1 Problem

An ostensibly simple problem is the motion of a snowball (or better, a cylinder/log) that rolls without slipping down a snowy slope, accumulating mass as it moves. A naïve approximation is that the cross section of the ball/log remains circular at all times (which implies that snow moves from the slope to be instantaneously distributed over the entire surface of the rolling object, thereby instantaneously acquiring kinetic energy, momentum and angular momentum).\(^1\) Show, that this (unphysical) assumption leads to different equations of motion via a force/torque analyses about the center or mass of the log and about its line of contact with the slope, as well as different ones based on energy conservation and a Lagrangian.

Does the motion have a simple, asymptotic (terminal) character?

Consider also the motion during the first full turn of rolling, for which a more accurate analysis can be given.

2 Solution

This problem is a variant on the many examples discussed by the author in [3].

We consider a cylindrical log that rolls down a slope of angle \(\alpha\) to the horizontal which is covered by a layer of snow of depth \(b\) (normal to the slope), as sketched in the figure below.

![Diagram of a rolling log on a snowy slope](image)

The rolling log (of mass density \(\rho\), the same as that of the snow) accumulates all of the snow that it encounters, without loss of energy to possible compaction of the snow. As such,

\(^1\)This assumption is tacitly made, for example, in [1] and sec. V.D of [2].
the shape of the cylinder is not quite circular, but an appealing approximation is that the
cylinder remains circular at all time, with radius \( r(t) \).

In this approximation, the mass of the log (of length \( l \)) is \( m = \pi r^2 l \), and the rate of
accumulation of mass is,

\[
\frac{dm}{dt} = 2\pi r l \frac{dr}{dt} = \rho bl v = \frac{mbv}{\pi r^2}, \quad \frac{dr}{dt} = \frac{bv}{2\pi r} = \frac{\omega b}{2\pi},
\]

where \( v = dx/dt \) is the speed of the center of the log down the slope, and \( \omega = v/r \) is the
angular velocity of the log.

### 2.1 Force/Torque Analysis

The force (component) \( F_x \) parallel to the slope (and uphill) is related to the \( x \)-component
of the momentum, \( p_x = m v \), of the log by,

\[
m \sin \alpha - F_x \frac{dp_x}{dt} = m \frac{dv}{dt} + v \frac{dm}{dt} = m \frac{dv}{dt} + \frac{mbv^2}{\pi r^2},
\]

where \( g \) is the acceleration due to gravity.

#### 2.1.1 Torque Analysis about the Center of Mass of the Log

In addition, the torque equation with respect to the center of mass of the log is,

\[
r F_x = \tau = \frac{dL}{dt} = \frac{d(I_C\omega)}{dt} = \frac{d(3mrv/2)}{dt} = \frac{3rv}{2} \frac{dm}{dt} + \frac{3mv}{2} \frac{dr}{dt} + \frac{mr}{2} \frac{dv}{dt}
\]

\[
= \frac{mbv^2}{2\pi r} + \frac{mbv^2}{4\pi r} + \frac{mr dv}{2},
\]

where \( \omega = v/r \) is related by

\[
\frac{dt}{dt} = \frac{7mbv^2}{4\pi r^2}, \quad a = \frac{dv}{dt} = \frac{2g}{3} \sin \alpha - \frac{7b}{6\pi r^2}. \tag{4}
\]

#### 2.1.2 Torque Analysis about the Line of Contact of the Log and Slope

We could also consider the torque equation with respect to the line of contact of the log with
the slope, which line is instantaneously at rest.

\[
r mg \sin \alpha = \tau_C = \frac{dL_C}{dt} = \frac{d(I_C\omega)}{dt} = \frac{d(3mrv/2)}{dt} = \frac{3rv}{2} \frac{dm}{dt} + \frac{3mv}{2} \frac{dr}{dt} + \frac{3mr}{2} \frac{dv}{dt}
\]

\[
= \frac{3mbv^2}{2\pi r} + \frac{3mbv^2}{4\pi r} + \frac{3mr dv}{2}, \tag{5}
\]

noting that the moment of inertia about the point of contact is \( I_C = I + mr^2 = 3mr^2/2 \). The
resulting equation of motion is (without need to consider the force at the line of contact),

\[
a = \frac{dv}{dt} = \frac{2g}{3} \sin \alpha - \frac{3b}{2\pi r^2}. \tag{6}
\]

\footnote{The normal force \( F_y \) is related by \( F_y = mg \cos \theta = \frac{dp_y}{dt} = \frac{d(mr^2/2)}{dt} \), but we don’t need to pursue
this, as \( F_y \) exerts no torque about the center of mass of the log (or about its line of contact with the slope).}
2.1.3 Comments

The two equation of motion, (4) and (6) differ, which alerts us to the possibility that the preceding analysis is not sufficiently accurate.

In the limit of no snow on the slope, \( b \to 0 \), both torque analyses yield the well known result that the acceleration of a solid cylinder which rolls without slip down a slope of angle \( \alpha \) is \( \frac{2}{3} g \sin \alpha \). And, if the correction to the acceleration in case of a slope with thickness \( b \) is proportional to that thickness, but independent of \( g \), then dimensional analysis tells us that the correction is proportional to \( bv^2/r^2 \). The task of a successful analysis of the motion is to identify the numerical coefficient of this term, which the torque analyses apparently fail to do in a convincing manner.

2.2 Energy Analysis

A different analysis can be based on the approximation that no energy is dissipated by the accumulation of snow on the rolling log, or by air resistance. Then, the mechanical energy, \( E = T + V \) is constant.

The kinetic energy \( T \) is related by,

\[
T = \frac{m}{2} \left[ v^2 + \left( \frac{dr}{dt} \right)^2 \right] + \frac{I\omega^2}{2} = \frac{3mv^2}{4} + \frac{m}{2} \left( \frac{dr}{dt} \right)^2 = \frac{3\pi \rho r^2 v^2}{4} + \frac{\rho lb^2 v^2}{8\pi}.
\]  

(7)

For the potential energy, we suppose that the log started from rest with a radius \( r_0 \) and mass \( m_0 = \pi \rho r_0^2 l \) and rolled distance \( x \) down the slope to its present position. During this time, it accumulated snow of volume \( blx \), such that the present radius \( r \) and mass \( m \) are related by,

\[
\pi r^2 = \pi r_0^2 + bx, \quad \frac{dr}{dx} = \frac{b}{2\pi r}, \quad m = m_0 + \rho blx
\]

(8)
in the approximation that the log is always circular. Then, relative to the origin, the initial potential energy \( V'_0 \), and the present potential energy \( V' \), of the log plus accumulated snow are,

\[
V'_0 = m_0 gr_0 \cos \alpha - \rho blx g \frac{x \sin \alpha + b \cos \alpha}{2}, \quad V' = mg(r \cos \alpha - x \sin \alpha).
\]  

(9)

Redefining the initial potential to be zero, the present potential energy \( V \) is,

\[
V = mg(r \cos \alpha - x \sin \alpha) + \rho blx g \frac{x \sin \alpha + b \cos \alpha}{2} - m_0 gr_0 \cos \alpha
\]

\[
= \rho lg \left( \frac{2\pi r^3 + b^2 x}{2} \right) \cos \alpha + \left( bx^2 - 2\pi r^2 x \right) \sin \alpha - m_0 gr_0 \cos \alpha.
\]  

(10)

In the approximation of conservation of mechanical energy \( E = T + V \), we have that,

\[
0 = \frac{3\pi \rho r^2 v^2}{4} + \frac{\rho lb^2 v^2}{8\pi} + \rho lg \left( \frac{2\pi r^3 + b^2 x}{2} \right) \cos \alpha + \left( bx^2 - 2\pi r^2 x \right) \sin \alpha - m_0 gr_0 \cos \alpha.
\]  

(11)
Taking the time derivative of the energy (and dividing by ρl), we obtain the equation of motion,

\[
0 = \frac{3\pi r^2 v}{2} \frac{dv}{dt} + \frac{3b v^3}{4} + \frac{b^2 v}{4\pi} \frac{dv}{dt} + \frac{g v \cos \alpha}{2} (3br + b^2) + g \sin \alpha (bv x - \pi r^2 v - bv x), \tag{12}
\]

\[
\frac{3\pi r^2}{2} \frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \pi r^2 g \sin \alpha - \frac{3b v^2}{4} - \frac{g \cos \alpha}{2} (3br + b^2), \tag{13}
\]

\[
\frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \frac{2g}{3} \sin \alpha - \frac{bv^2}{2\pi r^2} - \frac{g \cos \alpha}{3\pi r^2} (3br + b^2). \tag{14}
\]

If we neglect the small terms in \(b^2\), the equation of motion is,

\[
\frac{dv}{dt} = \frac{2g}{3} \sin \alpha - \frac{bv^2}{2\pi r^2} - \frac{gb}{\pi r} \cos \alpha, \tag{15}
\]

which disagrees with both eqs. (4) and (6), except in the limit of no snow, \(b \to 0\). Note the appearance in eq. (15) of the term proportional to \(gb/r\), which did not arise in the torque analyses.

### 2.3 Lagrangian Method

For completeness, we recall that the equation of motion can also be deduced from the Lagrangian, \(L = T - V\), although strictly this method is for the motion of a rigid body. Here, we take \(x\) as the single, independent coordinate, and note that \(\dot{x} = v\). Then, recalling eqs. (1) and (8),

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{16}
\]

\[
\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial x} \left( \frac{3\pi \rho l r^2}{2} + \frac{gb^2 v}{4\pi} \right), \tag{17}
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{3\pi \rho l r^2}{2} \frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) + \frac{3\rho lb v^2}{2} \tag{18}
\]

\[
\frac{\partial L}{\partial x} = -\frac{\rho l g}{2} [(3rb + b^2) \cos \alpha + (2bx - 2\pi r^2 + 2bx) \sin \alpha], \tag{19}
\]

\[
\frac{3\pi r^2}{2} \frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \pi r^2 g \sin \alpha - \frac{3b v^2}{2} - \frac{g \cos \alpha}{2} (3br + b^2), \tag{20}
\]

\[
\frac{dv}{dt} \left( 1 + \frac{b^2}{6\pi^2 r^2} \right) = \frac{2g}{3} \sin \alpha - \frac{bv^2}{2\pi r^2} - \frac{g \cos \alpha}{3\pi r^2} (3br + b^2). \tag{21}
\]

The equation of motion (21) differs slightly from eq. (14), as well as from eqs. (4) and (6), although all four equations agree in the limit that \(b \to 0\).

It appears that the approximation of the log as circular at all times as it rolls down a snowy slope does not lead to a consistent equation of motion.
2.4 Terminal Acceleration of the Rolling Log on the Snowy Slope

It is claimed in [1] that although the acceleration of the rolling log is not constant, it approaches a constant (terminal) value, such that the motion of the log is eventually similar to that of rolling on a slope without snow.\footnote{This result was obtained in an analysis that (tacitly) assumed conservation of mechanical energy. In practice, energy is not conserved in the rolling process, which is subject to various forms of energy dissipation that, in general, lead to a terminal velocity (zero terminal acceleration) of the motion. Analysis of these processes is beyond the scope of this note (and of [1]).}

The equation of motion used in [1] is based on a torque analysis about the center of mass, as in our sec. 2.1.1 above. Because this equation of motion was derived using the unphysical assumption that the ball/log had a circular cross section at all times, the inference of a nonzero terminal acceleration is doubtful.

We now present an approximate analysis (that will support the existence of a terminal acceleration) which avoids use of the assumption that the log has an exactly circular cross section, instead taking it to be only approximately circular.

The total kinetic energy \( T \) of the rolling log can be written as,\footnote{This form does not hold in the approximation that the log is always circular in cross section, as this requires instantaneous motion of snow from the line of contact to the entire surface of the log.}

\[
T = \frac{mv_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2},
\]

(22)

where all of the parameters vary with time. We define \( r_{cm} \) as the distance from the center of mass of the log to the line of contact of the log (which rolls without slipping) on the snowy slope. The mass and moment of inertia of the snow-covered log are approximately related by,

\[
m \approx \pi \rho r_{cm}^2 l, \quad \text{and} \quad I_{cm} \approx \frac{mr_{cm}^2}{2},
\]

(23)

where \( \rho \) is the mass density of the log, and \( l \) is its length. In addition, we approximate the center-of-mass velocity \( v_{cm} \) as,

\[
v_{cm} \approx \omega r_{cm},
\]

(24)

which holds exactly only if vectors \( v_{cm} \) and \( r_{cm} \) are perpendicular. Then, the kinetic energy is approximately,

\[
T \approx \frac{3mv_{cm}^2}{4},
\]

(25)

and its time derivative is approximately,

\[
\dot{T} \approx \frac{3\dot{m}v_{cm}^2}{4} + \frac{3mv_{cm}\dot{v}_{cm}}{2}.
\]

(26)

The rolling log accumulates mass from the snowy slope at rate,

\[
\dot{m} = \rho bv_{\text{contact}} \approx \rho bv_{cm} \approx \frac{mbv_{cm}}{\pi r_{cm}^2} \quad \text{(and} \quad \pi r_{cm}^2 \approx \pi r_0^2 + bx_{cm}),
\]

(27)
noting that the velocity of the line of contact of the log with the slope is approximately the same as the velocity of the center of mass of the log. We can also relate the rate of change \( \dot{m} \) of mass of the log to the rate of change \( \dot{r}_{cm} \) of its radius as approximately,

\[
\dot{m} = 2\pi \rho l r_{cm} \dot{r}_{cm},
\]

which together with eq. (27) implies that,

\[
\dot{r}_{cm} \approx \frac{b v_{cm}}{2\pi r_{cm}}.
\]

As the log rolls down the slope its gravitational potential energy \( V \) decreases at a rate approximately given by,

\[
\dot{V} \approx -mg v_{cm} \sin \alpha,
\]

where we have neglected the small rate of change of the potential energy associated with the changing mass of the log.

In the approximation of conservation of mechanical energy, we have that

\[
\dot{T} + \dot{V} = 0 \approx \frac{3mv_{cm}^2}{4} + \frac{3mv_{cm}\dot{v}_{cm}}{2} - mgv_{cm} \sin \alpha \approx \frac{3mv_{cm}}{2} \left( \dot{v}_{cm} - \frac{2g \sin \alpha}{3} + \frac{b v_{cm}^2}{2\pi r_{cm}^2} \right).
\]

Thus, we obtain an approximate equation of motion,

\[
\dot{v}_{cm} \approx \frac{2g \sin \alpha}{3} - \frac{b v_{cm}^2}{2\pi r_{cm}^2},
\]

which is similar to (but not the same as) the four equations of motion previously deduced using the unphysical assumption of a circular cross section of the log at all times. This gives some confidence that the form of these equations of motion has (approximate) physical relevance.

A clever suggestion in [1] is to take the time derivative of the equation of motion (32),

\[
\ddot{v}_{cm} \approx \frac{b v_{cm}^2}{2\pi r_{cm}^3} - \frac{b \dot{v}_{cm}}{2\pi r_{cm}^2} \approx \frac{b}{2\pi r_{cm}^2} \left( \frac{b v_{cm}^2}{2\pi r_{cm}^2} - \dot{v}_{cm} \right),
\]

recalling eq. (29), which indicates that the acceleration \( \dot{v}_{cm} \) takes on a constant (terminal) value,

\[
a_{\text{term}} \approx \frac{b v_{cm}^2}{2\pi r_{cm}^3}.
\]

Using this value in the equation of motion (32), we learn that the terminal acceleration is,

\[
a_{\text{term}} \approx \frac{g \sin \alpha}{3},
\]

which is 1/2 the acceleration of a solid cylinder that rolls without slipping on a slope without snow. That is, the log starts from rest on the snowy slope with acceleration \( 2g \sin \alpha / 3 \), but decelerates (while its velocity increases) until the acceleration is only \( g \sin \alpha / 3 \), after which the velocity increases linearly with time.

From eq. (32), we see that \( \dot{v}_{cm} \) would be zero if \( b v_{cm}^2 / 2\pi r_{cm}^2 = 2a_{\text{term}} \). For large times, we have that \( v_{cm} \approx a_{\text{term}} t \), \( x_{cm} \approx a_{\text{term}} t^2 / 2 \), and from eq. (31), \( \pi r_{cm}^2 \approx bx_{cm} \), such that \( b v_{cm}^2 / 2\pi r_{cm}^2 \approx a_{\text{term}} \), so \( \dot{v}_{cm} \) remains at \( a_{\text{term}} \) at large times, and never drops to zero (i.e., there is no terminal velocity in the present approximations).
3 Single Turn of a Rolling Log

We can make a more physical analysis, without the approximation that the rolling log is circular at all times, for the first full turn of rolling, as sketched below.

When the “center” of the log, at \((x, r)\), has moved distance \(x\) down the slope, an arc of angle \(\theta = x/r\) of snow has accumulated over a portion of the surface of the log, where \(r\) is the initial radius of the (initially circular) log.

We make the (unphysical) assumption that the snow within the circular arc of thickness \(b\) has angular velocity \(\omega\) about the line of contact at \((x, 0)\), which implies that the snow at \((x, -b)\) instantaneously takes on velocity \(\omega b\). While this is a small (unphysical) effect, it limits the accuracy of the analysis for large times. Of course, at the end of the first full turn of rolling, the “step” in the thickness of the snow on the rolling log encounters the snowy slope, and the analysis is not readily continued.

The initial mass of the log is,
\[
m_0 = \pi \rho r^2 l, \tag{36}
\]
and the mass of accumulated snow is, neglecting terms of order \(b^2\),
\[
m_s = \pi \rho [(r + b)^2 - r^2] l \frac{\theta}{2\pi} \approx \frac{\rho b l r \theta}{2} = m_0 \frac{b \theta}{2\pi r}, \tag{37}
\]
taking the radial thickness of the layer accumulated on the log to be \(b\), and the density \(\rho\) of the accumulated snow to be the same as that of the initial log. We also suppose that the thickness of the snow on the slope is \(b\), which implies that the density of the snow on the slope is slightly greater than \(\rho\).

Once the log has accumulated snow, its center of mass is not at the nominal center \((x, r)\) of the log. To analyze this, we first compute the center of mass \((x_s, y_s)\) coordinates of the accumulated snow, neglecting terms of order \(b^2\),
\[
m_{sx_s} = \int_r^{r+b} \int_0^\theta \rho l r' dr' d\theta' (x - r' \sin \theta') = m_s x - \rho l \frac{(r + b)^3 - r^3}{3} (1 - \cos \theta)
\]
\footnote{If the initial log had very low mass, it would not roll down the slope, as the snow must be lifted as it sticks to the log.}
\[ m_s x - \rho r^2 b (1 - \cos \theta) = m_s x - m_0 \frac{b}{\pi} (1 - \cos \theta), \quad (38) \]

\[ m_s y_s = \int_{r}^{r+b} \int_{0}^{\theta} \rho l r' dr' d\theta' (r - r' \cos \theta') = m_s r - \rho l (r + b)^3 - r^3 \frac{3}{3} \sin \theta \]

\[ \approx m_s r - \rho l t^2 b \sin \theta = m_s r - m_0 \frac{b}{\pi} \sin \theta. \quad (39) \]

The center of mass (cm) coordinates of the rolling log are then related by, to order \( b \),

\[ (m_0 + m_s)x_{cm} = m_0 x + m_s x_s = (m_0 + m_s)x - m_0 \frac{b}{\pi} (1 - \cos \theta), \quad (40) \]

\[ (m_0 + m_s)y_{cm} = m_0 r + m_s y_s = (m_0 + m_s)r - m_0 \frac{b}{\pi} \sin \theta, \quad (41) \]

and, noting from eq. (37) that \( m_0/(m_0 + m_s) \approx 1 - b\theta/\pi r \), the cm coordinates are,

\[ x_{cm} = x - \frac{b}{\pi} (1 - \cos \theta), \quad y_{cm} = r - \frac{b}{\pi} \sin \theta. \quad (42) \]

The velocity of the nominal center of the log is again \( v = dx/dt \), and the angular velocity of the log is again \( \omega = d\theta/dt = v/r \), while now the nominal radius \( r \) is constant during the first turn of rolling.

### 3.1 Torque Analysis

Because the \( x \)-coordinate of the center of mass is different from that of the line of contact of the log with the slope), a torque analysis about the center of mass would have to include the normal force, while a torque analysis about the nominal center of the log at \((x, r)\) would have to include a “coordinate” force acting on the center of mass. To avoid these complexities, we only consider a torque analysis about the line of contact.

For this, we need the moment of inertia \( I_C \) of the log (plus accumulated snow) about the line of contact,

\[ I_C = \frac{3m_0 r^2}{2} + \int_{r}^{r+b} \int_{0}^{\theta} \rho l r' dr' d\theta' (r^2 + r'^2 - 2rr' \cos \theta') \]

\[ = \frac{3m_0 r^2}{2} + \rho l \theta \left( \frac{r^3 (r + b)^2 - r^2}{2} + \frac{(r + b)^4 - r^4}{4} \right) - 2\rho l \frac{(r + b)^3 - r^3}{3} \sin \theta \]

\[ \approx \frac{3m_0 r^2}{2} + \frac{2m_0 br}{\pi} \theta - \frac{2m_0 br}{\pi} \sin \theta. \quad (43) \]

The torque \( \tau_C \) about the line of contact is due to the force of gravity, \( (m_0 + m_s)g \), where \( \mathbf{g} = \dot{x} g \sin \alpha - \dot{y} g \cos \alpha \), which acts at the center of mass, whose position relative to the line of contact is \( \dot{x}(x_{cm} - x) + \dot{y} y_{cm} \). The magnitude \( \tau_C \) of this torque is,

\[ \tau_C = |(\dot{x}(x_{cm} - x) + \dot{y} y_{cm}) \times (m_0 + m_s)(x g \sin \alpha - y g \cos \alpha)| \]

\[ = (m_0 + m_s)g |(x_{cm} - x)(- \cos \alpha) - y_{cm} \sin \alpha| \]

\[ \approx m_0 g \left( 1 + \frac{b \theta}{2\pi r} \right) \frac{b}{\pi} (1 - \cos \theta) \cos \alpha - \left( r - \frac{b}{\pi} \sin \theta \right) \sin \alpha | \]

\[ \approx m_0 g \left( 1 + \frac{b \theta}{2\pi r} \right) \frac{b}{\pi} (1 - \cos \theta) \cos \alpha - \left( r - \frac{b}{\pi} \sin \theta \right) \sin \alpha | \]
\[ \approx m_0 g \left[ \left( r + \frac{b\theta}{2\pi} - \frac{b}{\pi} \sin \theta \right) \sin \alpha - \frac{b}{\pi} \left( 1 - \cos \theta \right) \cos \alpha \right] \]

\[ \frac{dL_C}{dt} = \frac{d}{dt} I_C \omega = \frac{d}{dt} I_C v = \frac{dI_C}{dt} = \frac{I_C dv}{r} + \frac{v \, dI_C}{r} = \frac{I_C dv}{r} + \frac{2m_0 b v^2 (1 - \cos \theta)}{\pi r}. \] (44)

Hence, the equation of motion according to the torque analysis about the line of contact is,

\[ I_C \frac{dv}{dt} = m_0 g r \left[ \left( r + \frac{b\theta}{2\pi} - \frac{b}{\pi} \sin \theta \right) \sin \alpha - \frac{b}{\pi} \left( 1 - \cos \theta \right) \cos \alpha \right] - \frac{2m_0 b v^2 (1 - \cos \theta)}{\pi}. \] (45)

For \( b = 0 \), when \( I_C = 3m_0 r^2/2 \), eq. (45) reverts to the usual form \( dv/dt = 2g \sin \alpha/3 \).

### 3.2 Energy Analysis

The rolling log plus accumulated snow is instantaneously rotating with angular velocity \( \omega \) about the line of contact, so its kinetic energy is,\(^6\)

\[ T = \frac{I_C \omega^2}{2} = \frac{I_C v^2}{2r^2}, \]

\[ \frac{dT}{dt} = \frac{I_C v \, dv}{r^2} + \frac{v^2 \, dI_C}{2r^2} = \frac{I_C v \, dv}{r^2} + \frac{m_0 b v^3}{\pi r^2} (1 - \cos \theta). \] (46)

Relative to the origin, the initial gravitational potential energy \( V_0' \), and the present potential energy \( V' \), of the log plus accumulated snow are,

\[ V_0' = m_0 g r \cos \alpha - m_s g \frac{x \sin \alpha + b \cos \alpha}{2}, \] (47)

\[ V' = m_0 g (r \cos \alpha - x \sin \alpha) + m_s g (-x_s \sin \alpha + y_s \cos \alpha). \] (48)

Redefining the initial potential to be zero, the present potential energy \( V \) is, to order \( b \),

\[ V = -m_0 g x \sin \alpha + m_s g (-x_s \sin \alpha + y_s \cos \alpha) + m_s g \frac{x \sin \alpha + b \cos \alpha}{2} \]

\[ \approx -m_0 g x \sin \alpha - \left[ m_s x - m_0 \frac{b}{\pi} \left( 1 - \cos \theta \right) \right] g \sin \alpha \]

\[ + \left[ m_s r - m_0 \frac{b}{\pi} \sin \theta \right] g \cos \alpha + m_s g \frac{x \sin \alpha + b \cos \alpha}{2} \]

\[ = -m_0 g r \theta \sin \alpha - \left[ m_0 \frac{b \theta^2}{4\pi} - m_0 \frac{b}{\pi} \left( 1 - \cos \theta \right) \right] g \sin \alpha \]

\[ + \left[ m_0 \frac{b \theta}{2\pi} - m_0 \frac{b}{\pi} \sin \theta \right] g \cos \alpha. \] (49)

\[ -\frac{dV}{dt} = \frac{v}{r} m_0 g \sin \alpha + \frac{v}{r} m_0 \left[ \frac{b\theta}{2\pi} - \frac{b}{\pi} \sin \theta \right] g \sin \alpha - \frac{v}{r} m_0 \left[ \frac{b}{2\pi} - \frac{b}{\pi} \cos \theta \right] g \cos \alpha. \] (50)

Assuming that mechanical energy is conserved, \( dT/dt = -dV/dt \), and we arrive at the equation of motion,

\[ I_C \frac{dv}{dt} = m_0 g r \left[ \left( r + \frac{b\theta}{2\pi} - \frac{b}{\pi} \sin \theta \right) \sin \alpha - \frac{b}{\pi} \left( \frac{1}{2} - \cos \theta \right) \cos \alpha \right] - \frac{m_0 b v^2 (1 - \cos \theta)}{\pi}, \] (51)

which differs slightly from eq. (45).

\(^6\)We could not use eq. (46) when we assumed that the rolling log is always circular, as this implies that it is not a rigid body, but has instantaneous motion of snow over its entire surface.
3.3 Lagrangian Analysis

We now take the independent coordinate to be θ, with \( \dot{\theta} = \omega = v/r \). Then,

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \theta},
\]

\( (52) \)

\[
\frac{\partial L}{\partial \omega} = \frac{\partial T}{\partial \omega} = I_C \omega = \frac{I_C v}{r},
\]

\( (53) \)

\[
\frac{d}{dt} \frac{\partial L}{\partial \omega} = I_C \frac{dv}{dt} + \frac{2m_0 b v^2}{r} (1 - \cos \theta),
\]

\( (54) \)

\[
\frac{\partial L}{\partial \theta} = \frac{m_0 b v^2}{\pi r} (1 - \cos \theta) + m_0 g \sin \alpha \left( r + \frac{b}{2\pi} - \frac{b}{\pi} \sin \theta \right) - m_0 g \cos \alpha \left( \frac{b}{2\pi} - \frac{b}{\pi} \cos \theta \right),
\]

\( (55) \)

\[
I_C \frac{dv}{dt} = m_0 g r \left[ \left( r + \frac{b}{2\pi} - \frac{b}{\pi} \sin \theta \right) \sin \alpha - \frac{b}{\pi} \left( \frac{1}{2} - \cos \theta \right) \cos \alpha \right] - \frac{m_0 b v^2}{\pi} (1 - \cos \theta),
\]

\( (56) \)

which agrees with eq. (51) but not with eq. (45).

3.4 Comments

While two of our three analyses agree as to the equation of motion, physics is not democratic, and the accuracy of any of these equations of motion remains unclear.

References


[3] K.T. McDonald, Uncoiling on a Horizontal Surface (June 2, 2018),