Fields of a Uniformly Accelerated Charge

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(August 19, 2014; updated April 13, 2016)

1 Problem

Deduce the potentials, electromagnetic fields and Poynting vector of an electric charge \( q \) with rest mass \( m \) that moves parallel to a uniform external electric field \( \mathbf{E}_{\text{ext}} = E_0 \mathbf{x} \). Comment on the physical character of these fields in the idealized case of an infinite time domain for the motion.

2 Solution

This problem has a long history, in which everyone admires the mathematical elegance of the formal solution, but opinions differ as to the physical significance of the idealization that the motion began infinitely far in the past and will continue indefinitely. And, the notion of a static electric field of infinite spatial extent is also somewhat problematic.

The case of uniform acceleration during a finite time interval is discussed in Appendix A. Many people prefer to emphasize the charged particle rather than its fields, as reviewed in Appendix B.

2.1 The Motion

The force on the charge in the (inertial) lab frame is constant, \( \mathbf{F} = qE_0 \mathbf{x} \), which suggests that the motion is that for uniform acceleration. However, “uniform acceleration” cannot mean constant acceleration in the (inertial) lab frame, as this would eventually lead to faster-than-light motion. Rather, (following Born [1]) we note that for motion parallel to the electric field, the acceleration is uniform with respect to the instantaneous rest frame of the accelerated object, since the component of the electric field parallel to the motion is the same in this frame as in the lab frame.

Quantities in this frame will be designated with the superscript \( \star \).

From sec. 10 of Einstein’s first paper on relativity [2] we have that for acceleration parallel to the velocity \( \mathbf{v} \) of an object, the acceleration in the lab frame is related to that in the instantaneous rest frame according to

\[
\frac{dv}{dt} = (1 - v^2/c^2)^{3/2} \frac{dv^*}{dt^*},
\]

where \( c \) is the speed of light in vacuum. In this, two powers of \( \sqrt{1 - v^2/c^2} \equiv 1/\gamma \) come from the transformation of relative velocity, and another comes from time dilation.
For uniform acceleration \( a^* = dv^*/dt^* = qE^*/m = qE_0/m \) (in Gaussian units), eq. (1) can be integrated to find the velocity \( v \). Thus, the acceleration in the lab frame is related to that in the instantaneous rest frame according to

\[
\frac{v}{\sqrt{1 - v^2/c^2}} = \gamma v = at, \quad \text{and} \quad \frac{dx}{dt} = v = \frac{a^*t}{\sqrt{1 + a^{*2}t^2/c^2}}.
\]

(2)

supposing that \( v = 0 \) when \( t = 0 \). Integrating eq. (2), we obtain

\[
x = x_0 + \frac{c^2}{a^*} \left( \sqrt{1 + a^{*2}t^2/c^2} - 1 \right) = x_0 - \frac{c^2}{a^*} + \sqrt{\left( \frac{c^2}{a^*} \right)^2 + c^2t^2},
\]

(3)

where \( x_0 \) is the \( x \)-coordinate of the object at time \( t = 0 \).

We take \( x_0 = c^2/a^* \equiv b \), and write the motion as

\[
x(t) \equiv x_b = \sqrt{b^2 + c^2t^2}, \quad y = 0 = z.
\]

(4)

The (proper) time \( t^* \) on a clock carried by the accelerating object is related by

\[
dt^* = dt\sqrt{1 - v^2/c^2} = \frac{dt}{\sqrt{1 + a^{*2}t^2/c^2}},
\]

(5)

and hence,

\[
t^* = \frac{c}{a} \sinh^{-1} \left( \frac{a^*t}{c} \right), \quad t = \frac{c}{a^*} \sinh \left( \frac{a^*t^*}{c} \right).
\]

(6)

Using this, eqs. (2) and (4) can be rewritten as

\[
v = c \tanh \left( \frac{a^*t^*}{c} \right) = \frac{c^2t}{x_b}, \quad \text{and} \quad x_b = b \left( \cosh \left( \frac{a^*t^*}{c} \right) - 1 \right).
\]

(7)

As such, uniformly accelerated motion is often called “hyperbolic motion.”

\[1,2,3\]

### 2.2 The Potentials

The rest of this note largely follows the book of Schott (1912) \([6]\).

We compute the electromagnetic potentials \( V \) and \( A \) of the uniformly accelerated charge via the prescription of Liénard \([7]\) and Wiechert \([8]\),

\[
V(x,t) = \int \frac{q\delta(t' - t_r)}{R(t,t_r)} dt' = \frac{q}{R - \beta \cdot R}, \quad A(x,t) = \beta \cdot V,
\]

(9)

\[\text{1Hyperbolic motion appears to have been first discussed briefly by Minkowski [3], and then more fully by Born [1] and Sommerfeld [4].}

\[\text{2An extended object that is subject to the same uniform acceleration at all of its point is observed to have the same length in the lab frame at all times; there is no Lorentz contraction observed in the lab frame in the case of uniform acceleration of an extended object. See, for example, the Appendices of [5].}

\[\text{3For times such that } |at| \ll c, \text{ the position is well approximated by the Newtonian form}

\[
x \approx b + \frac{at^2}{2}, \quad (|at| \ll c).
\]

(8)
where

\[ \mathbf{R} = \mathbf{x}(t) - \mathbf{x}_b(t_r) = \left( x - \sqrt{b^2 + c^2t_r^2} \right) \hat{x} + \rho \hat{\rho}, \]  

(12)
is the position vector from the charge at the retarded time,

\[ t_r = t - \frac{R(t, t_r)}{c}, \]  

(13)
to the observer at position \( \mathbf{x} \) at the present time \( t \), \( \rho = \sqrt{y^2 + z^2} \),

\[ \beta_r = \frac{\mathbf{v}(t_r)}{c} = \frac{ct_r}{x_b(t_r)} \hat{x}. \]  

(14)

Equations (12) and (13) combine to give a quadratic equation in \( t_r \) with solution

\[ ct_r = \frac{ct(x^2 + b^2 + \rho^2 - c^2t^2) - xs}{2(x^2 - c^2t^2)}, \]  

(15)

where

\[ s = \sqrt{(x^2 + \rho^2 + b^2 - c^2t^2)^2 - 4b^2(x^2 - c^2t^2)} = \sqrt{(x^2 + \rho^2 - x_0^2)^2 + 4b^2\rho^2}, \]  

(16)

and the minus sign has been chosen for the term \( xs \) so that \( t_r < 0 \) when \( t = 0 \). Then,

\[ R = ct - ct_r = \frac{ct(x^2 - b^2 - \rho^2 - c^2t^2) + xs}{2(x^2 - c^2t^2)}, \]  

(17)

\[ x_b(t_r) = \sqrt{b^2 + c^2t_r^2} = \frac{x(x^2 + b^2 + \rho^2 - c^2t^2) - cts}{2(x^2 - c^2t^2)}, \]  

(18)

\[ \beta_r = \frac{ct_r}{x_b(t_r)} = \frac{ct(x^2 + b^2 + \rho^2 - c^2t^2) - xs}{x(x^2 + b^2 + \rho^2 - c^2t^2) - cts}, \]  

(19)

\[ R - \beta_r \cdot \mathbf{R} = R - \beta_r[x - x_b(t_r)] = ct - ct_r - \frac{ct_r}{x_b(t_r)}[x - x_b(t_r)] = ct - \beta_r x \]

\[ = \frac{s(x^2 - c^2t^2) + xs}{x(x^2 + b^2 + \rho^2 - c^2t^2) - cts}, \]  

(20)

\[ \int \frac{q\delta[f(t')]}{R} \, dt' = \int \frac{q\delta(f)}{R} \frac{df}{dt'} = \frac{q}{R \frac{df}{dt'}|_{f=0}}, \]  

(10)
so that \( f = 0 \) for \( t' = t - R/c = t_r \), and using eqs. (12) and (14) we find

\[ \frac{R \, df}{dt'} = \frac{R \, dR}{c \, dt'} = \frac{R \cdot \mathbf{v}}{c} = \frac{R \cdot \mathbf{x}_b(t')}{c \cdot dt'} = R - \mathbf{R} \cdot \mathbf{\beta}_r. \]  

(11)
and the potentials (9) can be written as
\[ V_{\text{Schott}} = \frac{q(x^2 + b^2 + \rho^2 - c^2t^2) - cts}{s(x^2 - c^2t^2)}, \quad A_{\text{Schott},x} = \frac{qct(x^2 + b^2 + \rho^2 - c^2t^2) - xs}{s(x^2 - c^2t^2)}. \] (21)

The potentials (and fields) are zero for \( x < -ct \).

These potentials appear to be singular at the planes \( x = \pm ct \).\(^6\) To see that the singularity occurs only for \( x = -ct \) we follow Schott in multiplying and dividing eq. (21) by \( x(x^2 + \rho^2 + b^2 - c^2t^2) + cts \) (or by \( ct(x^2 + \rho^2 + b^2 - c^2t^2) + xs \)) to find
\[ V_{\text{Schott}} = \frac{q(x^2 + \rho^2 + b^2 - c^2t^2)^2 + 4b^2c^2t^2}{s[x(x^2 + \rho^2 + b^2 - c^2t^2) + cts]}, \quad A_{\text{Schott},x} = \frac{4b^2c^2t^2 - (x^2 + \rho^2 + b^2 - c^2t^2)^2}{s[ct(x^2 + \rho^2 + b^2 - c^2t^2) + xs]}, \] (22)

Then, for \( x = \pm ct \) where \( s = b^2 + \rho^2 \) these become
\[ V_{\text{Schott}}(x = \pm ct) = \frac{q(x^2 + \rho^2 + b^2 - c^2t^2)^2 + 4b^2c^2t^2}{(x + ct)(\rho^2 + b^2)^2}, \quad A_{\text{Schott},x}(x = \pm ct) = \frac{4b^2c^2t^2 - (\rho^2 + b^2)^2}{(x + ct)(\rho^2 + b^2)^2}. \] (23)

which are singular only for the plane \( x = -ct \).

However, the potentials found above suffer from a defect apparently first noticed only in 1955 by Bondi and Gold \(^9\),\(^7\) that the corresponding electromagnetic fields do not satisfy Maxwell’s equations in the plane \( x + ct = 0 \). This can be attributed to the creation of the charged particle at \( t = -\infty \) with speed \( v_x = -c \) and with singular fields and potentials, while the Liénard-Wiechert forms (9) tacitly assume there is no singular behavior at early times. The defect can be remedied by expressing the singular behavior for \( x = -ct \) in terms of delta functions\(^8\),\(^9\)
\[ V(x = -ct) = -q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct), \quad A_x(x = -ct) = q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct). \] (29)

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\(^5\)These results are given on pp. 64-65 of [6].

\(^6\)The potentials are also singular at the location of the charge, \( x = x_b, \rho = 0 \). Close to the charge, the potentials are approximately those of a uniformly moving charge, as discussed further in sec. 2.4.

\(^7\)See also [10, 11, 12].

\(^8\)As discussed in eqs. (15)-(16) of [12], the retarded time associated with the plane \( x = -ct \) is \( t_r = -\infty \), and the retarded distance (12) can be (delicately) approximated for \( t_r = t' \rightarrow -\infty \) as
\[ R(t, t') = \left[ (x - \sqrt{b^2 + c^2t'^2})^2 + \rho^2 \right]^{1/2} \approx [(x + ct')^2 + b^2 + \rho^2]^{1/2} \approx -(x + ct') - \frac{b^2 + \rho^2}{2ct'}, \] (24)
so that the first form of eq. (9) gives the potential for \( x = -ct \), due to the contribution at \( t' = -\infty \), as
\[ V(x = -ct) = q \int_{t'=-\infty}^{t'} \frac{\delta(ct - R - ct')}{R/c} dt' \approx -q \int \delta \left( x + ct + \frac{b^2 + \rho^2}{2ct'} \right) \frac{dt'}{t'} = q \int \delta(\xi - \eta) \frac{d\eta}{\eta}, \] (25)
where \( \xi = x + ct, \eta = -(b^2 + \rho^2)/2ct' > 0 \) with \( dt'/t' = -d\eta/\eta \) and \( t' \rightarrow -\infty \) corresponding to the lower limit of the \( \eta \) integration. For fixed \( t \) we have \( dx = d\xi \), such that
\[ \int V(x = -ct) \, dx \approx q \int \delta(\xi - \eta) d\xi \frac{d\eta}{\eta} = q \int_{t'=-\infty}^{t'} \frac{d\eta}{\eta} = -q \ln \eta = -q \ln \frac{b^2 + \rho^2}{b^2} + q \ln \frac{2ct'}{b^2}. \] (26)

\(^9\)Since the retarded velocity associated with \( x = -ct \) is \(-c \dot{x}\), \( A_x(x = -ct) = -V(x = -ct) \). Hence, the
2.3 The Electromagnetic Fields

The electromagnetic fields follow from the potentials (21) according to
\[ E = -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t}, \quad B = \nabla \times A, \] (30)
such that the nonzero field components for \( x > -ct \) are
\[ E_x = q \frac{4b^2(x^2 - \rho^2 - x_b^2)}{s^3}, \quad E_\rho = q \frac{8b^2x \rho}{s^3}, \quad B_\phi = q \frac{8b^2ct \rho}{s^3} \quad (x > -ct), \] (31)
in cylindrical coordinates \((\rho, \phi, x)\) with \( \hat{\phi} = \hat{x} \times \hat{\rho} \).

The electric field lines \( E \) for times \( t = \mp b/c \), when the charge is at \( x = \sqrt{2}b \) are shown as solid lines in the figures below (from paper I of [10]), with the dashed lines being the Poynting vector \( S \) (sec. 2.5). The plane \( x = -ct \), on which the field lines appear to terminate, has moved to the left between figures (a) and (b).

Singular potentials at \( x = -ct \) can be written as
\[ V(x = -ct) = -q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct) + q \ln \frac{-2ct'}{b^2} \delta(x + ct), \] (27)
\[ A_x(x = -ct) = q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct) - q \ln \frac{-2ct'}{b^2} \delta(x + ct). \] (28)
The awkward terms in \( \ln(-2ct'/b^2) \) have no effect on the fields \( E \) and \( B \), and can be removed by the gauge transformation \( V \to V - \partial \Lambda/\partial ct, \ A_x \to A_x - \partial \Lambda/\partial x \) with \( \Lambda = q \ln(-2ct'/b^2) \delta(x + ct) \), leading at last to the forms (29). While the full potential (27) is positive (for positive \( q \)), the first, physical term is negative.
The figures suggest that the fields close to the charge resemble those of a uniformly moving charge, while away from the charge they curve towards the image charge at \( x = -x_b \). This will be verified in sec. 2.4.

Note that the pattern of field lines is as if there were a (negative) image charge at \( x = -x_b = -\sqrt{b^2 + c^2 t^2} \) (beyond the “event horizon”), as also shown in the left figure below (from p. 68 of [6], with \( ct = 4b/3 \)). Section 2.3.1 below will continue this theme.

In the plane \( x = -ct \), where the potentials (29) are singular, the fields are those of a singular wavefront,

\[
E_x = -q \frac{4b^2}{(b^2 + \rho^2)^2}, \quad E_\rho = 2q \frac{\rho}{b^2 + \rho^2} \delta(x + ct) = -B_\phi \quad (x = -ct),
\]

where the form for \( E_x \) is the limit of that in eq. (31) as \( x \to -ct \).

An electric field line of the accelerated charge for \( x > -ct \) does not “end” where it intercepts the plane \( x = -ct \), but has a kink there, and heads off within this plane to \( \rho = \infty \). The fields in this plane are essentially transverse, in contrast to those for \( x > -ct \), as emphasized in [11] from which the figure on the right above is taken.

The fields are zero for \( x < -ct \), and the (moving) plane \( x = -ct \) is a kind of “event horizon” in the limited sense that observers at \( x < -ct \) cannot be aware of the accelerated charge prior to time \( t = -x/c \). This contrasts with the case of a charge in uniform motion, whose field lines fill all space at all times.

The premise of uniformly accelerated motion for all times is that somehow the charge is brought into existence at \((x, y, z, t) = (\infty, 0, 0, -\infty)\) with initial velocity \( \mathbf{v} = -c \hat{x} \). The “initial” field lines are largely a “pancake” of transverse lines as in eq. (32), with \( E = B \) and \( \mathbf{E} \cdot \mathbf{B} = 0 \), as for “radiation” fields.\(^{10}\)

As time increases (from \( t = -\infty \)), these transverse “radiation” fields, in the plane \( x = -ct \), slowly pull away from the charge (which is at \( x_b = \sqrt{b^2 + c^2 t^2} \)), and the fields are nonzero in the region \( x \geq -ct \). For \( x > -ct \) the electric field lines are approximately those associated with a uniformly moving charge at \( x_b \) plus an image charge at \( -x_b \), while these lines bend into the plane \( x = -ct \) and the field is zero for \( x < -ct \).

\(^{10}\)Following an earlier discussion by Fermi [15], Weizsäcker [16] and Williams [17] noted that the electromagnetic fields of an electron in uniform relativistic motion are predominantly transverse, with \( \mathbf{E} \approx \mathbf{B} \) (in Gaussian units). This is very much like the fields of a plane wave, so one is led to regard a fast electron as carrying with it a cloud of virtual photons that it can shed (radiate) if perturbed. See also [18].
2.3.1 Electric Field in Bipolar Coordinates

The electric field lines for \( x > -ct \) (but not the scalar potential) lie on surfaces of constant coordinate in a bipolar coordinate system with foci at \( x = \pm x_b(t) \), illustrated in the figure on the next page.\(^{11}\) Following pp. 66–67 of [6], we define

\[
\begin{align*}
    x &= \frac{x_b \sinh \psi}{\cosh \psi - \cos \chi}, \quad \rho = \frac{x_b \sin \chi}{\cosh \psi - \cos \chi}.
\end{align*}
\]

(33)

For use below we note that

\[
\beta^2 = \frac{v^2}{c^2} = \frac{c^2 t^2}{b^2 + c^2 t^2}, \quad \text{so that} \quad 1 - \beta^2 = \frac{b^2}{x_b^2},
\]

(34)

and

\[
\begin{align*}
    s &= \sqrt{(x^2 + \rho^2 - x_b^2)^2 + 4b^2 \rho^2} \\
    &= \frac{x_b^2}{(\cosh \psi - \cos \chi)^2} \sqrt{[\sinh^2 \psi + \sin^2 \chi - (\cosh \psi - \cos \chi)^2]^2 + 4(1 - \beta^2) \sin^2 \chi (\cosh \psi - \cos \chi)^2} \\
    &= \frac{x_b^2}{(\cosh \psi - \cos \chi)^2} \sqrt{4 \cos^2 \chi (\cosh \psi - \cos \chi)^2 + 4(1 - \beta^2) \sin^2 \chi (\cosh \psi - \cos \chi)^2} \\
    &= \frac{2x_b^2}{\cosh \psi - \cos \chi} \sqrt{1 - \beta^2 \sin^2 \chi}.
\end{align*}
\]

(35)

\(^{11}\)The 3-dimensional coordinate system obtained by rotating the 2-dimensional bipolar coordinate system about the \( x \)-axis is called a bispherical coordinate system.
The so-called scale factors for bipolar coordinates are

\[ h_\psi = h_\chi = \frac{x_b}{\cosh \psi - \cos \chi}, \]  

(36)

and the unit vectors (with directions shown in the figure on the previous page) are

\[ \hat{\psi} = \frac{1}{h_\psi} \frac{\partial \mathbf{r}}{\partial \psi} = \frac{1 - \cosh \psi \cos \chi}{\cosh \psi - \cos \chi} \hat{x} - \frac{\sinh \psi \sin \chi}{\cosh \psi - \cos \chi} \hat{\rho}, \]

(37)

\[ \hat{\chi} = \frac{1}{h_\chi} \frac{\partial \mathbf{r}}{\partial \chi} = -\frac{\sinh \psi \sin \chi}{\cosh \psi - \cos \chi} \hat{x} - \frac{1 - \cosh \psi \cos \chi}{\cosh \psi - \cos \chi} \hat{\rho}, \]

(38)

where \( \mathbf{r} = x \hat{x} + \rho \hat{\rho} \).

Then, from

\[ \mathbf{E} = E_x \hat{x} + E_\rho \hat{\rho} = E_\psi \hat{\psi} + E_\chi \hat{\chi}, \]

(39)

we have that

\[
E_\psi = E_x \hat{x} \cdot \hat{\psi} + E_\rho \hat{\rho} \cdot \hat{\psi} = q \frac{4b^2(x^2 - \rho^2 - x_b^2)}{s^3} \frac{1 - \cosh \psi \cos \chi}{\cosh \psi - \cos \chi} - q \frac{8b^2 x_\rho}{s^3} \frac{\sinh \psi \sin \chi}{\cosh \psi - \cos \chi} 
\]

\[
= \frac{4qb^2 x_b^2}{s^3} \left( \sinh^2 \psi - \sin^2 \chi - (\cosh \psi - \cos \chi)^2(1 - \cosh \psi \cos \chi) \right) \frac{1 - \cosh \psi \cos \chi}{\cosh \psi - \cos \chi} - \frac{8qb^2 x_b^2}{s^3} \sinh^2 \psi \sin^2 \chi \frac{1}{\cosh \psi - \cos \chi} 
\]

\[
= -\frac{8qb^2 (1 - \cosh \psi \cos \chi)^2 + \sinh^2 \psi \sin^2 \chi}{s^3 (\cosh \psi - \cos \chi)^3} = -\frac{8qb^2 x_b^2}{s^3 (\cosh \psi - \cos \chi)^3}, \tag{40} 
\]

\[
E_\chi = E_x \hat{x} \cdot \hat{\chi} + E_\rho \hat{\rho} \cdot \hat{\chi} = -q \frac{4b^2(x^2 - \rho^2 - x_b^2)}{s^3} \frac{\sinh \psi \sin \chi}{\cosh \psi - \cos \chi} - q \frac{8b^2 x_\rho}{s^3} \frac{1 - \cosh \psi \cos \chi}{\cosh \psi - \cos \chi} 
\]

\[
= -\frac{4qb^2 x_b^2}{s^3} \left[ \sinh^2 \psi - \sin^2 \chi - (\cosh \psi - \cos \chi)^2 \sinh \psi \sin \chi \right] \frac{1 - \cosh \psi \cos \chi}{\cosh \psi - \cos \chi} - \frac{8qb^2 x_b^2 \sinh \psi \sin \chi(1 - \cosh \psi \cos \chi)}{s^3 (\cosh \psi - \cos \chi)^3} = 0. \tag{41} 
\]

This confirms that the electric field lines follow lines of constant \( \chi \), which are circles that pass through \( x = \pm x_b \), as if an image charge \( -q \) were at \( x = -x_b \) in addition to the actual charge \( q \) at \( x = x_b \). Of course, the physical electric field exists only for \( x \geq -ct > -x_b \), and only the field for \( x > -ct \) is described by eqs. (40)-(41).

The magnetic field (31) for \( x > -ct \), written in bipolar coordinates, is

\[ B_\phi = q \frac{8b^2 c t \rho}{s^3} = q \frac{\beta \sin \chi (\cosh \psi - \cos \chi)^2}{\gamma^2 x_b^2(1 - \beta^2 \sin^2 \chi)^{3/2}} = \beta \sin \chi E_\psi. \]

(42)

2.4 Potentials and Field Close to the Charge

To discuss the potentials (and fields) close to the charge we introduce the distance \( \mathbf{r} \) from the present position of the charge to the observation point, and the angle \( \theta \) between \( \mathbf{r} \) and
the positive $x$-axis,

$$\begin{align*}
r &= \sqrt{(x-x_b)^2 + \rho^2}, \\
x &= x_b + r \cos \theta, \\
\rho &= r \sin \theta.
\end{align*}$$

(43)

Then, from eq. (16), for small $r$ we have

$$\begin{align*}
s &= \sqrt{(-2xr \cos \theta + r^2 \cos^2 \theta)^2 + 4b^2r^2 \sin^2 \theta} \\& \approx 2r \sqrt{x_b^2 \cos^2 \theta + x_b^2(1 - \beta^2) \sin^2 \theta} \\& \approx 2r x_b \sqrt{1 - \beta^2 \sin^2 \theta},
\end{align*}$$

noting that $\beta^2 = \frac{v^2}{c^2} = \frac{c^2t^2}{x_b^2} = \frac{c^2t^2}{(b^2 + c^2t^2)}$, so that $1/\gamma^2 = 1 - \beta^2 = b^2/x_b^2$. Then, eq. (21) becomes (for small $r$ where $x^2 - c^2t^2 \approx x_b^2 - c^2t^2 = b^2$)

$$\begin{align*}
V &\approx \frac{2xb^2}{sb^2} = \frac{1}{r \sqrt{1 - \beta^2 \sin^2 \theta}}, \\
A_x &\approx \frac{2ctb^2}{sb^2} = \beta V,
\end{align*}$$

(45)

which are the potentials of a uniformly moving charge with velocity $\beta = \beta \hat{x}$. Similarly, from eq. (31),

$$\begin{align*}
E_x &\approx q \frac{4b^2(x^2 - x_b^2)}{s^3} \approx q \frac{8b^2x_b(x-x_b)}{8r^3x_b(1 - \beta^2 \sin^2 \theta)^{3/2}} = q \frac{x-x_b}{\gamma^2r^3(1 - \beta^2 \sin^2 \theta)^{3/2}}, \\
E_\rho &\approx q \frac{8b^2x_b \rho}{s^3} = q \frac{\rho}{\gamma^2r^3(1 - \beta^2 \sin^2 \theta)^{3/2}}, \\
B_\phi &\approx q \frac{8b^2ct \rho}{s^3} = q \frac{\beta \rho}{\gamma^2r^3(1 - \beta^2 \sin^2 \theta)^{3/2}} = \beta E_\rho, \\
B &\approx \beta \times E,
\end{align*}$$

(46, 47, 48)

which are the electromagnetic fields of a uniformly moving charge.

2.5 Poynting Vector

The Poynting vector for $x > -ct$ is, using the fields (40) and (42),

$$\begin{align*}
S_{x>-ct} &= \frac{cE \times B}{4\pi} + \frac{cE_\psi \hat{\psi} \times B_\phi \hat{\phi}}{4\pi} - \frac{cE_\psi B_\phi}{4\pi} \hat{\chi} = -\frac{cq^2\beta \left(\cosh \psi - \cos \chi\right)^4 \sin \chi}{4\pi \gamma^4 x_b^4 \left(1 - \beta^2 \sin^2 \chi\right)^3} \hat{\chi}.
\end{align*}$$

(49)

Lines of the Poynting vector for $x > -ct$ point along $\hat{\chi}$, following circular paths (which are orthogonal to the circular lines of the electric field), as shown in the figures on p. 5. The Poynting vector does not emanate from either the present or the retarded position of the charge! Furthermore, the Poynting vector is proportional to $\beta = v/c = ct/x_b$, and so vanishes for all $x > 0$ at the time $t = 0$ when the charge is instantaneously at rest.

The Poynting vector for $x = -ct$ is, using the fields eq. (31) extrapolated onto this plane (as suggested by Schott [6]),

$$\begin{align*}
S_{x=-ct, \text{Schott}} &= \frac{c}{4\pi} \left(E_x \hat{x} + E_\rho \hat{\rho} \right) \times B_\phi \hat{\phi} = \frac{c}{4\pi} (E_\rho B_\phi \hat{x} - E_x B_\phi \hat{\rho}) \\
&= -\frac{c}{4\pi} \frac{32q^2b^4 \rho}{(b^2 + \rho^2)^5} \left(\frac{2x \rho}{b^2 + \rho^2} \hat{x} + \hat{\rho}\right).
\end{align*}$$

(50)
The Poynting vector (50) vanishes at $x = 0$ at all times, such that no energy is transported across this plane (which is not crossed by the accelerated charge), so it would be inconsistent to have nonzero fields for $x < 0$ at $t > 0$. This defect is remedied by the forms (29) to the potentials for $x = -ct$, for which the Poynting vector (51) has a nonzero component in the $-x$ direction at all times. Using eqs. (32) one finds [9, 12]

$$S_{x=-ct} = \frac{c}{4\pi}(E_x \dot{x} + E_\rho \dot{\rho}) \times B_\phi \dot{\phi} = \frac{c}{4\pi}(E_\rho B_\phi \dot{x} - E_x B_\phi \dot{\rho})$$

$$= -\frac{cE_\rho^2}{4\pi} \dot{x} - \frac{c}{4\pi} \frac{8q^2\beta^2\rho^2}{(b^2 + \rho^2)^3} \delta(x + ct) \dot{\rho}, \quad (51)$$

where $S_x = -E_\rho^2c/4\pi = -uc$ and $u$ is the field energy density. That is, the flux of energy in the $-x$ direction in the plane $x = -ct$ is just the product of the energy density in that plane and its velocity. In eq. (51) the field energy density $u$ in the plane is infinite, but as shown in paper III of [10], the total field energy in the plane $x = -ct$ can be written at

$$U_{x=-ct} = \int u_{x=-ct} dVol = U_{-\infty} - \frac{2q^2a^2t}{3c^3}, \quad (52)$$

where $U_{-\infty}$ is the infinite field energy created along with the accelerated charge at $t = -\infty$. The field energy in the plane $x = -ct$ decreases with time and approaches zero as $t \to \infty$. The energy lost by the plane $x = -ct$ appears as an increase of the field energy in the region $x > -ct > 0$ for $t < 0$, and in the region $-ct < x < 0$ for $t > 0$.

Energy flows radially inward on the plane $x = -ct$, according to eq. (51). This flow can be said to exit the plane through its “bounding surface” $x = -ct^+$, leading to the increase of field energy in the region $x > -ct$ noted above.

### 2.6 Does a Uniformly Accelerated Charge “Radiate”?\(^{12}\)

That the Poynting vector (49) does not flow out from the charge is consistent with the fact that the “radiation reaction” force\(^{12}\) $F_{\text{rad react}} = (2q^2/3c^3)\dot{v}$ vanishes for uniform acceleration. This has led many people to conclude that a uniformly accelerated charge does not “radiate” \([1, 20, 21, 22, 23]\).\(^{13}\) On the other hand, the flux of energy associated with the Liénard-Wiechert fields (31) across a sphere of large radius $R$ at time $t' = t + R/c$ whose center is at the location of the charge at time $t$ is $2q^2a^2/3c^3$ [24], which indicates that it is reasonable to say that the accelerated charge does “radiate,” according to the so-called Sommerfeld criterion.\(^{14,15}\)

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\(^{12}\)For commentary by the author on the “radiation reaction,” see [19].

\(^{13}\)Many of the opinions on this issue are reviewed in [10], particularly paper II.

\(^{14}\)The fields of the electric dipole of charge $q$ at $x_b(t)$ and $-q$ at $-x_b(t)$ do not satisfy the Sommerfeld radiation condition, as noted in [1, 13].

\(^{15}\)As argued by Schott [25], in the case of uniform acceleration “the energy radiated by the electron is derived entirely from its acceleration energy; there is as it were internal compensation amongst the different parts of its radiation pressure, which causes its resultant effect to vanish.” This view is somewhat easier to follow if “acceleration energy” means energy stored in the near and induction zones of the electromagnetic field [10, 24, 26], as Schott was unaware of the transfer of energy from the plane $x = -ct$ into the region $x > -ct$. 
The author considers that the term “radiation” should be used wherever the Poynting vector is nonzero [14], and that a uniformly accelerated charge involves “radiation” even though the Poynting vector does not emanate from the charge.\textsuperscript{16,17,18}

2.7 Field Momentum and Electromagnetic Mass

If we suppose the charge $q$ is a spherical shell of radius $r_0$ when at rest, then when at velocity $v$ the shell is Lorentz contracted in the $x$-direction, and is an oblate spheroid of semiminor axis $r_0/\gamma$. On p. 69 of [6] Schott computes the electromagnetic field momentum outside that oblate spheroid to be $2q^2\gamma v/3r_0$ to lowest order, and identifies the electromagnetic mass as $2q^2/3r_0$, which value he attributes to Lorentz without reference.\textsuperscript{19}

A more complete calculation of the field energy and momentum of the fields of the accelerated charge is given in sec. 4, paper III of [10],

$$U_{x=-ct} = \frac{q^2\gamma}{2r_0} \left( 1 + \frac{v^2}{3c^2} \right) + \frac{2q^2\gamma av}{3} , \quad P_{x=-ct} = \frac{2q^2\gamma v}{3\gamma r_0} - \frac{2q^2\gamma av}{3} . \quad (53)$$

The field energy and momentum in the plane $x = -ct$ are\textsuperscript{20}

$$U_{x=-ct} = U_{-\infty} - \frac{2q^2\gamma av}{3c^3} , \quad P_{x=-ct} = -U_{-\infty}c \hat{x} + \frac{2q^2\gamma av}{3} , \quad (54)$$

and the total field energy and momentum are

$$U_{total} = U_{-\infty} + \frac{q^2\gamma}{2r_0} \left( 1 + \frac{v^2}{3c^2} \right) , \quad P_{total} = -U_{-\infty}c \hat{x} + \frac{2q^2\gamma v}{3\gamma r_0} . \quad (55)$$

The terms in $1/r_0$ can be interpreted (“renormalized”) as aspects of the energy and momentum of the particle. In this view, the field energy and momentum not associated with the

\textsuperscript{16}In the view that any nonzero Poynting vector is “radiation,” DC circuits with a battery and resistor involve “radiation” which flows from the battery to the resistor. Also, a charge with uniform velocity involves “radiation,” which is consistent of the virtual photon concept advocated by Fermi [15] and developed further by Weizsäcker [16] and Williams [17].

\textsuperscript{17}Teitelboim [27] has developed a Lorentz-invariant partition of the field energy-momentum tensor (of a single electric charge) into pieces he calls “bound” and “radiated”. In this view, a uniformly accelerated charge is a sink of bound energy-momentum and a source of radiated energy-momentum, with the fluxes of these two being equal and opposite close to the charge. See also paper II of [10].

\textsuperscript{18}See sec. 4.2 of [19] for commentary as to how Hawking-Unruh radiation (a quantum effect) by an accelerated charge supports the existence of “ordinary” radiation by that charge.

\textsuperscript{19}Probably the missing reference is to eq. (28) of [28], which considers the field momentum of a uniformly moving shell of charge. This had been considered earlier by J.J. Thomson in [29] for $v \ll c$, and in sec. 16 of [30] for arbitrary but constant $v$.

On p. 61 of [6] Schott refers to the “relativistic mass” $\gamma m$ as the “Lorentz mass.” This is likely a reference to the statement at the end of sec. 12 of [28], “the masses of all particles are influenced by a translation to the same degree as the electromagnetic masses of the electrons.” However, Lorentz distinguished between “longitudinal” and “transverse” masses, and the possible role of $\gamma m$ as “the” relativistic mass was not emphasized until 1912 (the publication year of [6]) by Tolman [31]. That Schott does not mention Einstein in this context is perhaps a precursor of the present trend [32] to deny the existence of “relativistic mass,” or at least that Einstein had anything to do with this concept.

\textsuperscript{20}Recall eq. (2) that $\gamma v = at$. 

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energy/momentum of the particle are constant, ending up all in the region $x > -ct$ “behind” the wavefront for large positive times.

The change with time of energy (and momentum) of the particle can be attributed to changes in the interference terms between the fields of the accelerated charge and the field $E_0 \hat{x}$ that accelerates the particle.\textsuperscript{21}

2.8 The Schott/Interaction Field Energy

A Appendix: Uniform Acceleration for a Finite Time Interval

References


\textsuperscript{21}For a computation of this in the low-velocity limit, see [33].


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