Snowball/Log Rolling down a Snowy Slope
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1 Problem

An ostensibly simple problem is the motion of a snowball (or better, a cylinder/log) that rolls
without slipping down a snowy slope, accumulating mass as it moves. A naïve approximation
is that the cross section of the ball/log remains circular at all times (which implies that
snow moves from the slope to be instantaneously distributed over the entire surface of the
rolling object, thereby instantaneously acquiring kinetic energy, momentum and angular
momentum).\(^1\) Show, that this (unphysical) assumption leads to different equations of motion
via a force/torque analyses about the center or mass of the log and about its line of contact
with the slope, as well as different ones based on energy conservation and a Lagrangian.

Does the motion have a simple, asymptotic (terminal) character?

Consider also the motion during the first full turn of rolling, for which a more accurate
analysis can be given.

2 Solution

This problem is a variant on the many examples discussed by the author in [3].

We consider a cylindrical log that rolls down a slope of angle \(\alpha\) to the horizontal which is
covered by a layer of snow of depth \(b\) (normal to the slope), as sketched in the figure below.

![Diagram](https://via.placeholder.com/150)

The rolling log (of mass density \(\rho\), the same as that of the snow) accumulates all of the
snow that it encounters, without loss of energy to possible compaction of the snow. As such,

\(^1\)This assumption is tacitly made, for example, in [1] and sec. V.D of [2].
the shape of the cylinder is not quite circular, but an appealing approximation is that the cylinder remains circular at all time, with radius \( r(t) \).

In this approximation, the mass of the log (of length \( l \)) is \( m = \pi pr^2l \), and the rate of accumulation of mass is,

\[
\frac{dm}{dt} = 2\pi pr^2 \frac{dr}{dt} = \rho b l v = \frac{mbv}{\pi r^2}, \quad \frac{dr}{dt} = \frac{bv}{2\pi r} = \frac{\omega b}{2\pi}, \tag{1}
\]

where \( v = \frac{dx}{dt} \) is the speed of the center of the log down the slope, and \( \omega = \frac{v}{r} \) is the angular velocity of the log.

### 2.1 Force/Torque Analysis

The force (component) \( F_x \) parallel to the slope (and uphill) is related to the \( x \)-component of the momentum, \( p_x = mv \), of the log by,\(^2\)

\[
m g \sin \alpha - F_x = \frac{dp_x}{dt} = m \frac{dv}{dt} + v \frac{dm}{dt} = m \frac{dv}{dt} + \frac{mbv^2}{\pi r^2}, \tag{2}
\]

where \( g \) is the acceleration due to gravity.

**2.1.1 Torque Analysis about the Center of Mass of the Log**

In addition, the torque equation with respect to the center of mass of the log is,

\[
r F_x = \tau = \frac{dL}{dt} = \frac{d(I_C \omega)}{dt} = \frac{d(mrv/2)}{dt} = \frac{rv}{2} \frac{dm}{dt} + \frac{mv}{2} \frac{dr}{dt} + \frac{mr}{2} \frac{dv}{dt},
\]

in the approximation that the moment of inertia of the (cylindrical) log about its axis is \( I = mr^2/2 \). Combining eqs. (2) and (3), we obtain an equation of motion,

\[
\frac{3}{2} \frac{mv}{dt} = mg \sin \alpha - \frac{7mbv^2}{4\pi r^2}, \quad a = \frac{dv}{dt} = \frac{2g}{3} \sin \alpha - \frac{7b}{6\pi r^2}. \tag{4}
\]

**2.1.2 Torque Analysis about the Line of Contact of the Log and Slope**

We could also consider the torque equation with respect to the line of contact of the log with the slope, which line is instantaneously at rest.

\[
r mg \sin \alpha = \tau_C = \frac{dL_C}{dt} = \frac{d(I_C \omega)}{dt} = \frac{d(3mr/2)}{dt} = \frac{3rv}{2} \frac{dm}{dt} + \frac{3mv}{2} \frac{dr}{dt} + \frac{3mr}{2} \frac{dv}{dt},
\]

noting that the moment of inertia about the point of contact is \( I_C = I + mr^2 = 3mr^2/2 \). The resulting equation of motion is (without need to consider the force at the line of contact),

\[
a = \frac{dv}{dt} = \frac{2g}{3} \sin \alpha - \frac{3b}{2\pi r^2}. \tag{6}
\]

\(^2\)The normal force \( F_y \) is related by \( F_y - mg \cos \theta = \frac{dp_y}{dt} = \frac{d(mr/2)}{dt} \), but we don’t need to pursue this, as \( F_y \) exerts no torque about the center of mass of the log (or about its line of contact with the slope).
2.1.3 Comments

The two equation of motion, (4) and (6) differ, which alerts us to the possibility that the preceding analysis is not sufficiently accurate.

In the limit of no snow on the slope, \( b \to 0 \), both torque analyses yield the well known result (reviewed in the Appendix) that the acceleration of a solid cylinder which rolls without slip down a slope of angle \( \alpha \) is \( \frac{2}{3}g \sin \alpha \). And, if the correction to the acceleration in case of a slope with thickness \( b \) is proportional to that thickness, but independent of \( g \), then dimensional analysis tells us that the correction is proportional to \( b v^2 / r^2 \). The task of a successful analysis of the motion is to identify the numerical coefficient of this term, which the torque analyses apparently fail to do in a convincing manner.

2.2 Energy Analysis

A different analysis can be based on the approximation that no energy is dissipated by the accumulation of snow on the rolling log, or by air resistance. Then, the mechanical energy, \( E = T + V \) is constant.

The kinetic energy \( T \) is related by,

\[
T = \frac{m}{2} \left( v^2 + \left( \frac{dr}{dt} \right)^2 \right) + \frac{I \omega^2}{2} = \frac{3mv^2}{4} + \frac{m}{2} \left( \frac{dr}{dt} \right)^2 = \frac{3\pi \rho r^2 v^2}{4} + \frac{\rho b^2 v^2}{8\pi}.
\]

(7)

For the potential energy, we suppose that the log started from rest with a radius \( r_0 \) and mass \( m_0 = \pi \rho r_0^2 l \) and rolled distance \( x \) down the slope to its present position. During this time, it accumulated snow of volume \( blx \), such that the present radius \( r \) and mass \( m \) are related by,

\[
\pi r^2 = \pi r_0^2 + bx, \quad \frac{\partial r}{\partial x} = \frac{b}{2\pi r}, \quad m = m_0 + \rho bx,
\]

(8)

in the approximation that the log is always circular. Then, relative to the origin, the initial potential energy \( V'_0 \), and the present potential energy \( V' \), of the log plus accumulated snow are,

\[
V'_0 = m_0gr_0 \cos \alpha - \rho blxg \frac{x \sin \alpha + b \cos \alpha}{2}, \quad V' = mg(r \cos \alpha - x \sin \alpha).
\]

(9)

Redefining the initial potential to be zero, the present potential energy \( V \) is,

\[
V = mg(r \cos \alpha - x \sin \alpha) + \rho blxg \frac{x \sin \alpha + b \cos \alpha}{2} - m_0gr_0 \cos \alpha = \rho lg \frac{(2\pi r^3 + b^2 x) \cos \alpha + (bx^2 - 2\pi r^2 x) \sin \alpha}{2} - m_0gr_0 \cos \alpha.
\]

(10)

In the approximation of conservation of mechanical energy \( E = T + V \), we have that,

\[
0 = \frac{3\pi \rho r^2 v^2}{4} + \frac{\rho b^2 v^2}{8\pi} + \rho lg \frac{(2\pi r^3 + b^2 x) \cos \alpha + (bx^2 - 2\pi r^2 x) \sin \alpha}{2} - m_0gr_0 \cos \alpha.
\]

(11)
Taking the time derivative of the energy (and dividing by \( \rho l \)), we obtain the equation of motion,

\[
0 = \frac{3\pi r^2 v}{2} \frac{dv}{dt} + \frac{3bv^3}{4} + \frac{b^2 v}{4\pi} \frac{dv}{dt} + \frac{g v \cos \alpha}{2} (3br + b^2) + g \sin \alpha (bv x - \pi r^2 v - bv x),
\]

\[
\frac{3\pi r^2}{2} \frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \pi r^2 g \sin \alpha - \frac{3bv^2}{4} - \frac{g \cos \alpha}{2} (3br + b^2),
\]

\[
\frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \frac{2g}{3} \sin \alpha - \frac{bv^2}{2\pi r^2} - \frac{g \cos \alpha}{3\pi r^2} (3br + b^2).
\]

If we neglect the small terms in \( b^2 \), the equation of motion is,

\[
\frac{dv}{dt} = \frac{2g}{3} \sin \alpha - \frac{bv^2}{2\pi r^2} - \frac{gb}{\pi r} \cos \alpha,
\]

which disagrees with both eqs. (4) and (6), except in the limit of no snow, \( b \to 0 \). Note the appearance in eq. (15) of the term proportional to \( gb/r \), which did not arise in the torque analyses.

### 2.3 Lagrangian Method

For completeness, we recall that the equation of motion can also be deduced from the Lagrangian, \( \mathcal{L} = T - V \), although strictly this method is for the motion of a rigid body. Here, we take \( x \) as the single, independent coordinate, and note that \( \dot{x} = v \). Then, recalling eqs. (1) and (8),

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x},
\]

\[
\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial T}{\partial v} = \frac{3\pi \rho l r^2 v}{2} + \frac{\rho lb^2 v}{4\pi},
\]

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{3\rho l r^2}{2} \frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) + \frac{3\rho lb v^2}{2},
\]

\[
\frac{\partial \mathcal{L}}{\partial x} = -\frac{\rho l g}{2} [(3rb + b^2) \cos \alpha + (2bx - 2\pi r^2 - 2bx) \sin \alpha],
\]

\[
\frac{3\pi r^2}{2} \frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \pi r^2 g \sin \alpha - \frac{3bv^2}{2} - \frac{g \cos \alpha}{2} (3rb + b^2),
\]

\[
\frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \frac{2g}{3} \sin \alpha - \frac{bv^2}{\pi r^2} - \frac{g \cos \alpha}{3\pi r^2} (3rb + b^2).
\]

The equation of motion (21) differs slightly from eq. (14), as well as from eqs. (4) and (6), although all four equations agree in the limit that \( b \to 0 \).

It appears that the approximation of the log as circular at all times as it rolls down a snowy slope does not lead to a consistent equation of motion.
2.4 Terminal Acceleration of the Rolling Log on the Snowy Slope

It is claimed in [1] that although the acceleration of the rolling log is not constant, it approaches a constant (terminal) value, such that the motion of the log is eventually similar to that of rolling on a slope without snow.3

The equation of motion used in [1] is based on a torque analysis about the center of mass, as in our sec. 2.1.1 above. Because this equation of motion was derived using the unphysical assumption that the ball/log had a circular cross section at all times, the inference of a nonzero terminal acceleration is doubtful.

We now present an approximate analysis (that will support the existence of a terminal acceleration) which avoids use of the assumption that the log has an exactly circular cross section, instead taking it to be only approximately circular.

The total kinetic energy $T$ of the rolling log can be written as,4

\[ T = \frac{mv_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2}, \]  

(22)

where all of the parameters vary with time. We define $r_{cm}$ as the distance from the center of mass of the log to the line of contact of the log (which rolls without slipping) on the snowy slope. The mass and moment of inertia of the snow-covered log are approximately related by,

\[ m \approx \pi \rho r_{cm}^2 l, \quad \text{and} \quad I_{cm} \approx mr_{cm}^2, \]  

(23)

where $\rho$ is the mass density of the log, and $l$ is its length. In addition, we approximate the center-of-mass velocity $v_{cm}$ as,

\[ v_{cm} \approx \omega r_{cm}, \]  

(24)

which holds exactly only if vectors $v_{cm}$ and $r_{cm}$ are perpendicular. Then, the kinetic energy is approximately,

\[ T \approx \frac{3mv_{cm}^2}{4}, \]  

(25)

and its time derivative is approximately,

\[ \dot{T} \approx \frac{3mv_{cm}^2}{4} + \frac{3mv_{cm}v_{cm}}{4}. \]  

(26)

The rolling log accumulates mass from the snowy slope at rate,

\[ \dot{m} = \rho bv_{contact} \approx \rho bv_{cm} \approx \frac{mbv_{cm}}{\pi r_{cm}^2} \quad (\text{and} \quad \pi r_{cm}^2 \approx \pi r_0^2 + bx_{cm}), \]  

(27)

3This result was obtained in an analysis that (tacitly) assumed conservation of mechanical energy. In practice, energy is not conserved in the rolling process, which is subject to various forms of energy dissipation that, in general, lead to a terminal velocity (zero terminal acceleration) of the motion. Analysis of these processes is beyond the scope of this note (and of [1]).

4This form does not hold in the approximation that the log is always circular in cross section, as this requires instantaneous motion of snow from the line of contact to the entire surface of the log.
noting that the velocity of the line of contact of the log with the slope is approximately the same as the velocity of the center of mass of the log. We can also relate the rate of change \( \dot{m} \) of mass of the log to the rate of change \( \dot{r}_{cm} \) of its radius as approximately,

\[
\dot{m} = 2\pi \rho l r_{cm} \dot{r}_{cm},
\]

which together with eq. (27) implies that,

\[
\dot{r}_{cm} \approx \frac{bv_{cm}}{2\pi r_{cm}}.
\]

As the log rolls down the slope its gravitational potential energy \( V \) decreases at a rate approximately given by,

\[
\dot{V} \approx -mgv_{cm} \sin \alpha,
\]

where we have neglected the small rate of change of the potential energy associated with the changing mass of the log.

In the approximation of conservation of mechanical energy, we have that

\[
\dot{T} + \dot{V} = 0 \approx \frac{3mv_{cm}^2}{4} + \frac{3mv_{cm} \dot{v}_{cm}}{2} - mgv_{cm} \sin \alpha \approx \frac{3mv_{cm}}{2} \left( \dot{v}_{cm} - \frac{2g \sin \alpha}{3} + \frac{bv_{cm}^2}{2\pi r_{cm}^2} \right).
\]

Thus, we obtain an approximate equation of motion,

\[
\dot{v}_{cm} \approx \frac{2g \sin \alpha}{3} - \frac{bv_{cm}^2}{2\pi r_{cm}^2},
\]

which is similar to (but not the same as) the four equations of motion previously deduced using the unphysical assumption of a circular cross section of the log at all times. This gives some confidence that the form of these equations of motion has (approximate) physical relevance.

A clever suggestion in [1] is to take the time derivative of the equation of motion (32),

\[
\ddot{v}_{cm} \approx \frac{bv_{cm} \dot{r}_{cm}}{2\pi r_{cm}^3} - \frac{b\dot{v}_{cm}}{2\pi r_{cm}^2} \approx \frac{b}{2\pi r_{cm}^2} \left( \frac{bv_{cm}^2}{2\pi r_{cm}^3} - \dot{v}_{cm} \right),
\]

recalling eq. (29), which indicates that the acceleration \( \dot{v}_{cm} \) takes on a constant (terminal) value,

\[
a_{term} \approx \frac{bv_{cm}^2}{2\pi r_{cm}^3}. \]

Using this value in the equation of motion (32), we learn that the terminal acceleration is,

\[
a_{term} \approx \frac{g \sin \alpha}{3},
\]

which is 1/2 the acceleration of a solid cylinder that rolls without slipping on a slope without snow. That is, the log starts from rest on the snowy slope with acceleration \( 2g \sin \alpha / 3 \), but
decelerates (while its velocity increases) until the acceleration is only $g \sin \alpha / 3$, after which the velocity increases linearly with time.

From eq. (32), we see that $\dot{v}_{cm}$ would be zero if $bv_{cm}^2/2\pi r_{cm}^2 = 2a_{term}$. For large times, we have that $v_{cm} \approx a_{term}t$, $x_{cm} \approx a_{cm}t^2/2$, and from eq. (31), $\pi r_{cm}^2 \approx bx_{cm}$, such that $bv_{cm}^2/2\pi r_{cm}^2 \approx a_{term}$, so $\dot{v}_{cm}$ remains at $a_{term}$ at large times, and never drops to zero (i.e., there is no terminal velocity in the present approximations).

The approximate character of the derivation of the equation of motion (32) does not exclude the existence of an additional small term, such as

$$-\frac{gb \cos \alpha}{\pi r},$$

that was found in secs. 2.2-3 above. As the radius $r$ of the rolling log grows with time, the effect of such a term becomes negligible, and the acceleration approaches a constant, terminal value, albeit this approach is slower than in the absence of the term (36).

### 3 Single Turn of a Rolling Log

We can make a more physical analysis, without the approximation that the rolling log is circular at all times, for the first full turn of rolling, as sketched below.

When the “center” of the log, at $(x, r)$, has moved distance $x$ down the slope, an arc of angle $\theta = x/r$ of snow has accumulated over a portion of the surface of the log, where $r$ is the initial radius of the (initially circular) log.

We make the (unphysical) assumption that the snow within the circular arc of thickness $b$ has angular velocity $\omega$ about the line of contact at $(x, 0)$, which implies that the snow at $(x, -b)$ instantaneously takes on velocity $\omega b$. While this is a small (unphysical) effect, it limits the accuracy of the analysis for large times. Of course, at the end of the first full turn of rolling, the “step” in the thickness of the snow on the rolling log encounters the snowy slope, and the analysis is not readily continued.
The initial mass of the log is,\(^5\)

\[ m_0 = \pi \rho r^2 l , \]  
and the mass of accumulated snow is, neglecting terms of order \(b^2\),

\[ m_s = \pi \rho [(r + b)^2 - r^2] l \frac{\theta}{2\pi} \approx \rho b l r \theta = \frac{m_0 b \theta}{\pi r}, \]  

(37)

(38)

taking the radial thickness of the layer accumulated on the log to be \(b\), and the density \(\rho\) of the accumulated snow to be the same as that of the initial log. We also suppose that the thickness of the snow on the slope is \(b\), which implies that the density of the snow on the slope is slightly greater than \(\rho\).

Once the log has accumulated snow, its center of mass is not at the nominal center \((x, r)\) of the log. To analyze this, we first compute the center of mass \((x_s, y_s)\) coordinates of the accumulated snow, neglecting terms of order \(b^2\),

\[ x_s = \int_r^{r+b} \int_0^\theta \rho l r' dr' d\theta' (x - r' \sin \theta') = m_s x - \rho l \frac{(r + b)^3 - r^3}{3} (1 - \cos \theta) \]
\[ \approx m_s x - \rho l r^2 b (1 - \cos \theta) = m_s x - \frac{m_0 b}{\pi} (1 - \cos \theta), \]

(39)

\[ y_s = \int_r^{r+b} \int_0^\theta \rho l r' dr' d\theta' (r - r' \cos \theta') = m_s r - \rho l \frac{(r + b)^3 - r^3}{3} \sin \theta \]
\[ \approx m_s r - \rho l r^2 b \sin \theta = m_s r - \frac{m_0 b}{\pi} \sin \theta. \]

(40)

The center of mass \((\text{cm})\) coordinates of the rolling log are then related by, to order \(b\),

\[ (m_0 + m_s)x_{\text{cm}} = m_0 x + m_s x_s = (m_0 + m_s)x - \frac{m_0 b}{\pi} (1 - \cos \theta), \]

(41)

\[ (m_0 + m_s)y_{\text{cm}} = m_0 r + m_s y_s = (m_0 + m_s)r - \frac{m_0 b}{\pi} \sin \theta, \]

(42)

and, noting from eq. (38) that \(m_0/(m_0 + m_s) \approx 1 - b \theta / \pi r\), the \(\text{cm}\) coordinates are,

\[ x_{\text{cm}} = x - \frac{b}{\pi} (1 - \cos \theta), \quad y_{\text{cm}} = r - \frac{b}{\pi} \sin \theta. \]

(43)

The velocity of the nominal center of the log is again \(v = dx/dt\), and the angular velocity of the log is again \(\omega = d\theta/dt = v/r\), while now the nominal radius \(r\) is constant during the first turn of rolling.

### 3.1 Torque Analysis

A torque analysis about the center of mass would include the torque due to the contact force \(F_C\) on the line of contact of the initial log with the snowy slope. Instead, we perform a

\(^5\)If the initial log had very low mass, it would not roll down the slope, as the snow must be lifted as it sticks to the log.
torque analysis in the lab frame about the point \( \mathbf{x}_C = (x, 0, 0) \), regarding this point as fixed when computing the angular momentum \( \mathbf{L}_C \) about this point. Then, the equation of motion is,\(^6\)

\[
\frac{d\mathbf{L}_C}{dt} = \mathbf{\tau}_C + m \mathbf{v}_{cm} \times \mathbf{v}_C, \quad \text{where} \quad \mathbf{L}_C = I_C \mathbf{\omega} = -I_C \mathbf{\omega} \hat{\mathbf{z}}. \tag{44}
\]

For this, we need the moment of inertia \( I_C \) (to order \( b/r \)) of the log (plus accumulated snow) about the line of contact,

\[
I_C = \frac{3m_0r^2}{2} + \int_{r}^{r+b} \int_{0}^{\theta} \rho l r' \, dr' \, d\theta' \left( r^2 + r'^2 - 2rr' \cos \theta' \right)
\]

\[
= \frac{3m_0r^2}{2} + \rho l \theta \left( \frac{r^2 (r + b)^2 - r^2}{2} + \frac{(r + b)^4 - r^4}{4} \right) - 2\rho l r \frac{(r + b)^3 - r^3}{3} \sin \theta
\]

\[
\approx \frac{3m_0r^2}{2} + \frac{2m_0r \theta}{\pi} - \frac{2m_0 r}{\pi} \sin \theta, \tag{45}
\]

\[
\frac{d\mathbf{L}_C}{dt} = \frac{d}{dt} I_C \mathbf{\omega} = \frac{d}{dt} \frac{I_C \mathbf{v}_C}{r} = \frac{I_C \mathbf{dv}_C}{r \, dt} + \frac{\mathbf{v} \, dI_C}{r \, dt} \approx \frac{I_C \mathbf{dv}_C}{r \, dt} + \frac{2m_0bv^2(1 - \cos \theta)}{\pi r} \tag{46}.
\]

The torque \( \mathbf{\tau}_C \) about the line of contact is due to the force of gravity, \((m_0 + m_s) \mathbf{g}\), where \( \mathbf{g} = g \sin \alpha \hat{\mathbf{x}} - g \cos \alpha \hat{\mathbf{y}} \), which acts at the center of mass, whose position relative to the line of contact is \((x_{cm} - x) \hat{\mathbf{x}} + y_{cm} \hat{\mathbf{y}}\),

\[
\mathbf{\tau}_C = (\dot{\mathbf{x}}(x_{cm} - x) + \dot{y} y_{cm}) \times (m_0 + m_s)(\mathbf{x} \sin \alpha - \mathbf{y} cos \alpha)
\]

\[
= (m_0 + m_s)g\left((x_{cm} - x)(- \cos \alpha) - y_{cm} \sin \alpha \right) \hat{\mathbf{z}}
\]

\[
\approx m_0g \left( 1 + \frac{b \theta}{\pi r} \right) \left[ \frac{b}{\pi} (1 - \cos \theta) \cos \alpha - \left( r - \frac{b}{\pi} \sin \theta \right) \sin \alpha \right] \hat{\mathbf{z}}
\]

\[
\approx -m_0g \left[ \left( r + \frac{b \theta}{\pi} - \frac{b}{\pi} \sin \theta \right) \sin \alpha - \frac{b}{\pi} (1 - \cos \theta) \cos \alpha \right] \hat{\mathbf{z}}. \tag{47}
\]

Also,

\[
m \mathbf{v}_{cm} \times \mathbf{v}_C = m \left( v + \frac{bu \sin \theta}{\pi r}, -\frac{bu \cos \theta}{\pi r} \right) \times (v, 0) = \frac{mbu^2 \cos \theta}{\pi r} \hat{\mathbf{z}} \approx \frac{m_0bu^2 \cos \theta}{\pi r} \hat{\mathbf{z}}. \tag{48}
\]

Hence, the equation of motion, according to the torque analysis (44) about the line of contact, is,

\[
I_C \frac{dv}{dt} = m_0gr \left[ \left( r + \frac{b \theta}{\pi} - \frac{b}{\pi} \sin \theta \right) \sin \alpha - \frac{b}{\pi} (1 - \cos \theta) \cos \alpha \right] - \frac{m_0bu^2(2 - \cos \theta)}{\pi} \hat{\mathbf{z}}. \tag{49}
\]

For \( b = 0 \), when \( I_C = 3m_0r^2/2 \), eq. (49) reverts to the usual form \( dv/dt = 2g \sin \alpha/3 \).

We could also perform a torque analysis about the point \( C \) of contact in the accelerated frame associated with the motion of this point down the snowy slope. The torque equation then has the form,

\[
\frac{d\mathbf{L}_C'}{dt} = \mathbf{\tau}_C + (\mathbf{x}_{cm} - \mathbf{x}_C) \times (-m \mathbf{a}_C), \quad \text{where} \quad \mathbf{L}_C' = \mathbf{L}_C - (\mathbf{x}_{cm} - \mathbf{x}_C) \times m \mathbf{v}_C. \tag{50}
\]

\(^6\)The equation of motion (44) corresponds to that described in footnote 7 of sec. A.3.2 below.
which includes the "fictitious" torque associated with the "fictitious" force that appears to act on the center of
mass of the cylinder, and the angular momentum $\mathbf{L}_P$ relative to the moving point $P$ is related lab-frame
quantities by (see, eqs. (80)-(82) below, or, for example, sec. 3 of [4], and the Appendix of [5]).

The "fictitious" torque is,

$$\mathbf{r} \times (-m \mathbf{a}_C) = \left( -\frac{b}{\pi} (1 - \cos \theta), r - \frac{b}{\pi} \sin \theta \right) \times \left( -m \frac{dv}{dt}, 0 \right) = m \frac{dv}{dt} \left( r - \frac{b}{\pi} \sin \theta \right) \hat{z},$$  \hspace{1cm} (51)

and,

$$\mathbf{r} \times m \mathbf{v}_C = \left( -\frac{b}{\pi} (1 - \cos \theta), r - \frac{b}{\pi} \sin \theta \right) \times (v, 0) = -mv \left( r - \frac{b}{\pi} \sin \theta \right) \hat{z},$$  \hspace{1cm} (52)

$$\frac{d}{dt} (\mathbf{r} \times m \mathbf{v}_C) = -m \frac{dv}{dt} \left( r - \frac{b}{\pi} \sin \theta \right) \hat{z} + \frac{mb^2 \cos \theta}{\pi r} \hat{z}. $$ \hspace{1cm} (53)

The equation of motion (50) leads to, using eqs. (46), (51) and (53),

$$I_C \frac{dv}{dt} = m_0 gr \left[ \left( r + \frac{b \theta}{\pi} \right) \sin \alpha - \frac{b}{\pi} (1 - \cos \theta) \cos \alpha \right] - m \frac{dv}{dt} \left( r - \frac{b}{\pi} \sin \theta \right)$$

$$-2 \frac{mb^2 (1 - \cos \theta)}{\pi} + m \frac{dv}{dt} \left( r - \frac{b}{\pi} \sin \theta \right) - \frac{mb^2 \cos \theta}{\pi r}$$

$$= m_0 gr \left[ \left( r + \frac{b \theta}{\pi} \right) \sin \alpha - \frac{b}{\pi} (1 - \cos \theta) \cos \alpha \right] - \frac{mb^2 (2 - \cos \theta)}{\pi},$$ \hspace{1cm} (54)

in agreement with eq. (49).

### 3.2 Energy Analysis

The rolling log plus accumulated snow is instantaneously rotating with angular velocity $\omega$ about the line of contact, so its kinetic energy is,$^7$

$$T = \frac{I_C \omega^2}{2} = \frac{I_C v^2}{2r^2}, \quad \frac{dT}{dt} = \frac{I_C v \frac{dv}{dt}}{r^2} + \frac{v^2 \frac{dI_C}{dt}}{2r^2} = \frac{I_C v \frac{dv}{dt}}{r^2} + \frac{m_0 bv^2}{\pi r^2} (1 - \cos \theta).$$  \hspace{1cm} (55)

Relative to the origin, the initial gravitational potential energy $V_0'$, and the present potential energy $V'$, of the log plus accumulated snow are,

$$V_0' = m_0 gr \cos \alpha - m_s g \frac{x \sin \alpha + b \cos \alpha}{2},$$ \hspace{1cm} (56)

$$V' = m_0 g (r \cos \alpha - x \sin \alpha) + m_s g (-x_s \sin \alpha + y_s \cos \alpha).$$ \hspace{1cm} (57)

Redefining the initial potential to be zero, the present potential energy $V$ is, to order $b$,

$$V = -m_0 gx \sin \alpha + m_s g (-x_s \sin \alpha + y_s \cos \alpha) + m_s g \frac{x \sin \alpha + b \cos \alpha}{2}$$

$$\approx -m_0 gx \sin \alpha - \left[ m_s x - m_0 \frac{b}{\pi} (1 - \cos \theta) \right] g \sin \alpha$$

$$+ \left[ m_s r - m_0 \frac{b}{\pi} \sin \theta \right] g \cos \alpha + m_s g \frac{x \sin \alpha + b \cos \alpha}{2}.$$ 

$^7$We could not use eq. (55) when we assumed that the rolling log is always circular, as this implies that
it is not a rigid body, but has instantaneous motion of snow over its entire surface.
\[
\approx -m_0gr\theta \sin \alpha - \left[ \frac{m_0b\theta^2}{2\pi} - m_0\frac{b}{\pi} (1 - \cos \theta) \right] g \sin \alpha \\
+ \left[ m_0\frac{b\theta}{\pi} - m_0\frac{b}{\pi} \sin \theta \right] g \cos \alpha.
\]  

(58)

\[-\frac{dV}{dt} = \frac{v}{r} m_0gr \sin \alpha + \frac{v}{r} m_0 \left[ \frac{b\theta}{\pi} - \frac{b}{\pi} \sin \theta \right] g \sin \alpha - \frac{v}{r} m_0 g \frac{b}{\pi} (1 - \cos \theta) \cos \alpha. \]  

(59)

Assuming that mechanical energy is conserved, \(dT/dt = -dV/dt\), and we arrive at the equation of motion,

\[
I_C \frac{dv}{dt} = m_0gr \left[ (r + \frac{b\theta}{\pi} - \frac{b}{\pi} \sin \theta) \sin \alpha - \frac{b}{\pi} (1 - \cos \theta) \cos \alpha \right] - \frac{m_0 bv^2 (1 - \cos \theta)}{\pi},
\]  

(60)

which differs slightly from eq. (49).

### 3.3 Lagrangian Analysis

We now take the independent coordinate to be \(\theta\), with \(\dot{\theta} = \omega = v/r\). Then,

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \theta},
\]  

(61)

\[
\frac{\partial L}{\partial \omega} = \frac{\partial T}{\partial \omega} = I_C \omega = \frac{I_C v}{r},
\]  

(62)

\[
\frac{d}{dt} \frac{\partial L}{\partial \omega} = \frac{I_C dv}{dt} + \frac{2m_0 bv^2}{\pi r} (1 - \cos \theta),
\]  

(63)

\[
\frac{\partial L}{\partial \theta} = \frac{m_0 bv^2}{\pi r} (1 - \cos \theta) + m_0 g \sin \alpha \left( r + \frac{b}{\pi} - \frac{b}{\pi} \sin \theta \right) - m_0 g \cos \alpha \frac{b}{\pi} (1 - \cos \theta),
\]  

(64)

\[
I_C \frac{dv}{dt} = m_0gr \left[ (r + \frac{b}{\pi} - \frac{b}{\pi} \sin \theta) \sin \alpha - \frac{b}{\pi} (1 - \cos \theta) \cos \alpha \right] - \frac{m_0 bv^2 (1 - \cos \theta)}{\pi},
\]  

(65)

which agrees with eq. (60) but not with eq. (49).

### 3.4 Comments

While two of our three analyses agree as to the equation of motion, physics is not democratic, and the accuracy of any of these equations of motion remains unclear.

### A Appendix: Cylinder Rolling on a Snowless Slope

As in the figure on p. 1, the coordinate \(x\) of the line of contact of the cylinder with the slope has nonzero acceleration \(a = dv/dt = d^2x/dt^2\). This is also the \(x\)-coordinate of the center of mass of the rolling log.

Force/torque analyses about points \(P\) that depend on the coordinate \(x(t)\) of the line of contact can be done either in the lab frame or in an accelerated frame based on point \(P\).
In the latter case, the analysis must include the “fictitious” (coordinate) force $-m\mathbf{a}_P$ which appears to act on the center of mass of the cylinder in the accelerated frame.

We first note that the moments of inertia for a solid cylinder of mass $m$ and radius $r$ are,

$$I_{cm} = \frac{mr^2}{2}, \quad I_{contact} = \frac{3mr^2}{2}. \quad (66)$$

Also, as in the figure on p. $x$ = distance along the slope to the point of contact

We consider rolling without slipping, such that the angular velocity $\omega$ of the rolling cylinder is related by,

$$\omega = \frac{v}{r}. \quad (67)$$

### A.1 Energy Analysis

The cylinder is in instantaneous rotation about the point of contact, so the kinetic energy $T$ (in the lab frame) is related by,

$$T = \frac{I_{contact} \omega^2}{2} = \frac{3mr^2 \omega^2}{4} = \frac{3mv^2}{4}. \quad (68)$$

The gravitational potential energy $V$ is related by,

$$V = -mgx \sin \alpha, \quad (69)$$

taking $V$ to be zero when $x = 0$.

The total mechanical energy $E = T + V$ is constant (neglecting possible dissipative interactions), so,

$$\frac{dE}{dt} = 0 = 3mva/2 - mgv \sin \alpha, \quad (70)$$

and the acceleration $a$ of the cylinder down the slope is,

$$a = \frac{2g}{3} \sin \alpha. \quad (71)$$

### A.2 Lagrangian Analysis

The equation of motion can also be deduced from the Lagrangian, $\mathcal{L} = T - V$. Here, we take $x$ as the single, independent coordinate, and note that $\dot{x} = v$. Then,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}, \quad (72)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial v} = \frac{3mv}{2}, \quad (73)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{3ma}{2}, \quad (74)$$

$$\frac{\partial \mathcal{L}}{\partial x} = mg \sin \alpha, \quad (75)$$

$$a = \frac{2g}{3} \sin \alpha. \quad (76)$$
A.3 Torque Analysis about the Line of Contact

A.3.1 Analysis in the Lab Frame

The lab-frame torque equation about the line of contact is simply,

\[ \tau_{\text{contact}} = rmg \sin \alpha = \frac{d}{dt} L_{\text{contact}} = \frac{d}{dt} (I_{\text{contact}} \omega) = \frac{d}{dt} \left( \frac{3mr^2 v}{2r} \right) = \frac{3mra}{2}, \]  

(77)

and hence,

\[ a = \frac{2g}{3} \sin \alpha. \]  

(78)

A.3.2 Analysis in the Accelerated Frame

When considering a torque analysis about a point \( x_P(t) \) that is accelerated in the lab frame, it is often tacitly assumed that the origin of the accelerated coordinate system at time \( t \) is at the point \( P \). However, we do not make this assumption here.

The \( x-y \) coordinates of the line of contact are \( x_P = (x, 0) \), so the lab-frame acceleration of the reference point is \( a_P = \dot{a} \hat{x} \). In the accelerated frame, the center of mass of the rolling cylinder is at rest, so its acceleration \( a'_{\text{cm}} \) in the accelerated frame is zero. That is, we can rewrite the lab-frame relation \( F_x + mg \sin \alpha = ma_{\text{cm}} \) as

\[ F_x + mg \sin \alpha - ma = ma'_{\text{cm}} = 0, \]  

(79)

where “fictitious” force \( -m a_P = -ma \hat{x} \) acts on the center of mass of the cylinder.

A torque analysis about a general point \( P \) in the accelerated frame can be written in the form,

\[ \tau'_P = \frac{dL'_P}{dt}, \]  

(80)

where the torque \( \tau'_P \) is the sum of the torque about \( P \) in the (inertial) lab frame and the “fictitious” torque associated with the “fictitious” force that appears to act on the center of mass of the cylinder,

\[ \tau'_P = \tau_P + (x_{\text{cm}} - x_P) \times (-ma_P), \]  

(81)

and the angular momentum \( L'_P \) relative to the moving point \( P \) is related lab-frame quantities by (see, for example, sec. 3 of [4], and the Appendix of [5]),

\[ L'_P = \int dm[(x - x_P) \times (v - v_P)] = L - x_P \times m v_{\text{cm}} - (x_{\text{cm}} - x_P) \times m v_P \]

\[ = L_{\text{cm}} + m(x_{\text{cm}} - x_P) \times (v_{\text{cm}} - v_P) = L_P - (x_{\text{cm}} - x_P) \times m v_P, \]  

(82)

where,

\[ L = \int dm x \times v = L_{\text{cm}} + x_{\text{cm}} \times m v_{\text{cm}}, \]  

(83)
is the (lab-frame) angular momentum about the origin, and,
\[ L_P = \int dm (x - x_P) \times v = L - x_P \times m v_{cm} = L_{cm} + (x_{cm} - x_P) \times m v_{cm}, \] (84)
is the angular momentum about point \( P \) regarding this point as fixed in the lab frame.8
For the present case where \( P = (x(t), 0) \) is on the line of contact, \( v_P = v_{cm} = v \), so,
\[ L'_P = L_{cm}, \quad L'_P = L_{cm} = I_{cm} \omega = \frac{mr^2 v}{2r} = \frac{mr v}{2}, \] (87)
and,
\[ \tau'_P = r m g \sin \alpha - rma. \] (88)
The torque equation (80) implies,
\[ \tau'_P = r m g \sin \alpha - rma = \frac{dL'_P}{dt} = \frac{mra}{2}, \] (89)
and hence,
\[ a = \frac{2g}{3} \sin \alpha, \] (90)
as found previously.

### A.4 Torque Analysis about the Center of Mass

#### A.4.1 Analysis in the Lab Frame

The force \( F \) that acts on the cylinder at the line of contact has components related by,
\[ F_x + m g \sin \alpha = ma, \quad F_y - m g = 0, \] (91)
The lab-frame torque equation about the center of mass is,
\[ \tau_{cm} = -r F_x = -rma + r m g \sin \alpha = \frac{d}{dt} L_{cm} = \frac{d}{dt} (I_{cm} \omega) = \frac{d}{dt} \left( \frac{mr v}{2} \right) = \frac{mra}{2}, \] (92)
\[ \frac{3mra}{2} = m r g \sin \alpha, \] (93)
and again,
\[ a = \frac{2g}{3} \sin \alpha. \] (94)

---

8In general, the torque equation for \( L_P \) of eq. (84) is not \( dL_P/dt = \tau_P \), but rather,
\[ \frac{dL_P}{dt} = \tau_P = \tau_P + m v_{cm} \times v_P. \] (85)
as discussed, for example, in sec. 11.8, p. 256 of [6] and sec. 2 of [4]. See also [7].
For the present case where \( P = (x(t), 0) \) is on the line of contact, \( v_P = v_{cm} = v \), and \( x_P \) is parallel to \( v_{cm} \), so \( \tau_P = \tau_P = r m g \sin \alpha \). Then, from eq. (84), \( L_P = L_{cm} + m v = 3mr v/2 \), and the torque equation (85) implies,
\[ \frac{dL_P}{dt} = \frac{d}{dt} \frac{3mr v}{2} = \frac{3mra}{2} \tau_P = r m g \sin \alpha, \quad \text{and hence,} \quad a = \frac{2g}{3} \sin \alpha, \quad \text{as found previously.} \] (86)
A.4.2 Analysis in the Accelerated Frame

In the accelerated frame defined by the position of the center of mass of the cylinder at point \( P = x_{cm} = (x, r) \), we have that \( x_P = x_{cm} \) and \( v_P = v_{cm} \). Then, according to eqs. (81) and (92),

\[
\tau'_P = \tau_P = \tau_{cm}, \quad \tau'_P = -rma + rmg \sin \alpha, \quad (95)
\]

and according to eq. (82),

\[
L'_P = L_{cm}. \quad (96)
\]

The torque equation \( \tau'_{cm} = dL'_{cm}/dt \) in the accelerated frame implies,

\[
\tau'_{cm} = -rma + rmg \sin \alpha = \frac{dL'_{cm}}{dt} = \frac{dL_{cm}}{dt} = \frac{mra}{2}, \quad (97)
\]

and again the equation of motion is,

\[
a = 2g \frac{3}{3} \sin \alpha. \quad (98)
\]

A.5 Torque Analysis about the Point \((x, 2r)\)

A.5.1 Analysis in the Lab Frame

The moment of inertia \( I_p \) about this point is the same, \( 3mr^2/2 \), as about the line of contact. The force \( F \) that acts on the cylinder at the line of contact has components related by eq. (91), and the lab-frame torque equation about \( P \) is,

\[
\tau_P = -2xF_x - rmg \sin \alpha = -2mar + mgr \sin \alpha. \quad (99)
\]

The lab-frame angular momentum about point \( P \) is,

\[
L_P = L_{cm} + (x_{cm} - x_P) \times m v_{cm}, \quad L_P = \frac{mrv}{2} - rmv = -\frac{mrv}{2} \quad (100)
\]

so the lab-frame torque equation is

\[
\tau_P = -2mar + mgr \sin \alpha = \frac{dL_P}{dt} = -\frac{mra}{2}, \quad (101)
\]

and again,

\[
a = \frac{2g}{3} \sin \alpha. \quad (102)
\]
A.5.2 Analysis in the Accelerated Frame

In the accelerated frame defined by the point \( P = (x, 2r) \), we have that and \( v_P = v \). Then, according to eqs. (81) and (99),

\[
\tau'_P = \tau_P + rma = -rma + rmg \sin \alpha,
\]

and according to eq. (82),

\[
L'_P = L_{cm}.
\]

The torque equation \( \tau'_P = dL'_P/dt \) in the accelerated frame implies,

\[
\tau'_P = -rma + rmg \sin \alpha = \frac{dL'_P}{dt} = \frac{dL_{cm}}{dt} = \frac{mra}{2},
\]

and again the equation of motion is,

\[
a = \frac{2g}{3} \sin \alpha.
\]

References


