Poincaré Potential for a Finite Solenoid

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1 Problem

Deduce the vector potential in the so-called Poincaré gauge for a (static) solenoid magnet of finite length that is encased in a shell of high permeability such that the external magnetic field is negligible.

2 Solution

This problem is an exercise to evaluate the vector potential in the so-called Poincaré gauge [1] in a simple example.

For examples with axial symmetry and no azimuthal magnetic field, the vector potential can only have an azimuthal component $A_\phi(r, z)$.[1] Then, Stokes theorem applied to a disk of radius r tells us that $A_\phi(r, z) = \Phi(r, z)/2\pi r$ where the magnetic flux is $\Phi(r, z) = \int_0^r 2\pi r\, dr\, B_z(r, z)$, which provides a more direct method of calculation than that explored in sec. 2.2 below. This relation must be obeyed by the Poincaré potential, so changing the origin must have no effect on this potential, as confirmed in sec. 2.2.1.

2.1 Infinite Solenoid

An infinite solenoid with uniform, static magnetic field $B_0\,\hat{\mathbf{z}}$ within radius R about the z-axis, and zero magnetic exterior field, is the curl of a vector potential $\mathbf{A}$ that has only an azimuthal component in cylindrical coordinates (r, ϕ, z),[2]

$$A_\phi = \frac{B_0}{2} \begin{cases} r & (r < R), \\ R^2/r & (r > R). \end{cases} \tag{1}$$

This potential is irrotational ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$) and so is the Coulomb gauge potential (as well as that in the Lorenz gauge, the Hamiltonian gauge and the Poincaré gauge [1]. The magnetic field (and the potential) are generated by azimuthal surface currents at radius R.

In anticipation of sec. 2.2, we can also consider an infinite solenoid for which the magnetic flux is returned in the region $R < r < R + d$. In this case, the magnetic field is purely axial,

[1] The vector potential could include a constant field $\mathbf{A}_0$, which is excluded if we add the additional constraint that the potential must vanish at infinity.

[2] In this note, the origin of the coordinate system is always take to lie on the axis of the solenoid. See [2] for the case when the origin is outside the solenoid.

with

$$B_z = B_0 \begin{cases} 1 & (r < R), \\ -R^2/[(R+d)^2 - R^2] & (R < r < R+d), \\ 0 & (r > R+d). \end{cases} \tag{2}$$

This field configuration can be generated by appropriate surface currents at radius R and $R + d$. An azimuthal vector potential for this field is

$$A_\phi = \frac{B_0}{2} \begin{cases} r & (r < R), \\ R^2[(R+d)^2 - r^2]/r[(R+d)^2 - R^2] & (R < r < R+d), \\ 0 & r > (R+d). \end{cases} \tag{3}$$

2.2 Finite Solenoid

A finite solenoid of length $2L$ and radius R can be encased in a shell of thickness $d \ll L, R$ of a high permeability material such that to a good approximation the magnetic field is uniform in the cavity defined by the shell and zero exterior to it. The field in the material of the shell can be approximated by simple forms that obey $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ outside the two tori ($R < r < R + d, L < |z| < L + d$, where I have no analytic expression for it. The following approximation should be good for all space outside a distance d from these tori.

$$\begin{aligned} B_z(|z| < L) &\approx B_0 \begin{cases} 1 & (r < R), \\ -R^2/[(R+d)^2 - R^2] & (R < r < R+d), \\ 0 & (r > R+d), \end{cases} \\ B_z(L < |z| < L+d) &\approx B_0 \begin{cases} 1 + (L - |z|)/d & (r < R), \\ \text{not specified} & (R < r < R+d), \\ 0 & (r > R+d), \end{cases} \\ B_z(|z| > L+d) &\approx 0, \end{aligned} \tag{4}$$

and

$$\begin{aligned} B_r(|z| < L) &\approx 0, \\ B_r(L < |z| < L+d) &\approx B_0 \begin{cases} \pm r/2d & (r < R), \\ \text{not specified} & (R < r < R+d), \\ 0 & (r > R+d), \end{cases} \\ B_r(|z| > L+d) &\approx 0. \end{aligned} \tag{5}$$

The currents that generate this field follow from the fourth Maxwell equation, $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}/\mu_0$, and are purely azimuthal in view of the azimuthal symmetry of $\mathbf{B}$. From these the vector potential could be calculated in the Coulomb gauge, which is identical to the potential in the Lorenz gauge and in the Hamiltonian gauge (where the scalar potential V is zero; see sec. 8 of [1]) for static examples with zero charge density, such as the present case.

Here, we estimate the vector potential in the so-called Poincaré gauge (see sec. 9A of [1] and [3, 4]),[3] where

$$V(\mathbf{x},t) = -\mathbf{x}\cdot\int_0^1 du\,\mathbf{E}(u\mathbf{x},t), \qquad \mathbf{A}(\mathbf{x},t) = -\mathbf{x}\times\int_0^1 u\,du\,\mathbf{B}(u\mathbf{x},t). \tag{6}$$

In examples like the present with zero charge density the Poincaré gauge scalar potential is zero. The divergence of the Poincaré gauge vector potential is

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \mathbf{x}\cdot\int_0^1 u\,du\,\boldsymbol{\nabla}\times\mathbf{B}(u\mathbf{x},t) = \mathbf{x}\cdot\int_0^1 u\,du\left(\mu_0\mathbf{J}(u\mathbf{x},t) + \frac{1}{c^2}\frac{\partial\mathbf{E}(u\mathbf{x},t)}{\partial t}\right). \tag{7}$$

In static examples with only azimuthal currents we have that $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ in the Poincaré gauge, so the Poincaré vector potential in the present case is the same as that in the Coulomb, Hamiltonian and Lorenz gauges. It is simplest to calculate this potential using eq. (6), which involves integration along rays emanating from the origin. Only if that ray comes close to the tori where the field $\mathbf{B}$ is not specified in eqs. (4)-(5) will the calculation be a poor approximation.

The vector potential is the same as that of eq. (3) for points inside the solenoid, inside the material of the shell at $R < r < R + d$, and outside the shell in the central region where $|z|/r < L/R$. In the forward truncated cone where $z/r > L/R$ and also $z > L + d$, we have that

$$\begin{aligned} I_z &= \int_0^1 u\,du\,B_z(u\mathbf{x}) \approx \int_0^{L/z} u\,du\,B_0 + \int_{L/z}^{(L+d)/z} u\,du\,B_0\left(1 + \frac{L - uz}{d}\right) \\ &= \frac{B_0}{6}\frac{3L^2 + 3dL + d^2}{z^2}, \end{aligned} \tag{8}$$

$$I_\phi = 0, \tag{9}$$

$$I_r = \int_0^1 u\,du\,B_r(u\mathbf{x}) \approx \int_{L/z}^{(L+d)/z} u\,du\,\frac{B_0 ur}{2d} = \frac{B_0 r(3L^2 + 3dL + d^2)}{6z^3}. \tag{10}$$

The vector potential has only the azimuthal component

$$A_\phi = rI_z - zI_r \approx 0, \tag{11}$$

and so the magnetic field in the forward cone is zero. A similar calculation shows that the magnetic field in the backward truncated cone is zero as well.

2.2.1 Effect of a Change of Origin

We see from the definition (6) of the Poincaré potentials that they are affected by the choice of origin of the coordinate system. Consider the case that the center of the solenoid is at $z_0 > L + d$, such that the solenoid and the permeable shall are entirely at $z > 0$. Then,

[3]The Poincaré gauge is also called the multipolar gauge [5].

the potential can be nonzero only for $z > z_0 - L - d$ and inside the cone of half angle $\tan^{-1}[R/(z_0 - L - d]$ about the positive z-axis.

For a point inside the solenoid we now have that

$$
\begin{aligned}
I_z &= \int_0^1 u\,du\,B_z(u\mathbf{x}) \approx \int_{(z_0-L)/z}^1 u\,du\,B_0 + \int_{(z_0-L-d)/z}^{(z_0-L)/z} u\,du\,B_0 \frac{uz - z_0 + L + d}{d} \\
&= \frac{B_0}{6}\frac{3z^2 - 3(z_0 - L)^2 + 3d(z_0 - L) - d^2}{z^2}\,, \qquad (12) \\
I_\phi &= 0, \qquad (13) \\
I_r &= \int_0^1 u\,du\,B_r(u\mathbf{x}) \approx \int_{(z_0-L-d)/z}^{(z_0-L)/z} u\,du\,\frac{-B_0 ur}{2d} = -\frac{B_0 r}{6}\frac{3(z_0 - L)^2 - 3(z_0 - L)d + d^2}{z^3} \qquad (14)
\end{aligned}
$$

The vector potential has only the azimuthal component

$$
A_\phi = rI_z - zI_r \approx \frac{B_0 r}{2}\,, \qquad (15)
$$

as when the origin was at the center of the solenoid.

References

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