1 Problem

Obtain a Legendre series expansion for the potential inside a resistive bead of radius $a$ and conductivity $\sigma$ when a current $I$ enters at one pole through a fine wire, and leaves through the other pole via a similar fine wire. Define the potential as $\phi = 0$ on the equator.

Refer to the expansion of $1/r$ given, for example, in eq. (3.38) of [1], to show that

$$\phi(r) = \phi(r, \theta, \varphi) = \frac{I}{2\pi \sigma} \left[ \frac{1}{r_1(r)} - \frac{1}{r_2(r)} + \frac{1}{2} \int_0^r \left( \frac{1}{r_1(r')} - \frac{1}{r_2(r')} \right) \frac{dr'}{r'} \right],$$

(1)

where $r_{1,2}(r')$ is the distance from the “north” (“south”) pole to the point $r' = (r', \theta, \varphi)$ in spherical coordinates. The integrals can be found in tables if desired.

Suppose the wires have radius $b \ll a$, and their surfaces of contact with the bead are equipotentials, to show that the resistance of the bead is that of a piece of wire roughly $b$ long, if that wire also had conductivity $\sigma$.

Hint: Express the radial current density at $r = a$ in terms of delta functions, $\delta(\cos \theta - 1)$ and $\delta(\cos \theta + 1)$.

Also deduce the potential outside the bead, and the surface charge density thereon, supposing that the fine wires are perfect conductors connected to perfect conducting hemispheres of radii $d > a$, which in turn are connected to a ring battery of voltage drop $2V$, where $V$ is the potential at the “north” pole of the bead.
2 Solution

2.1 Potential inside the Resistive Bead

Although current is flowing inside the resistive bead, its interior remains electrically neutral to a very good approximation.\(^1\) Hence, the electromagnetic scalar potential \(\phi\) satisfies Laplace’s equation, \(\nabla^2 \phi = 0\).

We analyze the problem in spherical coordinates \((r, \theta, \varphi)\), with the origin at the center of the bead of radius \(a\), and \(\theta = 0\) and \(\pi\) at the points of contact with the wires. The problem has axial symmetry, so \(\phi\) will be independent of \(\varphi\). We require the potential to be well behaved at the origin, so it can be expressed in a Legendre series,

\[
\phi(r < a) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos \theta). \tag{2}
\]

The convention that \(\phi = 0\) at the equator, \(\theta = \pi/2\), implies that \(A_n = 0\) for \(n\) even. Therefore, we can write

\[
\phi(r < a) = \sum_{n \text{ odd}} A_n \left(\frac{r}{a}\right)^n P_n(\cos \theta). \tag{3}
\]

To complete the solution inside the bead, we need a boundary condition on the potential \(\phi\) at the surface of the sphere \(r = a\). We know that the radial component of the current density, \(J_r\) is zero at the surface, except for the contact points where the current enters and exits. Since \(\mathbf{J} = \sigma \mathbf{E} = -\sigma \nabla \phi\), we obtain a condition on the derivative of the potential at the boundary,

\[
\frac{\partial}{\partial r} \phi(r = a^-) = -E_r(r = a^-) = -\frac{J_r(r = a^-)}{\sigma}. \tag{4}
\]

In the limit of very fine wires, the current density \(J_r(r = a^-)\) is zero except at the poles, so we can express it in terms of Dirac \(\delta\) functions. The current \(dI\) that crosses an annular region on the surface of the bead of angular extent \(d\cos \theta\) centered on angle \(\theta\) is given by

\[
dI = 2\pi a^2 J_r(a^-, \theta) d\cos \theta. \tag{5}
\]

Current \(I\) enters at \(\cos \theta = 1\), and exits at \(\cos \theta = -1\). Hence, the form

\[
J_r(a^-, \theta) = \frac{I}{2\pi a^2} [\delta(\cos \theta - 1) + \delta(\cos \theta + 1)]. \tag{6}
\]

describes the entrance and exit currents upon integration of eq. (5).

Combining eqs. (3)-(4) and (6), we have

\[
\sum_{n \text{ odd}} n A_n \frac{a}{a} P_n(\cos \theta) = \frac{I}{2\pi a^2 \sigma} [\delta(\cos \theta - 1) - \delta(\cos \theta + 1)]. \tag{7}
\]

\(^1\)For a discussion of the slight departure from electrical neutrality of current-carrying conductors, see [2].
As usual, to evaluate the Fourier coefficients \( A_n \), we multiply eq. (7) by \( P_n(\cos \theta) \) and integrate over \( d\cos \theta \) to find, recalling that \( \int_{-1}^{1} P_m(x)P_n(x) \, dx = 2\delta_{mn}/(2n+1) \),

\[
\frac{nA_n}{a} \int_{-1}^{1} P_m^2(\cos \theta) \, d\cos \theta = \frac{2nA_n}{(2n+1)a} = \frac{2I}{2\pi a^2\sigma}.
\] (8)

Thus, the Legendre series expansion for the potential is

\[
\phi(r < a, \theta) = \frac{I}{2\pi a\sigma} \sum_{n \text{ odd}} \left(2 + \frac{1}{n}\right) \left(\frac{r}{a}\right)^n P_n(\cos \theta).
\] (9)

### 2.2 Current Density inside the Bead

Inside the resistive bead, the current density \( \mathbf{J} \) is given by

\[
\mathbf{J} = \sigma \mathbf{E} = -\sigma \frac{\partial \phi}{\partial r} \hat{r} - \sigma \frac{\partial \phi}{r \partial \theta} \hat{\theta} = -\sigma \frac{\partial \phi}{\partial r} \hat{r} + \sigma \sin \theta \frac{\partial \phi}{\partial \cos \theta} \hat{\theta}.
\] (10)

In the midplane, \( (\theta = \pi/2) \), the current density has only a \( \theta \)-component \((-z\)-component),

\[
J_z(r, \theta = \pi/2) = -\frac{I}{2\pi ar} \sum_{n \text{ odd}} \frac{2n+1}{n} \left(\frac{r}{a}\right)^n \frac{dP_n(0)}{d\cos \theta} = -\frac{I}{2\pi ar} \sum_{n \text{ odd}} (2n+1) \left(\frac{r}{a}\right)^n P_{n-1}(0),
\] (11)

using eq. (13).

The total current across the midplane is

\[
I_z(\theta = \pi/2) = 2\pi \int_0^a J_z(r, \theta = \pi/2) \, dr = -\frac{I}{a} \int_0^a \sum_{n \text{ odd}} (2n+1) \frac{r^n}{a^n} P_{n-1}(0) \, dr
\]

\[
= -I \sum_{n \text{ odd}} \frac{2n+1}{n+1} P_{n-1}(0) = -I,
\] (15)

as expected, based on numerical evaluation of eq. (15) for terms up to \( n = N \) as shown in the figure below (due to Boris Ivetić).

---

2 Some useful relations among the Legendre polynomials are, from eqs. (3.29) and (3.31) of [1],

\[
(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n+1}(x),
\] (12)

\[
(1-x^2)\frac{dP_n(x)}{dx} = n[P_{n-1}(x) - xP_n(x)],
\] (13)

\[
\int_{-1}^{1} xP_n(x)P_{m-1}(x) \, dx = \frac{2n}{(2n+1)(2n-1)} \delta_{mn}.
\] (14)

where \( P'_n(x) = dP_n(x)/dx \).
The current density along the axis of the bead is

\[ J_z(r, \theta = 0) = -\frac{I}{2\pi a^2} \sum_{n \text{ odd}} (2n + 1) \left( \frac{r}{a} \right)^{n-1} = -\frac{I}{2\pi a^2} \left( 3 + 7\frac{r^2}{a^2} + 11\frac{r^4}{a^4} + \cdots \right), \]  

(16)

which diverges at the poles \( r = a, \theta = 0, \pi \), while that on the surface of the bead is

\[ J_\theta(a^-, \theta) = -\frac{I \sin \theta}{2\pi a^2} \sum_{n \text{ odd}} \frac{2n + 1}{n} P_n'(\cos \theta) \]

\[ = -\frac{I}{2\pi a^2 \sin \theta} \sum_{n \text{ odd}} (2n + 1)[P_{n-1}(\cos \theta) - \cos \theta P_n(\cos \theta)], \]  

(17)

using eq. (13), which also is ill behaved at the poles.\(^3\)

These divergences result from the unphysical assumption that the wires have zero radius. For wires of finite radius \( b \), the series (9) for the potential will be cut off at large \( n \), as discussed further below, such that all fields and current densities are finite.

### 2.3 Closed Form for the Interior Potential

To express the series (9) in closed form, we utilize the expansion for the distance \( r_1 \) between the points \( (a, 0, \varphi) \) and \( r = (r, \theta, \varphi) \) given in eq. (3.38) of [1],

\[ \frac{1}{r_1} = \frac{1}{a} \sum_{n=0}^{\infty} \left( \frac{r}{a} \right)^n P_n(\cos \theta), \]

(19)

Similarly, the distance \( r_2 \) between the points \( (a, \pi, \varphi) \) and \( (r, \theta, \varphi) \) is

\[ \frac{1}{r_2} = \frac{1}{a} \sum_{n=0}^{\infty} \left( \frac{r}{a} \right)^n P_n(\cos(\theta - \pi)) = \frac{1}{a} \sum_{n=0}^{\infty} \left( \frac{r}{a} \right)^n P_n(-\cos \theta) = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left( \frac{r}{a} \right)^n P_n(\cos \theta). \]

(20)

\(^3\)For \( x = \cos \theta \approx 1, \sin \theta = \sqrt{1 - x^2} \approx \sqrt{2(1 - x)} \) and \( P_n(x) = \sum_{m=0}^{n}[C_m^n]^2(x - 1)^{n-m}(1 + x)^m/2^n \approx 1 - n^2(1 - x)/2, \) so that eq. (17) leads to

\[ J_\theta(a^-, x \approx 1) \approx -\frac{I}{2\pi a^2} \frac{\sqrt{1 - x}}{2\sqrt{2}} \sum_{n \text{ odd}} (4n^2 - 1). \]  

(18)
Hence,
\[
\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} \sum_{n \text{ odd}} \left( \frac{r}{a} \right)^n P_n(\cos \theta).
\] (21)

It follows that, on integration along the radius from the origin to the point \((r, \theta)\),
\[
\int_0^r \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \frac{dr'}{r'} = \frac{2}{a} \sum_{n \text{ odd}} \frac{1}{n} \left( \frac{r}{a} \right)^n P_n(\cos \theta).
\] (22)

Then, eqs. (9) and (21)-(22) combine to give the alternative form (1) for \(\phi\),
\[
\phi(r < a, \theta, \varphi) = \frac{I}{2\pi\sigma} \left[ \frac{1}{r_1} - \frac{1}{r_2} \right] + \frac{1}{2} \int_0^r \left( \frac{1}{r_1'} - \frac{1}{r_2'} \right) \frac{dr'}{r'}
\] (1)

where \(r' = (r', \theta, \varphi)\).

As we approach the “north” pole, \(r_1 \to 0\), the first term in eq. (1) dominates. Similarly, near the “south” pole, \(r_2 \to 0\), the second term dominates (we claim; details in eqs. (32)-(33) below). That is, the potential \(\phi\) diverges at the poles for the case of very fine wires.

### 2.4 Formal Solution for Wires of Radius \(b\)

This section was suggested by Boris Ivetić.

We can obtain a solution for wires of radius \(b \ll a\), which make contact with the bead over a spherical cap of angle \(\alpha = \sin^{-1} b/a \approx b/a\), if we suppose that the radial electric field on this cap is given by
\[
E_r(a^-, |\cos \theta| > \cos \alpha \approx 1 - b^2/2a^2) = \frac{J_r}{\sigma} \approx \pm \frac{I}{\pi b^2\sigma},
\] (23)

which is a good approximation for small \(b/a\). As before, the radial electric field at the surface of the bead is zero outside the region of contact with the wires,
\[
E_r(a^-, |\cos \theta| < \cos \alpha) = 0.
\] (24)

Now, eqs. (3)-(4), together with the boundary conditions (23)-(24), lead to the relation
\[
\sum_{n \text{ odd}} \frac{nA_n}{a} P_n(\cos \theta) = \pm \frac{I}{\pi b^2\sigma} \left\{ \begin{array}{ll}
1 & (|\cos \theta| > \cos \alpha), \\
0 & (|\cos \theta| < \cos \alpha).
\end{array} \right.
\] (25)

Multiplying by \(P_n\) and integrating with respect to \(x = \cos \theta\) yields, using eq. (12),
\[
\frac{2nA_n}{(2n+1)a} = \frac{2I}{\pi b^2\sigma} \int_{\cos \alpha}^1 P_n(x) \, dx = \frac{2I}{(2n+1)\pi b^2\sigma} \int_{\cos \alpha}^1 (P_{n+1}'(x) - P_{n-1}'(x)) \, dx,
\] (26)

\[
A_n = \frac{Ia}{n\pi b^2\sigma} (P_{n-1}(\cos \alpha) - P_{n+1}(\cos \alpha)).
\] (27)
However, this formal solution does not readily provide insight as to how the divergent behavior of the potential, illustrated in sec. 2.2, is avoided.\textsuperscript{4,5} So, we return to discussion of the forms (1) and (9) in sec. 2.5.

2.5 Ohm’s Law for Wires of Radius $b$

When considering actual wires of radius $b \ll a$, we suppose that our solution (1) based on wires of zero radius, holds away from the region of contact between the wire and the bead. Indeed, we expect that the potential close to the wire, and outside the resistive bead, to be constant in planes perpendicular to the axis of the wire, so that the interface between the wire and the bead is an equipotential. This cuts off the formal divergence in eq. (1) near the poles.

In this way, the potential at the upper interface is obtained from (1) on putting $r_1 = b$ and neglecting all but the first term,

$$\phi_{\text{interface}} \equiv V \approx \frac{I}{2\pi \sigma b}.$$  \hfill (29)

The potential difference across the bead is twice this;

$$\Delta V = 2V \approx \frac{I}{\pi \sigma b} = \frac{b}{\sigma \pi b^2} \equiv IR.$$  \hfill (30)

Thus, the effective resistance of the bead is

$$R \approx \frac{1}{\pi \sigma b} = \frac{b}{\sigma \pi b^2},$$  \hfill (31)

which is also the resistance of a piece of wire of radius $b$, length $b$, and conductivity $\sigma$.

To verify the claim that the first term of eq. (1) dominates for small $r_1$, we consider the point $(r, \theta) = (a - b, 0)$ for $b \ll a$. Then, the first term of eq. (1) is $1/b$, and the second term is $1/(2a - b)$ which is negligible compared to the first. Inside the integral term of eq. (1), we have $R_1 = a - r$ and $R_2 = a + r$, so the integral is

$$\int_{a-b}^{a-b} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \ln r = \int_{0}^{a-b} \frac{2}{a^2 - r^2} \ln r = \frac{1}{a} \ln \frac{2a - b}{b} \approx \frac{1}{a} \ln \frac{2a}{b}. \hfill (32)$$

The ratio of the integral term to the first term of (1) is therefore,

$$\frac{b}{2a} \ln \frac{2a}{b}, \hfill (33)$$

which goes to zero as $b$ becomes small.

\textsuperscript{4}For large enough $n$, $P_n(x)$ is oscillatory on the interval $[\cos \alpha, 1]$, so the integral $\int_{\cos \alpha}^{1} P_n(x) \, dx$ goes to zero and the potential remains finite near the poles.

\textsuperscript{5}In case of finite wires, $b/a = 0.03$, we can use the potential found above to obtain the current density along the axis of the bead,

$$J_z(r, \theta = 0, \pi) = -\frac{I}{\pi b^2} \sum_{n \text{ odd}} (P_{n-1}(\cos \alpha) - P_{n+1}(\cos \alpha)) \left( \frac{r}{a} \right)^{n-1}. \hfill (28)$$

At the poles, $r = a$, $\theta = 0$, $\pi$, all terms in the series (28) cancel except for the very first, with $P_0(\cos \alpha) = 1$, which implies that $J_z(a, \theta = 0, \pi) = -I/\pi b^2$, as expected for lead wires of radius $b$. 

6
2.6 Magnetic Field and Poynting Vector

The power dissipated by the resistive bead is, according to eq. (31),

$$ P = I^2 R \approx \frac{I^2}{\pi \sigma b} . $$

As a check on the solution (9) for the potential, we consider whether the dissipated power equals the integral of the Poynting flux, \( S = (c/4\pi)E \times B \) (in Gaussian units), normal to the surface of the bead, where \( c \) is the speed of light in vacuum.

For this, we need the magnetic field \( B \) at \( r = a \), due to the electric currents in the problem. This field is azimuthal, because of the azimuthal symmetry of the problem. Then, the magnetic field at the surface of the resistive bead follows easily from Ampère’s law,

$$ 2\pi r B_\phi(r = a^-) = 2\pi a \sin \theta B_0 = -\frac{4\pi}{c} I, \quad B_\phi(r = a^-) = -\frac{2I}{ca \sin \theta} . $$

The radial component of the Poynting vector also depends on the electric field component,

$$ E_\theta(r = a^-) = -\frac{1}{a} \frac{\partial \phi(r = a^-)}{\partial \theta} . $$

Then, the radial component of the Poynting vector at the surface of the bead is

$$ S_r(r = a^-, \theta) = \frac{c}{4\pi} E_\theta(r = a^-) B_\phi(r = a^-) = \frac{I}{2\pi a^2} \frac{1}{\sin \theta} \frac{\partial \phi(r = a^-)}{\partial \theta} = -\frac{I}{2\pi a^2} \frac{\partial \phi(r = a^-)}{\partial \cos \theta} = -\frac{I}{2\pi a^2} \frac{I}{2\pi a \sigma} \sum_{n \text{ odd}} \left( 2 + \frac{1}{n} \right) P'_n(\cos \theta) . $$

The integral of the radial component of the Poynting vector over the surface of the bead is

$$ P_{\text{into bead}} = -2\pi a^2 \int_{-1}^{1} S_r(r = a^-, \theta) d \cos \theta = \frac{I^2}{2\pi a \sigma} \sum_{n \text{ odd}} \left( 2 + \frac{1}{n} \right) \int_{-1}^{1} P'_n(\cos \theta) d \cos \theta $$

$$ = \frac{I^2}{\pi a \sigma} \sum_{n \text{ odd}} \left( 2 + \frac{1}{n} \right) \approx \frac{b}{a} I^2 R \sum_{n \text{ odd}} \left( 2 + \frac{1}{n} \right) . $$

Formally, the result (38) diverges, which corresponds to infinite power dissipation at the points of contact of the wires with the bead, in the limit of zero radius of these wires. For wires of finite radius \( b \), the power dissipated is finite, \( P = I^2 R \) for resistance \( R \) as approximated in eq. (31), but then the formal solutions (1) and (9) are only approximate. Since the sum of the first \( N \) terms of the series \( \sum_{n \text{ odd}} (2 + 1/n) \) is roughly \( N \), we infer that the form (9) for the potential inside the bead in case of wires of radius \( b \) is a reasonable approximation if we keep only the first \( N \approx a/b \) terms.

2.7 Potential outside the Resistive Bead

We now consider the problem outside the bead, for which one model is that the wires are perfect conductors extending from \( r = a \) to distance \( d \), where they are attached to perfectly
conducting hemispheres of radius \( d \), with a ring-shaped battery of potential difference \( 2V \) between them at location \((r, \theta) = (d, \pi/2)\).

In the region \( a < r < d \) and outside the wires at \( \theta = 0 \) and \( \pi \), the potential obeys \( \nabla^2 \phi = 0 \), is azimuthally symmetric, and symmetric about the plane \( \theta = \pi/2 \), so it can be expanded as

\[
\phi(a < r < d) = \sum_{n \text{ odd}} \left[ B_n \left( \frac{r}{a} \right)^n + C_n \left( \frac{a}{r} \right)^{n+1} \right] P_n(\cos \theta).
\] (39)

Continuity of the potential at \( r = a \) requires, recalling eqs. (3) and (8), that

\[
B_n + C_n = A_n = \frac{(2n+1)I}{2\pi an} = \frac{(2n+1)bV}{an}.
\] (40)

The constant potential \( V \) on the upper hemisphere requires that

\[
\phi(r = d, 0 < \cos \theta < 1) = V = \sum_{n \text{ odd}} \left[ B_n \left( \frac{d}{a} \right)^n + C_n \left( \frac{a}{d} \right)^{n+1} \right] P_n(\cos \theta),
\] (41)

and the constant potential \( V \) on the upper wire requires that

\[
\phi(a < r < d, \theta = 0) = V = \sum_{n \text{ odd}} \left[ B_n \left( \frac{r}{a} \right)^n + C_n \left( \frac{a}{r} \right)^{n+1} \right].
\] (42)

If we multiply eq. (41) by \( P_n(\cos \theta) = P_n(x) \) and integrate over \( x \) from 0 to 1, we obtain (using Wolfram Alpha with integrate Legendre \( P(n,x) \) from \( x=0 \) to \( 1 \)),

\[
V \int_0^1 P_n(x) \, dx = \frac{\sqrt{\pi} V}{2\Gamma(1 - \frac{n}{2}) \Gamma(\frac{n+1}{2})} \equiv K_n V = \frac{1}{2n+1} \left[ B_n \left( \frac{d}{a} \right)^n + C_n \left( \frac{a}{d} \right)^{n+1} \right],
\] (43)

\( K_1 = 1/2, \ K_3 = -1/8, \ K_5 = 1/16, \ K_7 = -5/128, \ K_9 = 7/256, \ K_{11} = -21/1024, \ ... \)

Combining this with eq. (40), we find

\[
B_n = \frac{(2n+1)V}{(d/a)^n - (a/d)^{n+1}} \left[ K_n - \frac{b}{an} \left( \frac{a}{d} \right)^{n+1} \right], \quad C_n = \frac{(2n+1)V}{(d/a)^n - (a/d)^{n+1}} \left[ \frac{b}{an} \left( \frac{d}{a} \right)^n - K_n \right].
\] (44)

\[\text{Jackson uses Rodrigues’ formula and integration by parts \( n \) times to find}\]

\( K_n = (-1/2)^{(n-1)/2}(n-2)!/2!(n+1)/2! \), his eq. (3.26) \[1\].
It is not obvious how well the $B_n$ and $C_n$ of eq. (44) satisfy the condition (42), but a numerical example suggests that they do so. For example, suppose that $d = 2a$, $b/a = 0.03$ and $V = 1$. Then, using only the first six terms of eq. (42) yields the following plot.\(^7\)

The potential outside the perfectly conducting hemispheres, which are held at potentials $\pm V$, is given in eq. (3.36) of [1]. In our notation,

$$\phi(r > d) = V \sum_{n \text{ odd}} (2n + 1) K_n \left( \frac{d}{r} \right)^{n+1} P_n(\cos \theta).$$

\[(45)\]

### 2.8 Surface Charge Density on the Resistive Bead

We also infer that the surface $r = a$ of the bead supports electric charge density

$$\zeta(\theta) = \frac{E_r(r = a^+) - E_r(r = a^-)}{4\pi} = -\frac{\partial}{\partial r} [\phi(r = a^+) - \phi(r = a^-)]$$

$$= \sum_{n \text{ odd}} \frac{-nB_n + (n + 1)C_n + nA_n}{a} P_n(\cos \theta) = \sum_{n \text{ odd}} \frac{(2n + 1)C_n}{a} P_n(\cos \theta).$$

This illustrates the general result that current-carrying conductors (of finite conductivity $\sigma$) have nonzero surface charge density, as needed to shape the electric field $E = J/\sigma$ which drives the current inside the conductor [3].

For completeness, we compute the linear charge density $\lambda$ on the lead wires, and the surface charge density $\zeta$ on the inside and outside surfaces of the hemispheres at $r = d$,

$$\lambda(a < r < d, \theta = 0, \pi) = 2\pi b \zeta = \frac{b}{2} E_\theta = -\frac{b}{2r} \frac{\partial \phi}{\partial \theta}$$

$$= \frac{b}{2r} \sin \theta \sum_{n \text{ odd}} \left[ B_n \left( \frac{r}{a} \right)^n + C_n \left( \frac{a}{r} \right)^{n+1} \right] P_n'(\cos \theta)$$

$$\approx \pm \frac{b^2}{2r^2} \sum_{n \text{ odd}} \left[ B_n \left( \frac{r}{a} \right)^n + C_n \left( \frac{a}{r} \right)^{n+1} \right] P_n'(1)$$

\(^7\)In the limit that $d \gg a$, $B_n \to 0$, $C_n \to A_n$, and eq. (42) would imply that $V \to 0$. That is, to satisfy all three constraints (39), (41) and (42) on the potential $\phi(a < r < d)$, we need $B_n$ nonzero, and $d/a$ finite.
\[
\zeta(r = d^-) = -\frac{E_r(r = d^-)}{4\pi} = \frac{1}{4\pi} \sum_{n \text{ odd}} n B_n \left( \frac{d}{a} \right)^n - (n + 1) C_n \left( \frac{d}{a} \right)^{n+1} \right] P_n(\cos \theta),
\]
where \( P_n'(1) = dP_n(x = 1)/dx = n(n + 1)/2, \) and \( \sin \theta \approx b/r \) for points on the surface of the lead wires.

\section{Appendix: Resistive Spherical Shell}

This Appendix was suggested by Boris Ivetić.

We can also consider the case of a resistive spherical shell of outer radius \( a \) and thickness \( \epsilon \ll a \).

\subsection*{A.1 Surface Current and Total Resistance of the Shell}

The surface current density \( K \) (per unit length) on a resistive shell has only a \( \theta \)-component, related by conservation of charge flowing across rings of circumference \( 2\pi a \sin \theta \) at angles \( \theta \) as

\[
K_\theta(\theta) = \frac{I}{2\pi a \sin \theta}.
\]

To estimate the electrical resistance \( R \) of the spherical shell, we note that an annulus of extent \( a \, d\theta \) about angle \( \theta \), with circumference \( C = 2\pi a \sin \theta \), has resistance \( dR = a \, d\theta / \sigma_S C = d\theta / 2\pi \sigma_S \sin \theta \) to the surface current \( K_\theta \). To find the total resistance, we integrate \( dR \) from \( \theta = b/a \) to \( \pi - b/a \), supposing the current enters and exits the shell through wires of radius \( b \ll a \). Then,

\[
R \approx 2 \int_{b/a}^{\pi/2} dR = \frac{1}{\pi \sigma_S} \int_{b/a}^{\pi/2} \frac{d\theta}{\sin \theta} \approx -\frac{1}{\pi \sigma_S} \ln \frac{b}{2a},
\]

which is very large for small \( b/a \), in contrast to the result (31) for a solid resistive bead (of volume conductivity \( \sigma \)).

\subsection*{A.2 Potential inside the Shell}

The resistive shell supports charge densities on both of its surfaces, which in the limit of zero thickness \( \epsilon \) we suppose is a single surface density \( \zeta(a, \theta) \). Then, the potential for \( r < a - \epsilon \approx a \) (as well at that for \( a < r < d \) and \( r > d \)) obeys \( \nabla^2 \phi = 0 \), so we can again seek a potential
in spherical coordinates that is independent of azimuth $\varphi$, and symmetric about the plane $\theta = 0$, with the form (3),

$$\phi(r < a) = \sum_{n \text{ odd}} A_n \left( \frac{r}{a} \right)^n P_n(\cos \theta).$$

(3)

This surface current is driven by the electric field $E_\theta$ inside the resistive shell according to Ohm’s law, which we take to have the form

$$K_\theta = \sigma_S E_\theta(r = a^-) = -\sigma \frac{\partial \phi(r = a^-)}{\partial \theta} = \sigma_S \sin \theta \frac{\partial \phi(r = a^-)}{\partial \cos \theta} = \sigma_S \sin \theta \sum_{n \text{ odd}} A_n P'_n(\cos \theta). \quad (52)$$

where $\sigma_S$ is the surface conductivity.\(^8\) Then, recalling eq. (50) and using eq. (13),

$$(1 - \cos^2 \theta) \sum_{n \text{ odd}} A_n P'_n(\cos \theta) = \sum_{n \text{ odd}} n A_n [P_{n-1}(\cos \theta) - \cos \theta P_n(\cos \theta)] = \frac{I}{2\pi a \sigma_S}. \quad (53)$$

To find $A_1$, we simply integrate eq. (53) with respect to $\cos \theta$ from $-1$ to 1,

$$A_1 \int_{-1}^{1} (1 - \cos^2 \theta) d \cos \theta = \frac{4A_1}{3} = \frac{I}{\pi a \sigma_S}, \quad A_1 = \frac{3I}{4\pi a \sigma_S}. \quad (54)$$

For $n > 1$, we multiply eq. (53) by $P_{n-1}(\cos \theta)$ and integrate to find, using eq. (13),

$$\frac{2n(n+1)}{(2n-1)(2n+1)} A_n = \frac{2(n-1)(n-2)}{(2n-1)(2n-3)} A_{n-2}, \quad \frac{(n-1)(n-2)(2n+1)}{n(n+1)(2n-3)} A_{n-2}. \quad (55)$$

Thus, $A_3 = 7A_1/18$, $A_5 = 11A_1/45$, $A_7 = 5A_1/28$, ..., with $A_n \approx (1 - 2/n)A_{n-2}$ for large $n$.

The potential is divergent at the poles of the shell, which divergence is suppressed in practice by the finite radius $b$ of the lead wires, which implies that $A_n \to 0$ for $n$ large enough that $P_n(\cos \theta)$ is oscillatory on the interval $0 < \theta < b/a$.

References


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\(^8\)Strictly, we should seek a potential inside the shell, for $a - \epsilon < r < a$, that leads to zero radial electric field $E_r$ inside this shell. However, $E_\theta$ is continuous across the surface of the shell, so to the extent that we restrict our attention to $E_\theta$ in and around the shell, we can use the potential for $r < a - \epsilon$ in eq. (52).