No Bootstrap Spaceships via Magnets in Electric Fields

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Abstract

Contrary to what is sometimes claimed about the example of Shockley and James, Phys. Rev. Lett. **18**, 876 (1967), the existence of "hidden" mechanical momentum is consistent with momentum conservation in the example treated in that paper. Only with "hidden" momentum is it possible to respect the center-of-energy theorem, and avoid "bootstrap spaceships".

1 Introduction

A curiosity of the theory of classical electromagnetism is that some configurations of stationary electric charges and currents have nonzero electromagnetic-field momentum,

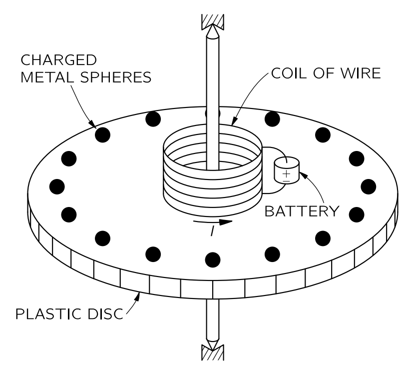
$$\mathbf{P}_{\rm EM} = \int \mu_0 \mathbf{E} \times \mathbf{B}\, d{\rm Vol} = \int \frac{\mathbf{S}}{c^2}\, d\mathbf{Vol}, \qquad (1)$$

where $\mathbf{S}$ is the electromagnetic-energy-flux vector introduced by Poynting (1883) [1] and $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum, as first noted by J.J. Thomson (1904) [2, 3, 4].[1] See also [8]-[42].

[1]The curious character of the Poynting vector was mentioned by Heaviside (1887) [5] (see also p. 94 of [6]), where he considered a uniformly magnetized sphere with a uniform electric surface charge. Here, lines of the Poynting vector $\mathbf{S}$ (Heaviside's $\mathbf{W}$) flow in circles outside the sphere, about its magnetic axis. As Heaviside remarked: *What is the use of it? On the other hand, what harm does it do?*

In 1963, Feynman posed the now-famous disk paradox related to field angular momentum in sec. 17-4 of [7]. This paradox was perhaps inspired by Heaviside's remarks, as well as a comment of J.J. Thomson, p. 348 of [2], about angular momentum in the electromagnetic field, $\mathbf{L}_{\rm EM} = \int \mathbf{r} \times (\mu_0 \mathbf{E} \times \mathbf{B})\, d{\rm Vol}$.

An insulating disk has electric charge around its rim and is initially at rest. This disk is coaxial with a solenoid magnet that initially has nonzero current, and the disk is free to rotate with respect to the solenoid.



If at some time the current in the solenoid goes to zero, the decreasing magnetic flux in the solenoid induces an azimuthal electric field that causes the charged disk to rotate. The "paradox" is that this behavior appears to violate conservation of angular momentum.

At the end of sec. 27-6, Feynman gave a verbal resolution of the paradox: *Do you remember the paradox we described in Section 17-4 about a solenoid and some charges mounted on a disc? It seemed that when the current turned off, the whole disc should start to turn. The puzzle was: Where did the angular momentum come from?*

The answer is that if you have a magnetic field and some charges, there will be some angular momentum in the field. It must have been put there when the field was built up. When the field is turned off, the angular momentum is given back. So the disc in the paradox would start rotating. This mystic circulating flow of energy, which at first seemed so ridiculous, is absolutely necessary. There is really a momentum flow. It is needed to maintain the conservation of angular momentum in the whole world.

If this field momentum later vanished, it might seem that the matter of the system would acquire "overt" momentum, and a nonzero center-of-mass velocity, although the system was initially at rest [2]. Such behavior was argued as impossible in [8] (1949), and has been called a "bootstrap spaceship" [16]. Comments in 1964 [9] (and in 1966 [11]) that a magnet in an electric field could be a "bootstrap spaceship" led to the development of the concept of "hidden" mechanical momentum by Shockley and James [13] (1967); for a review, see [42].

A more formal argument against "bootstrap spaceships" is based on the center-of-energy theorem [10, 15, 18], that the total momentum is zero for an isolated system whose center of mass/energy is at rest. This understanding supports the view that "hidden mechanical momentum" [13] is contained in the electric currents in many examples with nonzero electromagnetic-field momentum; for reviews, see [37, 39].[2] Nonetheless, there remain a few skeptics as to "hidden" momentum and the center-of-energy theorem [43, 44, 45], as well as proponents of "bootstrap spaceships" [46, 47, 48].

2 Shockley's Example

Shockley and James (1967) [13] considered the example shown in Fig. 1 (perhaps inspired by the Feynman disk paradox [7]), in which two electric charges $\pm Q$ at $(\pm x, 0, 0)$ are attached to a nonconducting rod that is also attached to the axle of two oppositely charged disks which rotate in opposite directions, creating magnetic dipole $\mathbf{m} = (0, 0, m)$.[3] The "pillbox" surrounding the disks is nonconducting.

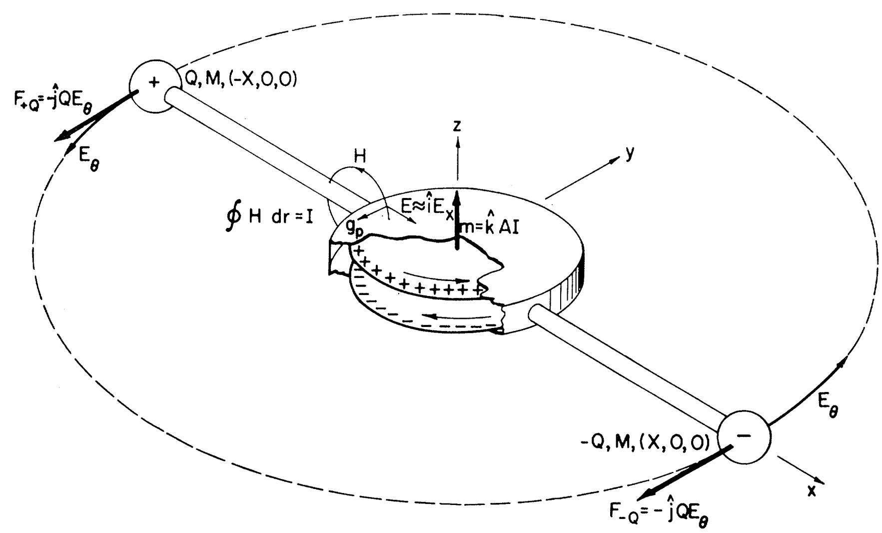


Figure 1: A magnetic dipole $\mathbf{m} = m\,\hat{\mathbf{k}}$ in an electric field $\mathbf{E} = E\,\hat{\mathbf{i}}$, as considered in [13]. The forces $\mathbf{F}_{\pm Q} = -QE_\theta\,\hat{\mathbf{j}}$ shown on the charges $\pm Q$ exist when the magnetic moment $\mathbf{m}$ drops to zero, and azimuthal electric field $\mathbf{E}_\theta$ is induced by the changing magnetic field.

[2]For an example in which the "hidden" mechanical momentum is more abstractly related by $\mathbf{P}_{\rm hid,mech} = \mathbf{P}_{\rm mech} - m_{\rm mech}\mathbf{v}_{\rm cm,mech} = -m_{\rm mech}\mathbf{v}_{\rm cm,mech}$, see sec. 2.4 of [29].

[3]In the notation of [13], the electric field at the origin is $\mathbf{E} = (E, 0, 0)$, but in [45] this field is called $2\mathbf{E}$.

2.1 Disassembly of the System

Shockley and James considered a possible disassembly of this system, in which the magnetic moment $\mathbf{m}$ dropped slowly to zero,[4] which induced an azimuthal electric field $\mathbf{E}_\theta$, which in turn exerted forces $\mathbf{F}_{\pm Q} = -QE_\theta\,\hat{\mathbf{j}}$ on each of the two electric charges $\pm Q$. Shockley and James computed that the momentum impulse given to the system by the forces $\mathbf{F}_{\pm Q}$ is,

$$\Delta\mathbf{P} = \varepsilon_0\mu_0\,\mathbf{E}\times\mathbf{m} = \frac{\mathbf{E}\times\mathbf{m}}{c^2}\,, \tag{2}$$

where c is the speed of light in vacuum. If this momentum were "overt", and the system ended with motion in the $-y$ $(-\hat{\mathbf{j}})$ direction, it would be a "bootstrap spaceship".

Shockley and James also computed that the assembled system contains electromagnetic-field momentum,[5]

$$\mathbf{P}_{\rm EM} = \int \mu_0\mathbf{E}\times\mathbf{B}\,d{\rm Vol} = \varepsilon_0\mu_0\,\mathbf{E}\times\mathbf{m} = \frac{\mathbf{E}\times\mathbf{m}}{c^2}\,. \tag{3}$$

It would be a violation of the center-of-energy theorem if the assembled system, whose center of mass/energy is at rest, had nonzero total momentum. Therefore, Shockley and James argued that the assembled system contains a "hidden" mechanical momentum,

$$\mathbf{P}_{\rm hid} = -\mathbf{P}_{\rm EM} = \varepsilon_0\mu_0\mathbf{m}\times\mathbf{E} = \frac{\mathbf{m}\times\mathbf{E}}{c^2}\,. \tag{4}$$

They added that when the magnetic dipole $\mathbf{m}$ dropped to zero, the changing "hidden" mechanical momentum was associated with "hidden momentum (pseudo) forces" equal and opposite to the forces $\mathbf{F}_{\pm Q}$, such the total "force" on (and the total momentum of) the system is zero at all times, and the system remains always at rest.

2.2 Assembly of the System, I

In [43] and [45], the system of Shockley and James was considered during assembly, rather than disassembly, supposing that the magnetic moment $\mathbf{m}$ (called $\boldsymbol{\mu}$ in [43] and $\mathbf{M}$ in [45]) was created when the charges $\pm Q$ were far from the origin, and then the charges were brought in to their assembled positions. It was stated in [43] and [45] that during the assembly, the magnetic moment, and the center of mass of the charges $\pm Q$, would be held fixed by external forces.

The supposition in [43] and [45] that external forces would be needed to hold the system at rest during assembly implies that it would be a "bootstrap spaceship" if left free.

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[4]Radiation would be negligible while the magnetic moment changed, in that the radiation by N evenly spaced charges rotating with velocity v is suppressed by a factor $N(v/c)^{2N}$ compared to the (weak) magnetic dipole radiation when $N = 1$, as noted in [49, 50].

[5]This was first computed on p. 347 of [2] (1904), and also noted just before eq. (3) in [45], where the electric field was called $2\mathbf{E}$ and the magnetic moment was called $\mathbf{M}$. In Gaussian units, $\mathbf{P}_{\rm EM} = (c/4\pi)\int\mathbf{E}\times\mathbf{B}\,d{\rm Vol} = \mathbf{E}\times\mathbf{m}/c$, as in eq. (25) of [43].

2.2.1 Assembly with No External Forces

It is more instructive to consider, along with Shockley and James [13], that the charges $\pm Q$ slide on a rod, and that there are no external forces on the system. Then, there must be mechanical forces between the rod and the charges to bring them in from "infinity", and to hold them in place in their assembled position. These forces are internal, and sum to zero.

Since the rod is connected to the axle of the counter-rotating, oppositely charged disks, there can be equal-and-opposite, internal, mechanical forces between these two objects.

As the charges $\pm Q$ move inwards, their magnetic field exerts a net $\mathbf{I} \times \mathbf{B}$ force in the $+y$ direction on the charges on the counter-rotating disks. In addition, the magnetic field of the magnetic dipole exerts a Lorentz force on the moving charges $\pm Q$, which force is also in the $+y$ direction. That is, the total internal force $\mathbf{F}_{\rm int}$ on the system is nonzero (which never occurs in an all-mechanical system [51]). The total mechanical momentum added to the system by these forces during the assembly was computed in eqs. (5) and (10) of [45] to be,

$$\Delta \mathbf{P}_{\rm mech} = \int \mathbf{F}_{\rm int}\, dt = -\varepsilon_0 \mu_0\, \mathbf{E} \times \mathbf{m} = -\mathbf{P}_{\rm EM}, \qquad (5)$$

in the notation of [13], or $-\varepsilon_0 \mu_0\, 2\mathbf{E} \times \mathbf{M}$ in the notation of [45].

Again, the electromagnetic-field momentum of the assembled system is given by eq. (3), so the sum of eqs. (3) and (5) is zero, *i.e.*, the total momentum of the system is zero.

If the mechanical momentum (5) were "overt", then both the charges $\pm Q$ and the rotating, charged disks would be in motion in the $+y$ direction, and the system would be a "bootstrap spaceship".

This is avoided in that the momentum (5) is not "overt", but rather is "hidden", according to eq. (4). The center of mass of the system remains at rest at all times, without the need for any external forces to accomplish this.

2.3 Assembly of the System, II

In a different assembly scenario, the electric charges $\pm Q$ are brought to their assembled positions while the magnetic moment is still zero, and then the oppositely charged disks are set in (opposite) rotations until the desired magnetic moment $\mathbf{m}$ is achieved. Again, we suppose that there are no external forces on any part of the system during assembly, and that the rod supporting the charges $\pm Q$ is connected to the axle of the oppositely charged, counter-rotating disks,

While the magnetic field of the rotating, charged disks is increasing, an electric field is induced that exerts forces in the $+y$ direction on the charges $\pm Q$. This electric field was computed in eq. (40) of [45] (for $\theta = \pi/2$ and $\phi = 0,\ \pi$). There is no force on the magnetic dipole during the assembly, so the momentum transferred to the system during assembly by the internal forces is,

$$\Delta \mathbf{P}_{\rm mech} = \int \mathbf{F}_{\rm int}\, dt = -\varepsilon_0 \mu_0 \mathbf{E} \times \mathbf{m} = -\mathbf{P}_{\rm EM}, \qquad (6)$$

as computed in eq. (44) of [45], but given in the notation of [13].

While this momentum is equal and opposite to the field momentum, eq. (3), if it were an "overt" mechanical momentum, the system would be a "bootstrap spaceship".

As in the previous assembly scenario, this is avoided by the momentum (6) being a "hidden" momentum, eq. (4).

2.4 Force and Pseudoforce

In general, a system of (relativistic) mass m and center-of-mass velocity $\mathbf{v}_{\rm cm}$ can have both "overt" mechanical momentum $m\mathbf{v}_{\rm cm}$ and "hidden" mechanical momentum $\mathbf{P}_{\rm hid}$, where in many cases the latter is associated with electric currents that are in an electric field (or matter currents in a gravitational field [25]). The total mechanical momentum is $m\,\mathbf{v}_{\rm cm} + \mathbf{P}_{\rm hid}$, and the equation of motion of the system can be written as,

$$\mathbf{F}_{\rm total} = \frac{d\mathbf{P}_{\rm mech,\ total}}{dt} = \frac{d}{dt}(m\,\mathbf{v}_{\rm cm} + \mathbf{P}_{\rm hid}) = m\,\mathbf{a}_{\rm cm} + \frac{d\mathbf{P}_{\rm hid}}{dt}\,, \tag{7}$$

where $\mathbf{a}_{\rm cm} = d\mathbf{v}_{\rm cm}/dt$ is the acceleration of the center of mass of the system. It is suggestive to rewrite this as,

$$m\,\mathbf{a}_{\rm cm} = \mathbf{F}_{\rm total} - \frac{d\mathbf{P}_{\rm hid}}{dt} = \mathbf{F}_{\rm ext} + \mathbf{F}_{\rm int} - \frac{d\mathbf{P}_{\rm hid}}{dt}\,, \tag{8}$$

where the term $-d\mathbf{P}_{\rm hid}/dt$ is not strictly a force in the usual Newtonian sense, but can be called a pseudoforce, as in [13]. The second form of eq. (8) acknowledges that in electromechanical examples the total internal force $\mathbf{F}_{\rm int}$ can be nonzero.

In the assembly scenarios of secs. 2.2-3 above, the external force on the system of charges $\pm Q$ and rotating, charged disks is zero, $\mathbf{F}_{\rm ext} = 0$, while the total internal force was found to obey,

$$\mathbf{F}_{\rm int} = -\frac{d}{dt}\,(\varepsilon_0\mu_0\mathbf{E}\times\mathbf{m}) = -\frac{d\mathbf{P}_{\rm EM}}{dt} = \frac{d\mathbf{P}_{\rm hid}}{dt}\,, \tag{9}$$

recalling eqs. (3)-(6). Thus, from eq. (8), $m\,\mathbf{a}_{\rm cm} = 0$ during these assemblies, and the velocity of the center of mass of the system remains zero at all times.

Both [43] and [45] missed that no external forces are needed in the assembly or disassembly of the system in Shockley's example, by supposing that the pseudoforce $-d\mathbf{P}_{\rm hid}/dt$ does not exist, and that its effect must instead be provided by an external force.

2.4.1 A Variant on Assembly II of the System

It may be of interest to consider a variant of the second assembly scenario (see sec. 2.3 above), in which the rod supporting the charges $\pm Q$ is not connected to the axle of the counter-rotating, oppositely charged disks.

In this case, the mechanical momentum (6) would be transferred only to the charges $\pm Q$ and their supporting rod,

$$\mathbf{P}_{\rm mech,\pm Q} = -\varepsilon_0\mu_0\mathbf{E}\times\mathbf{m} = -\mathbf{P}_{\rm EM}, \tag{10}$$

and associated with this "overt" momentum, the charges plus rod would take on a nonzero velocity in the $+y$ direction. There is no (Newtonian) force on the magnetic dipole, so it might seem that it would remain at rest. But, if this happened, the system would be a "bootstrap spaceship", whose center of mass would be in motion, although initially at rest.

To resolve this paradox, we can apply the argument of eq. (8) to the magnetic dipole rather than to the whole system,

$$m_{\rm dipole}\,\mathbf{a}_{\rm dipole} = \mathbf{F}_{\rm dipole} - \frac{d\mathbf{P}_{\rm hid,dipole}}{dt} = -\frac{d\mathbf{P}_{\rm hid}}{dt}\,, \tag{11}$$

since in this scenario the Newtonian forces on the magnetic dipole are zero, while the "hidden" momentum of the system is entirely in the electric currents of the magnetic dipole. Then, the time integral of eq. (11) yields,

$$m_{\rm dipole}\,\mathbf{v}_{\rm dipole} = \mathbf{P}_{\rm mech,dipole} = -\mathbf{P}_{\rm hid} = \mathbf{P}_{\rm EM}, \tag{12}$$

where we use a result from sec. 4.1.4 of [39] that the "hidden" mechanical momentum and the electromagnetic-field momentum are equal and opposite for quasistatic systems even if they are not at rest. The total "overt" momentum of this system is zero,

$$\mathbf{P}_{\rm overt} = \mathbf{P}_{\rm mech,\pm Q} + m_{\rm dipole}\,\mathbf{v}_{\rm dipole} = -\mathbf{P}_{\rm EM} + \mathbf{P}_{\rm EM} = 0, \tag{13}$$

and the total momentum of the system is also zero,

$$\mathbf{P}_{\rm total} = \mathbf{P}_{\rm overt} + \mathbf{P}_{\rm EM} + \mathbf{P}_{\rm hid} = 0, \tag{14}$$

as expected for an isolated system that initially had zero momentum.

2.4.2 Conducting Magnet

If the magnetic dipole were a (perfectly) conducting magnet, it would have no electromagnetic-field momentum when in an electric field, as first noted in [17]. This system would also have no "hidden" mechanical momentum. Furthermore, during assembly of this system subject to no external forces, there would be no net internal force on the system of the form in eqs. (5) and (6), as hold for a nonconducting magnet made from counter-rotating, oppositely charged disks.[6] These effects are the result of the "shielding" of the interior of a (perfect) conductor from any electric field. Then, from eq. (8) we also have that $m\mathbf{a}_{\rm cm} = 0$ during the assembly (or disassembly) of a conducting magnetic dipole in an electric field that was initially at rest.

3 A Variant of Shockley's Example

A small variant of Shockley's example was considered in sec. 5 of [45], where the electric field was provided by a shell with electric charge density proportional to $\cos\theta_y$, yielding a uniform electric field in the y direction inside the shell, which also contained the magnetic

[6]There is an electromagnetic force on the conducting magnetic dipole, $\mathbf{F}_{\rm int,\mathbf{m}} = -\varepsilon_0\mu_0\, d(\mathbf{E}\times\mathbf{m})/dt$, as computed in [19], contrary to a claim in [44]. This is also an equal and opposite force $\mathbf{F}_{{\rm int},Q_i}$ on the charges Q_i that generate the electric field $\mathbf{E}$. Here, the quantity $\varepsilon_0\mu_0\mathbf{E}\times\mathbf{m}$ does not have the significance of an electromagnetic-field momentum, or the negative of a "hidden" mechanical momentum, as both of these latter quantities are zero in the case of a (perfectly) conducting magnet.

dipole **m**. The only assembly scenario considered was that in which the magnetic dipole was initially zero, and was created by spinning up a pair of oppositely charged disks with opposite rotations. This scenario is equivalent to that discussed in sec. 2.3 above (or that of sec. 2.4.1 if the shell and magnetic dipole were not connected to one another), where the system was found to develop no net "overt" momentum, while containing equal-and-opposite electromagnetic-field momentum and "hidden" mechanical momentum, and the total momentum remained zero at all times.

4 Summary

We have shown that the existence of "hidden" mechanical momentum leads to momentum conservation during the assemblage of the systems considered in [45], contrary to what was claimed there. We also discussed that the nonexistence of "hidden" momentum would violate the center-of-energy theorem in some cases (and would imply the existence of "bootstrap spaceships"). We conclude that the existence of "hidden" mechanical momentum is necessary for a consistent treatment of these, and other, electromechanical examples in which electric currents with nonzero magnetic moments are subject to electric fields.

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