to find the stream function in spherical coordinates,

$$\frac{\partial\Psi}{\partial\theta} = r^2\sin\theta\, u_r = r^2 v\left(1 - \frac{a^3}{r^3}\right)\cos\theta\sin\theta, \tag{17}$$

$$\Psi = \frac{v}{2}\left(1 - \frac{a^3}{r^3}\right) r^2\sin^2\theta = \frac{v}{2}\left(1 - \frac{a^3}{r^3}\right)\varrho^2. \tag{18}$$

As a check, u_θ of eq. (10) indeed follows from the last of eq. (16).

A particular streamline obeys

$$\left(1 - \frac{a^3}{r^3}\right) r^2\sin^2\theta = \left(1 - \frac{a^3}{r^3}\right)\varrho^2 = \varrho_\infty, \tag{19}$$

where ϱ_∞ is the distance between the streamline and the z-axis far from the sphere.

The square of the velocity (10) is

$$u^2 = v^2\left[1 - \frac{a^3(3\cos^2\theta - 1)}{r^3} + \frac{a^6(3\cos^2\theta + 1)}{4r^6}\right]. \tag{20}$$

Finally, we can use Bernoulli's equation along a streamline to find

$$P(r,\theta,\phi) + \frac{\rho u^2}{2} - \rho g(H - r\sin\theta\cos\phi) = P_\infty + \frac{\rho v^2}{2} - \rho g(H - \varrho_\infty\cos\phi) = P_0 + \frac{\rho v^2}{2}, \tag{21}$$

$$\begin{aligned} P(r,\theta,\phi) &= P_0 + \frac{\rho v^2}{2}\left[\frac{a^3(3\cos^2\theta - 1)}{r^3} - \frac{a^6(3\cos^2\theta + 1)}{4r^6}\right] + \rho g(H - \varrho_\infty\cos\phi) \\ &= P_0 + \frac{\rho v^2}{2}\left[\frac{a^3(3\cos^2\theta - 1)}{r^3} - \frac{a^6(3\cos^2\theta + 1)}{4r^6}\right] \\ &\quad + \rho g H - \rho g\left(1 - \frac{a^3}{r^3}\right) r^2\sin^2\theta\cos\phi. \end{aligned} \tag{22}$$

At the stagnation point $(r,\theta,\phi) = (a,0,0)$ the pressure is that found earlier in eq. (1).

2.2 Solution in the Rest Frame of the Fluid Far from the Sphere

2.2.1 The Unsteady Bernoulli Equation

In the rest frame of the fluid far from the sphere the latter moves with velocity $-v\,\hat{\mathbf{z}}$, and the fluid velocity near the sphere is time dependent. As such, Bernoulli's equation does not apply here, since this equation assumes steady flow (in the frame where Bernoulli's equation is to be applied).[4]

A so-called unsteady Bernoulli equation can be deduced when the flow is not steady, starting from Euler's equation (sec. 2 of [3]) (which assumes that viscosity is negligible),

$$\rho \frac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla} P + \rho \mathbf{g}, \tag{23}$$

where the convective derivative appropriate for a fluid element that moves with velocity $\mathbf{u}$ is

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}. \tag{24}$$

Multiplying eq. (23) by a small increment $d\mathbf{s}$ along a streamline (such that $d\mathbf{s}$ is parallel to $\mathbf{u}$), we have

$$\rho \frac{d\mathbf{u}}{dt}\cdot d\mathbf{s} = \rho \frac{\partial u}{\partial t} ds + \rho u \frac{\partial u}{\partial s} ds = -\boldsymbol{\nabla} P \cdot d\mathbf{s} + \rho \mathbf{g}\cdot d\mathbf{s}, \tag{25}$$

$$\rho \frac{\partial u}{\partial t}\, ds + \rho u\, du = \rho \frac{\partial u}{\partial t}\, ds + \frac{\rho}{2}\, du^2 = -dP - \rho g dy, \tag{26}$$

where du and dP are differentials along the streamline, and we suppose that the y axis is vertical and upwards. We integrate eq. (26) from points 1 to 2 along a streamline to find (for an incompressible fluid),

$$\rho \int_1^2 \frac{\partial u}{\partial t}\, ds + \frac{\rho}{2}(u_2^2 - u_1^2) = P_1 - P_2 - \rho g(y_2 - y_1), \tag{27}$$

$$P_1 + \frac{\rho u_1^2}{2} + \rho g y_1 = P_2 + \frac{\rho u_2^2}{2} + \rho g y_2 + \rho \int_1^2 \frac{\partial u}{\partial t}\, ds. \tag{28}$$

Equation (28) is the unsteady Bernoulli equation,[5] which reduces to the classic Bernoulli equation for steady flow ($\partial \mathbf{u}/\partial t = 0$).

2.2.2 Pressure at the Stagnation Point

To deduce the pressure at the stagnation point $z = a - vt$ of the moving sphere, we recall eq. (10) and write the fluid velocity along z-axis (a streamline) in the rest frame of the sphere as

$$\mathbf{u}^\star(0,0,z^\star) = v\,\hat{\mathbf{z}}\left(1 - \frac{a^3}{z^{\star 3}}\right), \tag{29}$$

[4]See, for example, sec. 5 of [3]. For a dramatic example of when Bernoulli's equation is and isn't applicable, see [4].

[5]The unsteady Bernoulli equation differs from the so-called extended Bernoulli equation [5].

in cylindrical coordinates $(\varrho, \phi, z^\star)$ where $z^\star = z + vt$ is the z-coordinate in the rest frame of the sphere. The velocity in the rest frame of the distant fluid is then given by

$$\mathbf{u}(0,0,z,t) = \mathbf{u}^\star(0,0,z^\star) - v\,\hat{\mathbf{z}} = -\frac{a^3 v}{z^{\star 3}}\,\hat{\mathbf{z}} = -\frac{a^3 v}{(z+vt)^3}\,\hat{\mathbf{z}}, \qquad \frac{\partial \mathbf{u}(0,0,z,t)}{\partial t} = \frac{3a^3 v^2}{(z+vt)^4}\,\hat{\mathbf{z}}. \tag{30}$$

Taking point 2 to be at $z = \infty$ along the z-axis, where $P_2 = P_0 + \rho g H$, the pressure at the stagnation point 1, $(\varrho, \phi, z) = (0, 0, a)$ at time $t = 0$, is given by the unsteady Bernoulli equation (28) as

$$P_1 + \frac{\rho v^2}{2} - \rho g H = P_2 - \rho g H + \rho \int_a^\infty \frac{3a^3 v^2}{z^4}\,dz = P_0 + \rho v^2. \tag{31}$$

Thus,

$$P_1 = P_0 + \rho g H + \frac{\rho v^2}{2}\,, \tag{32}$$

as found in eq. (1).

Of course, it is advantageous to use the steady Bernoulli equation when possible.

References

[1] G.G. Stokes, *On some Cases of Fluid Motion*, Trans. Camb. Phil. Soc. **7**, 105 (1842), http://physics.princeton.edu/~mcdonald/examples/fluids/stokes_tcps_8_105_43.pdf

[2] K.T. McDonald, *The Helmholtz Decomposition and the Coulomb Gauge* (Apr. 17, 2008), http://physics.princeton.edu/~mcdonald/examples/helmholtz.pdf

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[4] K.T. McDonald, *Is Bernoulli's Equation Relativistically Invariant?* (Sept. 13, 2009), http://physics.princeton.edu/~mcdonald/examples/bernoulli.pdf

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