Electromagnetic Angular Momentum of a Rotating Cylindrical Shell of Charge

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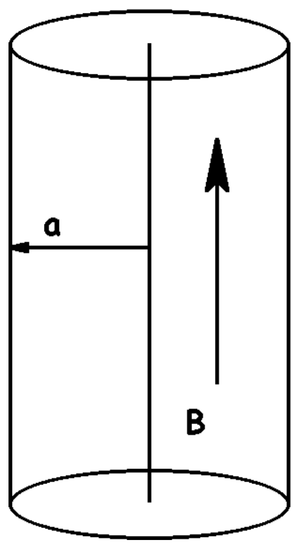
1 Problem

Discuss the electromagnetic field momentum of a cylindrical shell of radius a that carries surface charge density λ per unit length along it axial direction when the cylinder rotates with angular velocity ω about its axis. In addition there can be charge density λ_0 per unit length at rest along the axis of the cylinder.

This problem is based on a query by Michael Romalis, May 20, 2015. The geometry is that of the Feynman cylinder paradox [1].

2 Solution

We take the z-axis to be that of the cylinder.



The electric field is radial in a cylindrical coordinate system (r, ϕ, z), and given in Gaussian units by

$$\mathbf{E} = \begin{cases} \frac{2\lambda_0}{r}\,\hat{\mathbf{r}} & (r < a), \\ \frac{2(\lambda_0+\lambda)}{r}\,\hat{\mathbf{r}} & (r > a). \end{cases} \tag{1}$$

The rotating cylindrical shell of charge produces a solenoidal magnetic field, which is given in the approximation of an infinite cylinder as

$$\mathbf{B} = \begin{cases} \frac{2\lambda\omega}{c}\,\hat{\mathbf{z}} & (r < a), \\ 0 & (r > a). \end{cases} \tag{2}$$

However, since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ everywhere, the lines of magnetic field form closed loops and there is a weak magnetic field outside a physical, finite solenoid.

The angular momentum stored in the electromagnetic fields is nominally given by

$$\mathbf{L}_{\rm P} = \int \mathbf{r} \times \mathbf{p}_{\rm EM}\, d{\rm Vol}, = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol}, \tag{3}$$

which we call the Poynting form,[1] where the density of momentum stored in the electromagnetic fields is

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\,. \tag{4}$$

For an infinite solenoid, with nominally zero magnetic field outside, this form gives an unphysical result for the field angular momentum, as discussed in [6]. Rather, the field momentum of long solenoids is more expediently computed via the (Maxwell) form based on the vector potential in the Coulomb gauges (which was interpreted by Maxwell as electromagnetic momentum [7]).

$$\mathbf{L}_{\rm M} = \int \mathbf{r} \times \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol}. \tag{5}$$

The equivalence of the two forms (3) and (5) for bounded, quasistatic charge and current distributions was perhaps first shown in [8], and is demonstrated in sec. 2.2 of [9].

The vector potential in the present example can be written as

$$\mathbf{A} = \begin{cases} \frac{rB(r<a)}{2}\,\hat{\boldsymbol{\phi}} = \frac{r\lambda\omega}{c}\,\hat{\boldsymbol{\phi}} & (r < a), \\ \frac{a^2 B(r<a)}{2r}\,\hat{\boldsymbol{\phi}} = \frac{a^2\lambda\omega}{cr}\,\hat{\boldsymbol{\phi}} & (r > a). \end{cases} \tag{6}$$

Hence, the electromagnetic field angular momentum is

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \frac{\rho \mathbf{A}}{c}\, d{\rm Vol} = a\,\hat{\mathbf{r}} \times \lambda \frac{aB(r<a)\hat{\boldsymbol{\phi}}}{2c} = \frac{\lambda a^2 \omega}{c^2}\hat{\mathbf{z}}, \tag{7}$$

for any value of the charge density λ_0 along the axis.

2.1 Do the Electric and Magnetic Field Lines Rotate as the Cylinder Rotates?

An appealing view of electric field lines is that they begin/end on electric charges, such that if charges are in motion so are the electric fields lines associated with them. Hence, when $\lambda_0 = 0$ or $-\lambda$, we readily interpret the radial electric field lines of eq. (1) as rotating with angular velocity ω.

In contrast, magnetic field lines always form close loop, as magnetic charges do not exist (as far as we know). Hence, it is less clear that the magnetic field lines rotate along with the charged cylinder. Indeed, Faraday's view (secs. 218 and 220 of [10], and sec. 3090 of [11]) was that the magnetic field lines do not rotate in this case.[2]

[1]The form (3) is based on the Poynting vector **S** [2], and was first given by J.J. Thomson [3, 4, 5].
[2]For a review of this issue, see sec. 2 of [12].

If we follow Einstein [13] in supposing that the density $u = (E^2+B^2)/8\pi$ of energy in the electric and magnetic fields corresponds to density u/c^2 of effective mass, and also suppose that this energy density rotates along with the charged cylinder, then there are densities of momentum and angular momentum associated with the fields. In particular, the angular momentum per unit length associated with the rotating electric field lines when $\lambda_0 = -\lambda$ is

$$\mathbf{L}_E = \int \mathbf{r} \times \left(\frac{E^2}{8\pi c^2} \boldsymbol{\omega} \times \mathbf{r} \right) d\text{Area} = \int_0^a \frac{r^2\omega}{8\pi c^2} \left(\frac{2\lambda}{r} \right)^2 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^2 a^2 \omega}{2c^2} \hat{\mathbf{z}} \qquad (\lambda_0 = -\lambda), \quad (8)$$

and that associated with the magnetic field (if it rotates) is

$$\mathbf{L}_B = \int \mathbf{r} \times \left(\frac{B^2}{8\pi^2} \boldsymbol{\omega} \times \mathbf{r} \right) d\text{Area} = \int_0^a \frac{r^2\omega}{8\pi c^2} \left(\frac{2\lambda\omega}{c} \right)^2 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^4 a^2 \omega^3}{2c^4} \hat{\mathbf{z}},. \quad (9)$$

In contrast, the field angular momentum per unit length was computed in eq. (7) to be $\lambda^2 a^2 \omega\, \hat{\mathbf{z}}/c^2$.

The supposed contribution (9) to the field angular momentum due to the possibly rotating magnetic field lines does not have the same functional form as the "standard" result (7), which reinforces Faraday's view that the magnetic fields lines are not actually rotating in this case.

On the other hand, the result (8) obtained by assuming that the electric field lines do rotate is 1/2 of the "standard" result (7). This suggests that there is some validity to regarding the rotating electric field as carrying momentum and angular momentum with it.

However, when we consider the case that the wire along the axis has zero charge density rather than $-\lambda$, we see that the interpretation of the rotating electric field as carrying angular momentum is doubtful.

We noted above that the most reliable computation of the field angular momentum associated with a long/infinite solenoid is via the vector potential as in eq (7), independent of the value of the charge density λ_0 on the wire.

In particular, if $\lambda_0 = 0$, then the electric field is zero for $r < a$, and $E_r = 2\lambda/r$ for $r > a$, and the field angular momentum associated with the rotating electric field lines is infinite,[3]

$$\mathbf{L}_E = \int \mathbf{r} \times \left(\frac{E^2}{8\pi c^2} \boldsymbol{\omega} \times \mathbf{r} \right) d\text{Area} = \int_a^\infty \frac{r^2\omega}{8\pi c^2} \left(\frac{2\lambda}{r} \right)^2 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^2(\infty^2 - a^2)\omega}{2c^2} \hat{\mathbf{z}}. \quad (10)$$

Note also that the velocity of rotation of the electric field lines is $v = \omega r$ at radius r, which exceeds the speed of light for $r > c/\omega$. Hence, the interpretation of the rotating field energy density $u = E^2/8\pi$ as being associated with an effective, rotating mass density $E^2/8\pi c^2$ is doubtful for $r > c/\omega$.

We conclude that the form (3), or better (5), should be used for computation of the field angular momentum, rather than supposing that the rotating electric field lines can be associated with a rotating, effective mass density $E^2/8\pi c^2$.

[3]In this case the Poynting form (3) for the field angular momentum is nominally zero, which illustrates a limitation of this form for infinite solenoids [6].

A Rotating Spherical Shell of Charge

The case of a spherical shell of radius a with uniform surface density of electric charge in rotation at angular velocity ω about, say, the z-axis has been considered in [14]. For total charge Q the electric field in the lab frame is, in spherical coordinates (r, θ, ϕ),

$$\begin{aligned} \mathbf{E}(r < a) &= 0, \\ \mathbf{E}(r > a) &= \frac{Q}{r^2}\hat{\mathbf{r}}\,, \end{aligned} \tag{11}$$

independent of the angular velocity ω, while inside the shell the magnetic field is uniform, and outside the shell it has the form of a point magnetic dipole $\mathbf{m} = Qa^2\omega\,\hat{\mathbf{z}}/3c$,

$$\begin{aligned} \mathbf{B}(r < a) &= \frac{2\mathbf{m}}{a^3} = \frac{2Q\omega}{3ac}\,\hat{\mathbf{z}}, \\ \mathbf{B}(r > a) &= \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} = \frac{Qa^2\omega}{3cr^3}(2\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}). \end{aligned} \tag{12}$$

The vector potential can be written as

$$\begin{aligned} \mathbf{A}(r < a) &= \frac{r\sin\theta B(r<a)}{2}\,\hat{\boldsymbol{\phi}} = \frac{r\sin\theta Q\omega}{3ac}\,\hat{\boldsymbol{\phi}}, \\ \mathbf{A}(r > a) &= \frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\sin\theta Q a^2\omega}{3cr^2}\,\hat{\boldsymbol{\phi}}, \end{aligned} \tag{13}$$

so the field angular momentum can be computed from the Maxwell form (5) as

$$\begin{aligned} \mathbf{L}_{\rm EM} &= \int \mathbf{r}\times\frac{\varrho\mathbf{A}}{c}\,d{\rm Vol} = \int a\hat{\mathbf{r}}\times\frac{Q}{4\pi a^2}\frac{\sin\theta Q\omega}{3c^2}\,\hat{\boldsymbol{\phi}}\,d{\rm Area} = \frac{Q^2\omega}{12\pi ac^2}\int_{-1}^{1}\sin^2\theta\,2\pi a^2\,d\cos\theta\,\hat{\mathbf{z}} \\ &= \frac{2Q^2a\omega}{9c^2}\,\hat{\mathbf{z}} \end{aligned} \tag{14}$$

If we suppose that the electric and magnetic field lines rotate along with the spherical shell, the field angular momentum associated with the moving field energy density would be

$$\mathbf{L}_E = \int \mathbf{r}\times\left(\frac{E^2}{8\pi c^2}\boldsymbol{\omega}\times\mathbf{r}\right)d{\rm Vol} = 2\pi\int_a^\infty r^2\,dr\int_{-1}^{1}d\cos\theta\,\frac{r^2\omega\sin^2\theta}{8\pi c^2}\frac{Q^2}{r^4}\,\hat{\mathbf{z}} = \frac{Q^2(\infty - a)\omega}{3c^2}\,\hat{\mathbf{z}}, \tag{15}$$

and that associated with the magnetic field is

$$\begin{aligned} \mathbf{L}_B &= \int \mathbf{r}\times\left(\frac{B^2}{8\pi c^2}\boldsymbol{\omega}\times\mathbf{r}\right)d{\rm Vol} = 2\pi\int_0^a r^2\,dr\int_{-1}^{1}d\cos\theta\,\frac{r^2\omega\sin^2\theta}{8\pi c^2}\frac{4Q^2\omega^2}{9a^2c^2}\,\hat{\mathbf{z}} \\ &\quad +2\pi\int_a^\infty r^2\,dr\int_{-1}^{1}d\cos\theta\,\frac{r^2\omega\sin^2\theta}{8\pi c^2}\frac{Q^2a^4\omega^2}{9c^2r^6}(3\cos^2\theta+1)\,\hat{\mathbf{z}} = \frac{37Q^2a^3\omega^3}{105c^4}\,\hat{\mathbf{z}}. \end{aligned} \tag{16}$$

Again, it seems to be a bad model that the field lines rotate and carry field energy with them that can be associated with a moving, effective mass density.

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