# Hydrogen-Atom Wavefunctions as Radiationless Modes

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## 1 Problem

Show that if the quantum probability current for hydrogen-atom wavefunctions is interpreted as an electric current, the implication is that this current does not emit radiation (*i.e.*, is a so-called radiationless mode).

## 2 Solution

After the discovery of the electron by J.J. Thomson [1], it was natural to consider that electrons are constituents of atoms, but if electrons in an atom are small electric charges moving in circular orbits, they would emit electromagnetic radiation and the radius of the orbit would quickly collapse to zero.[1] In 1903, Thomson pointed out [3, 4] that a uniform ring of electric charge, rotating about its symmetry axis at constant angular velocity, does not emit electromagnetic radiation although the moving charge is subject to a centripetal acceleration. This was the first discovery of a "classical radiationless mode".[2] Thomson was inspired by this result to propose his "plum pudding" model of an atom in which the positive electric charge was spread out in a continuous fashion, perhaps around rings of negative electrons.

Little survives of this model today, except that the quantum probability current,

$$\mathbf{J}_{\rm prob} = \frac{i\hbar}{2M}(\Psi \boldsymbol{\nabla} \Psi^\star - \Psi^\star \boldsymbol{\nabla} \Psi), \tag{1}$$

of the wavefunction of an electron of mass $M$ in, say, a hydrogen atom flows in rings, such that if $e\mathbf{J}_{\rm prob}$ is interpreted as a classical electric current,[3] this current emits no radiation.[4] That, is the hydrogen-atom wavefunctions can be mapped onto classical radiationless states.

[1]See, for example, [2].

[2]It was soon noted, most crisply by Ehrenfest [5], that a uniform spherical shell of electric charge can have arbitrary radial acceleration without emitting any radiation. See also, [6].

[3]The notion of the probability current (1) has its origins in Schrödinger's interpretation of $e\,|\Psi|^2$ as electric charge density, and $e\mathbf{J}_{\rm prob}$ as the electric current density. See sec. 7 of [7], which slightly predates Born's interpretation [8] of $|\Psi|^2$ as a probability density, based on considerations of scattering rather than of the character of atomic states (where Schrödinger's interpretation has considerable appeal).

[4]This conclusion appears on pp. 138-139 of [7], which has been translated as:

> To given an example in the conservative one-electron problem ... the current density ... generally is a stationary current distribution. ... We may in a certain sense speak of a return to electrostatic and magnetostatic models. In this way the lack of radiation in the normal state would, indeed, find a startlingly simple explanation.

To see this, note that a hydrogen-atom wavefunction can be written in a spherical coordinate system $(r, \theta, \phi)$ centered on the proton as[5]

$$\Psi_{nlm} = R_{nl}(r) P_l^m(\cos\theta)\, e^{im\phi}, \tag{2}$$

where the associated Legendre polynomial $P_l^m$ can be written as

$$P_l^m(\cos\theta) = \sin^{|m|}\theta \left(\frac{d}{d\cos\theta}\right)^{|m|} P_l(\cos\theta), \tag{3}$$

and $P_l$ is a Legendre polynomial. Then, since the gradient operator in spherical coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\,, \tag{4}$$

the probability current associated with the wavefunction $\Psi_{nlm}$ is

$$\mathbf{J}_{\mathrm{prob},nlm} = \frac{m\hbar\,|\Psi_{nlm}|^2}{Mr\sin\theta}\,\hat{\boldsymbol{\phi}}. \tag{5}$$

This is well behaved for all $\theta$, since $|\Psi_{nlm}|^2 \propto \sin^{2|m|}\theta$ according to eqs. (2)-(3). The probability current $\mathbf{J}_{\mathrm{prob},nlm}$ is purely azimuthal, time-independent, and nonzero for nonzero (integer) index $m$ (which labels the $z$-component of the orbital angular momentum of the atomic electron). So, if we interpret $e\mathbf{J}_{\mathrm{prob},nlm}$ as the electric current density associated with the atomic electric, this is a superposition of steady ring currents, which do not emit electromagnetic radiation.

The spread-out character of the quantum electron in an atom implies that the wavefunctions are radiationless states for all values of indices $n$, $m$, and $l$ (and not just for the ground state $\Psi_{100}$). A classical atom with current density $e\mathbf{J}_{\mathrm{prob},nlm}$ would not radiate/decay, and there would be no classical explanation for the observed spectral lines of excited atomic states.

## 3 Comments

There is a long history of attempts to explain what are now considered to be "quantum" concepts by "classical" notions. The absence of radiation by the ground state of atoms is one of the phenomena that has attracted such attention. Prior to the initial development of a quantum view of atoms by Bohr [11], various attempts besides that of Thomson [3] were made to accommodate radiationless modes into the "classical" electron theory of small, charged spherical shells [12, 13, 14, 15, 16, 17, 18, 19]. Even after the success of the "new" quantum theory of the 1920's in explaining atomic states as stationary radiationless modes with a semiclassical interpretation via the probability current density (or Schrödinger's electric current density), people have continued to seek purely classical models of electrons that can have radiationless orbital motion [20, 21, 22, 23, 24, 25].[6]

[5] The essence of eq. (2) appears in Schrödinger's first paper on wave mechanics [9].

[6] An "outsider science" effort, http://brilliantlightpower.com/wp-content/uploads/theory/Volume1.djvu, presents in overwhelming detail a theory in which it is claimed the wavefunctions (2) must have radial

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functions that are delta-functions, corresponding to superpositions of current rings of a single radius that somehow interpenetrate, to be radiationless. From this misunderstanding come claims of hydrogen-atom states with binding energy greater than a Rydberg.

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