The Relativity of Acceleration

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1 Problem

Acceleration is not often considered in special relativity, but it can be, as in the following problem.

A set of objects, initially at rest in an inertial frame at positions $x_i = (x_{i,0}, 0, 0)$, where $x_{i,0} = i L$, begin accelerating at a constant value $a = (a, 0, 0)$ at time $t'_0 = 0$ in a second inertial frame that has velocity $v = (v, 0, 0)$ with respect to the first. Both $a$ and $v$ are positive.

What are the subsequent histories of these objects according to observers in these two frames? Can the acceleration be large enough that these objects collide with one another?

2 Solution

A constant acceleration $a$ cannot be sustained indefinitely, as the speed $a\Delta t$ of such an object in the first frame would eventually exceed the speed of light $c$, which is impossible. So, this problem concerns only time intervals $\Delta t$ small enough that all speeds are less than $c$.

While the objects start accelerating simultaneously in the second frame, they do not start simultaneously in the first frame, due to the so-called relativity of simultaneity. This problem will illustrate that when the acceleration is constant in the first frame this is not so in the second frame, which effect could be called the relativity of acceleration.

Since all motion in this problem involves only the $x$-coordinates, it is sufficient to consider the Lorentz transformations of only the $x$-coordinates and time between the two frames,

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\begin{align*}
x' &= \gamma(x - vt), \\
t' &= \gamma(t - vx/c^2), \\
x &= \gamma(x' + vt'), \\
t &= \gamma(t' + vx'/c^2),
\end{align*}
$$

where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$. (3)

The theory of relativity is based, in part, on the premise that there is only a single “reality”, but different observers can describe this differently.¹ For example, the start of acceleration of object $i$ is an “event”, described by coordinates $(x_{0,i}, t_{0,i})$ in the first frame and coordinates $(x'_{0,i}, t'_{0,i})$ in the second frame. These two sets of coordinates are related by the Lorentz transformations (1)-(2). However, there is only one object with index $i$, not different objects in different frames.

A collision between two objects, say $i$ and $i - 1$ would be an “event”, described by coordinates $(x_{i,i-1}, t_{i,i-1})$ in the first frame and coordinates $(x'_{i,i-1}, t'_{i,i-1})$ in the second frame.

¹This premise contrasts with that of the “multiverse” interpretation of quantum theory in which different observers experience different events in different universes.
Again, these two sets of coordinates should be related by the Lorentz transformations (1)-(2). However, there is only one collision between two objects, not different collisions in different frames. That is, if a collision occurs in one frame, it occurs in all, or if there is no collision in one frame there is no collision in any other.

In the present example, we will find no collisions in either frame, by somewhat different arguments.

2.1 History in the First Frame during the Acceleration

Before time $t'_0 = 0$ in the second frame, all objects $i$ are at rest, at $x_{i,0} = iL$, in the first frame.

These objects start accelerating in the first frame at times $t_{i,0}$ that can be determined from the time transformation of eq. (1),

$$t_{i,0} = \frac{vx_{i,0}}{c^2} + \frac{t'_0}{\gamma} = \frac{vx_{i,0}}{c^2} + \frac{i vL}{c^2}. \quad (4)$$

The motion of object $i$ for time $t > t_{i,0}$ in the first frame is given by,

$$v_i = a(t-t_{i,0}), \quad x_i = x_{i,0} + \frac{a}{2}(t-t_{i,0})^2 = iL + \frac{a}{2} \left( t - \frac{i vL}{c^2} \right)^2 \quad \left( t_{i,0} < t < t_{i,0} + \frac{c}{a} \right), \quad (5)$$

where this motion is restricted to times $t < t_{i,f} = t_{i,0} + c/a$ such that $v_i < c$.

Under the assumption of constant acceleration $a$ in the first frame, object $i$ (if it did not collide with any other object) would reach the speed of light (after infinite expenditure of energy by the accelerating mechanism) at the “final” time $t_{i,f}$ and “final” position $x_{i,f}$ related by,

$$t_{i,f} = t_{i,0} + \frac{c}{a} = \frac{ivL}{c^2} + \frac{c}{a}, \quad x_{i,f} = x_{i,0} + \frac{a}{2}(t_{i,f} - t_{i,0})^2 = iL + \frac{c^2}{2a}. \quad (6)$$

2.1.1 Do the Objects Collide with One Another?

However, the motion (5) actually holds only if object $i$ does not collide with either of its nearest neighbors $i-1$ and $i+1$ while accelerating.

Objects $i$ and $i-1$ collide at time $t_{i;i-1}$ related by $x_i(t_{i;i-1}) = x_{i-1}(t_{i;i-1})$,

$$iL + \frac{a}{2} \left( t_{i;i-1} - \frac{ivL}{c^2} \right)^2 = (i-1)L + \frac{a}{2} \left( t_{i;i-1} - \frac{(i-1)vL}{c^2} \right)^2, \quad (7)$$

$$t_{i;i-1}^2 - \frac{2ivLt_{i;i-1}}{c^2} + \frac{i^2v^2L^2}{c^4} - t_{i;i-1}^2 + \frac{2(i-1)vLt_{i;i-1}}{c^2} - \frac{(i-1)^2v^2L^2}{c^4} = \frac{2L}{a}, \quad (8)$$

$$t_{i;i-1}^2 - \frac{2ivLt_{i;i-1}}{c^2} + \frac{(2i-1)v^2L^2}{c^4} = \frac{2L}{a}, \quad (9)$$

$$t_{i;i-1} = \left( \frac{2i-1}{2c^2} \right) \frac{c^2}{av} = t_{i,0} + \frac{c^2}{av} - \frac{vL}{2c^2} = t_{i-1,0} + \frac{c^2}{av} + \frac{vL}{2c^2}. \quad (10)$$

\(^2\)If the acceleration $a$ were negative, the time interval of acceleration would be (at most) $c/|a|$. \]
The velocity of object $i-1$ at the time of the collision is,

$$v_{i-1}(t_{i;1}) = a(t_{i;1} - t_{i-1,0}) = \frac{c^2}{v} + \frac{avL}{2c^2} > c,$$

(11)

which is greater than the speed $c$ of light. This is impossible, and hence collisions between the objects $i$ do not occur for $a$ and $v$ both positive.\(^3\)

It could be that $a$ is positive but $v$ is negative, say $v = -u$ where $u$ is positive. In this case, eq. (10) would be,

$$t_{i;1} = t_{i,0} - \frac{c^2}{au} + \frac{uL}{2c^2} = t_{i-1,0} - \frac{c^2}{au} - \frac{uL}{2c^2},$$

(13)

which would imply that object $i-1$ collided at a time before the acceleration started. Furthermore, eq. (12) indicates that the velocity of object $i$ at the time of the supposed collision would be negative and with magnitude larger than $c$. That is, collisions do not occur for positive $a$ and negative $v$.

If $a$ is negative and $v$ is positive, then according to eq. (10), object $i$ collided at a time before the acceleration started (and according to eq. (12), the velocity of object $i$ exceeded $c$ at the time of that collision), so collisions do not occur for this case.

Finally, if $a$ and $v$ were both negative, then according to eq. (12) the velocity of object $i$ at the time of the collision would be negative, with magnitude greater than $c$, so collisions do not occur for this case.

In sum, collisions do not occur for any combination of $a$ and $v$ (for any value of the distance $L$ between adjacent objects).

### 2.2 History in the Second Frame during the Acceleration

In the second frame, all objects $i$ start accelerating at time $t_{i,0}' = 0$, and positions $x_{i,0}'$ that can be computed from the $x$-transformation of eq. (2),

$$x_{i,0}' = \frac{x_{i,0}}{\gamma} - v t_{i,0}' = \frac{x_{i,0}}{\gamma} = \frac{iL}{\gamma},$$

(14)

which we recognize as an example of the Lorentz contraction of objects in motion with respect to an observer (here in the second frame).\(^4\) The initial velocity of object $i$ in the

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3For completeness, we note that the velocity of object $i$ at the time of the collision is,

$$v_i(t_{i;i-1}) = a(t_{i;i-1} - t_{i,0}) = \frac{v^2}{v} - \frac{avL}{2c^2}.$$

(12)

This could be less that $c$, so consideration only of the velocity of object $i$ would not rule out the collisions.

4We could also have used the $x$-transformation of eq. (1),

$$x_{i,0}' = \gamma(x_{i,0} - vt_{i,0}) = \gamma \left(iL - \frac{ivL}{c^2}\right) = \gamma iL \left(1 - \frac{v^2}{c^2}\right) = \frac{iL}{\gamma},$$

(15)

or the time-transformation of eq. (2),

$$t = \gamma(t' + vx'/c^2), \quad \frac{vx_{i,0}'}{c^2} = \frac{t_{i,0}}{\gamma} - t_{i,0}', \quad x_{i,0}' = \frac{c^2}{v} \gamma c^2 = \frac{iL}{\gamma}.$$  (16)
The two computations (20) and (22) of the supposedly constant acceleration at negative \( x \) which contradiction indicates that this acceleration is not actually constant. Also, the position \( \text{if} \) determine the time dependence of the acceleration \( a \). Acceleration that is constant in one inertial frame is not constant in another.

2.2.1 Is the Acceleration of Object \( i \) Constant in the Second Frame?

If the motion object \( i \) in the second frame were that of constant acceleration \( a'_i \), its velocity \( v'_i(t') \) would have the time dependence during the interval \([t'_i,0, t'_i,f] = [0, t'_i,f],\)

\[
v'_i(t') = v'_i,0 + a'_i(t' - t'_i,0) = -v + a'_i t',
\]

and in particular, recalling eq. (18),

\[
v'_i(t'_i,f) = c = -v + a'_i t'_i,f, \quad a'_i = \frac{c + v}{\gamma c - v/2}.
\]

Also, the position \( x'_i(t') \) during this time interval would be related by,

\[
x'_i(t') = x'_i,0 - vt' + \frac{a'_i t'^2}{2} = \frac{i L}{\gamma} - vt' + \frac{a'_i t'^2}{2},
\]

and in particular,

\[
x'_i(t'_i,f) = \frac{i L}{\gamma} + \frac{\gamma c}{a} (\frac{c}{2} - v) = \frac{i L}{\gamma} - v \gamma \frac{c}{a} (\frac{c}{2} - v) + \frac{a'_i \gamma^2}{2 a^2} (\frac{c}{2} - v)^2, \quad a'_i = \frac{a}{\gamma^3 (c - v/2)^2}.
\]

The two computations (20) and (22) of the supposedly constant acceleration \( a'_i \) disagree, which contradiction indicates that this acceleration is not actually constant.

Thus, we have a first lesson in the relativity of acceleration: Acceleration that is constant in one inertial frame is not constant in another.

We must now examine the motion of the objects in the second frame in more detail, to determine the time dependence of the acceleration \( a'_i(t') \).

2.2.2 Transformation of the History \((x_i, t_i)\) from the First to the Second Frame

The history of object \( i \) in the first frame when accelerating is given in eq. (5),

\[
x_i(t_i) = x_{i,0} + \frac{a}{2} (t_i - t_{i,0})^2 = i L + \frac{a}{2} \left( t_i - \frac{i v L}{c^2} \right)^2 \quad \left( t_{i,0} = \frac{i v L}{c^2} < t_i < t_{i,0} + \frac{c}{a} \right).
\]

It is “amusing” that for \( v > c/2 \), the “final” position in the second frame for the object with \( i = 0 \), is at negative \( x' \), while its “final” position in the first frame is at positive \( x \).
The transformation of this history into the second frame can be made using eq. (1),
\[
\frac{x_i'}{\gamma} = x_i - vt_i = iL + \frac{a}{2} \left( t_i - \frac{ivL}{c^2} \right)^2 - vt_i, \quad (24)
\]
\[
\frac{t_i'}{\gamma} = t_i - \frac{vx_i}{c^2} = t_i - \frac{v}{c^2} \left[ iL + \frac{a}{2} \left( t_i - \frac{ivL}{c^2} \right)^2 \right]. \quad (25)
\]

To convert eq. (24) to a history of the form \(x_i'(t_i')\) we need to know \(t_i\) as a function of \(t_i'\), which we can obtain from eq. (25),
\[
\frac{a t_i^2}{2} - \frac{c^2 t_i}{v} - \frac{2iavL t_i}{c^2} + iL + \frac{i^2 a v^2 L^2}{2c^4} + \frac{c^2 t_i'}{\gamma v} = 0,
\]
\[
t_i^2 - 2 \left( \frac{c^2}{av} + \frac{i vL}{c^2} \right) t_i + \frac{2iL}{a} + \frac{i^2 a v^2 L^2}{c^4} + \frac{2c^2 t_i'}{\gamma av} = 0, \quad (26)
\]
\[
t_i = \frac{c^2}{av} + i vL/c^2 \pm \sqrt{\left( \frac{c^2}{av} + \frac{i vL}{c^2} \right)^2 - \left( \frac{2iL}{a} + \frac{i^2 a v^2 L^2}{c^4} + \frac{2c^2 t_i'}{\gamma av} \right)}.
\]

Note that in arriving at eq. (27), we have divided by the acceleration \(a\) in the first frame, so if this acceleration is zero, this relation is not valid.

Since \(t_i = ivL/c^2\) when \(t_i' = 0\), we use the negative square root root,
\[
t_i = \frac{ivL}{c^2} + \frac{c^2}{av} \left( 1 - \sqrt{1 - \frac{2avt_i'}{\gamma c^2}} \right), \quad (28)
\]
\[
\left( t_i - \frac{ivL}{c^2} \right)^2 = \frac{2c^4}{a^2 v^2} \left( 1 - \sqrt{1 - \frac{2avt_i'}{\gamma c^2}} - \frac{avt_i'}{\gamma c^2} \right). \quad (29)
\]

Using this in eq. (24), we obtain the history of object \(i\) in the second frame,
\[
\frac{x_i'}{\gamma} = i L + \frac{c^4}{av^2} - \frac{c^4}{av^2} \sqrt{1 - \frac{2avt_i'}{\gamma c^2}} - \frac{c^2 t_i'}{\gamma v} - i \frac{v^2 L}{c^2} - \frac{c^2}{a} + \frac{c^2}{a} \sqrt{1 - \frac{2avt_i'}{\gamma c^2}}
\]
\[
= \left( 1 - \frac{v^2}{c^2} \right) \left[ iL + \frac{c^4}{av^2} \sqrt{1 - \frac{2avt_i'}{\gamma c^2}} \right] - \frac{c^2 t_i'}{\gamma v}, \quad (30)
\]
\[
x_i'(t_i') = \frac{1}{\gamma} \left[ iL + \frac{c^4}{av^2} \sqrt{1 - \frac{2avt_i'}{\gamma c^2}} \right] - \frac{c^2 t_i'}{\gamma v}. \quad (31)
\]

Taking time derivatives of eq. (31), we obtain the velocity and acceleration of object \(i\) in the second frame,
\[
v_i(t_i') = -\frac{c^2}{\gamma^2 v \sqrt{1 - \frac{2avt_i'}{\gamma c^2}}} - \frac{c^2}{v}, \quad v_i'(0) = -v, \quad v_i'(t_{i,f}) = c \quad \text{for} \quad t_{i,f} = \frac{\gamma}{a} \left( c - \frac{v}{2} \right), \quad (32)
\]
\[
a_i(t_i') = \frac{a}{\gamma^2 \left( 1 - \frac{2avt_i'}{\gamma c^2} \right)^{3/2}}. \quad (33)
\]
The velocity $v'_i$ of eq. (32), as observed in the second frame, begins at $-v$ at $t' = 0$ when the acceleration starts, as expected, and reaches the speed of light at time $t'_{i,f}$ as previously found in eq. (18). The acceleration $a'_i$ is the same for all objects $i$ as observed in the second frame, but is time dependent, while the acceleration $a$ in the first frame is constant. This illustrates the “relativity of acceleration” as discussed in sec. 2.2.1 above.

2.2.3 No Collisions in the Second Frame

A consequence of eqs. (31) and (33) is that adjacent objects maintain constant separation $L/\gamma$ during their acceleration as observed in the second frame, and hence do not collide, in agreement with the argument of sec. 2.1.1 above.

This problem was suggested by David Seppala.