

Alternative Forms of the Poynting Vector

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1 Introduction

A prominent feature of Maxwell’s vision of *A Dynamical Theory of the Electromagnetic Field* [7] is that the electromagnetic field contains energy, whose volume density u is, in Gaussian units,

$$u = \frac{\mathbf{E} \cdot \mathbf{D}}{8\pi} + \frac{\mathbf{B} \cdot \mathbf{H}}{8\pi}, \quad (1)$$

for linear media where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, with ϵ and μ being the (relative) permittivity and permeability of the medium, and c is the speed of light in vacuum.^{1,2,3}

Subsequently, Poynting (1883) [13] promoted the vision that the flux of energy in the electromagnetic field is described by the vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} \quad (\text{Poynting}), \quad \text{where} \quad \mathbf{E} \cdot \mathbf{J} = -\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{S} \quad (2)$$

is the rate of work done by electromagnetic fields on electric current density \mathbf{J} .

Perhaps because the implications of eqs. (1)-(2) are sometimes counterintuitive,^{4,5} there has been ongoing doubt as to the physical interpretation of these relations.

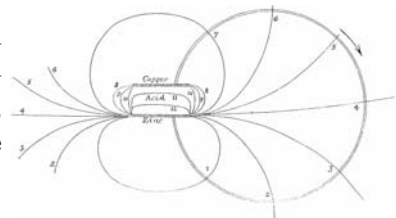
¹In sec. 74 of [7], Maxwell stated: *In speaking of the Energy of the field ... I wish to be understood literally ... The only question is, Where does it reside? On the old theories it resides in the electrified bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance. On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves.*

In Art. 631 of his *Treatise* [9], Maxwell wrote: *Hence, the electrostatic (potential) energy of the whole field will be the same if we suppose that it resides in every part of the field where electrical force (\mathbf{E}) and electrical displacement (\mathbf{D}) occur, instead of being confined to places where free electricity is found. In (vector) language (and in Gaussian units) it is $\mathbf{E} \cdot \mathbf{D}/8\pi$. And in Art. 636 he wrote: *According to our hypothesis, we assume the kinetic energy to exist wherever there is magnetic force, that is, in general, in every part of the field. The amount of this energy per unit of volume is $\mathbf{B} \cdot \mathbf{H}/8\pi$, and this energy exists in some form of motion in every portion of space.**

²That the magnetic-field-energy density can be written as $\mathbf{B} \cdot \mathbf{H}/8\pi$ was first deduced by Maxwell in (1856), p. 63 of [5], via a transformation of the magnetic potential energy. Yet, Maxwell called the transformed energy “kinetic”.

³Maxwell’s comments about electromagnetic field energy contrast with his characterization of “mechanical” potential energy in Art. 97 of [11], where he stated: *Rankine [4] introduced the term Potential Energy—a very felicitous expression, since it not only signifies the energy which the system has not in actual possession, but only has the power to acquire.*

⁴One of the first examples of counterintuitive behavior was given by Poynting in Fig. 3 of [13], that the flow of energy from a battery to a resistive wire loop connected to it is not through the wire, but across the “empty” space between the battery and segments of the wire. See also, for example, [83].



⁵Another example was given by Heaviside on p. 94 of [31], where he considered a uniformly magnetized

An early expression of such doubt was by Heaviside (1887) [19], despite his having independently deduced (1884) that the vector \mathbf{S} of eq. (2) can be considered to represent the flow of energy in the electromagnetic field [15, 16, 17]. Heaviside’s comment, p. 93 of [31], was based on an awareness that his and Poynting’s derivation of eq. (2) actually only determine $\nabla \cdot \mathbf{S}$:⁶ *If \mathbf{S} be the vector energy-current density, we may add to it another vector, say \mathbf{s} , provided \mathbf{s} have no convergence anywhere (i.e., $\nabla \cdot \mathbf{s} = 0$). The existence of \mathbf{s} is possible, but there does not appear to be any present means of finding out whether it is real, and how it is to be expressed.*

Among later doubters, the most notable is Feynman.^{7,8}

In the rest of this note we review the alternatives to the Poynting vector that have been proposed, after some comments (sec. 2) about electromagnetic field momentum and angular momentum.

2 Electromagnetic Momentum and Angular Momentum

2.1 Field Momentum

Maxwell enunciated a conception of electromagnetic momentum in sec. 57 of [7] as,

$$\mathbf{P}_{\text{EM}}^{(\text{Maxwell})} = \int \frac{\varrho \mathbf{A}^{(C)}}{c} d\text{Vol}, \quad (3)$$

where ϱ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [100]). Maxwell

sphere with a uniform electric surface charge. Here, lines of the Poynting vector \mathbf{S} (Heaviside’s \mathbf{W}) flow in circles outside the sphere, about its magnetic axis. As Heaviside remarked: *What is the use of it? On the other hand, what harm does it do?* See Feynman’s answer in sec. 3.9.7 below.

⁶Heaviside credited J.J. Thomson for pointing this out. See sec. 280, p. 313 of [29].

⁷Feynman stated, sec. 27-4 of [61]: *All we did was to find a possible “u” and a possible “S”. How do we know that by juggling the terms around some more we couldn’t find another formula for “u” and another formula for “S”? The new \mathbf{S} and the new u would be different, but they would still satisfy (the second of eq. 2 of this paper). It’s possible. It can be done, but the forms that have been found always involve various derivatives of the field (and always with second-order terms like a second derivative or the square of a first derivative). There are, in fact, an infinite number of different possibilities for u and \mathbf{S} , and so far no one has thought of an experimental way to tell which one is right! People have guessed that the simplest one is probably the correct one, but we must say that we do not know for certain what is the actual location in space of the electromagnetic field energy. So we too will take the easy way out and say that the field energy is given by eq. (1). Then the flow vector \mathbf{S} must be given by eq. (2).*

⁸This note concerns the notion of flow of field energy in classical electromagnetism. In quantum theory one can also consider the flow of field energy, but here one is led to speak of the probability amplitude and probability density of the flow. For example, in a classical analysis of Young’s double-slit experiment, lines of the Poynting vector (2) go through one slit or the other, whereas in the quantum view a single photon has a nonzero amplitude to go through both slits (unless it is observed at one of the slits).

Although the quantum Poynting vector is subject to “uncertainty” in its interpretation, this has not led to suggestions of alternative forms for it.

regarded the vector potential \mathbf{A} at the location of an electric charge q as providing a measure, $q\mathbf{A}/c$, of electromagnetic momentum and an interpretation of Faraday's electrotonic state (secs. 60-61 of [2]). That Faraday associated with some kind of momentum with this state is hinted in sec. 1077 of [3].

However, the form (3) seems to associate the momentum with charges rather than with fields.

In 1888, Heaviside, p. 330 of [23], inferred from the Maxwell stress tensor that the volume force density could be written as,⁹

$$\mathbf{f} = \frac{d}{dt} \frac{\mathbf{S}}{c^2} = \frac{d}{dt} \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} \quad (\text{Heaviside}), \quad (4)$$

without explicitly identifying $\mathbf{S}/c^2 = \mathbf{E} \times \mathbf{H}/4\pi c$ as the field-momentum density.¹⁰

In 1891, Thomson noted [24] that a sheet of electric displacement \mathbf{D} (parallel to the surface) which moves perpendicular to its surface with velocity \mathbf{v} must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\nabla \times \mathbf{H} = (1/c) \partial \mathbf{D} / \partial t$.¹¹ Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [24],

$$\mathbf{p}_{\text{EM}}^{(\text{Thomson})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}. \quad (5)$$

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [29], adding the remark (p. 9) that the momentum density (5) is closely related to the Poynting vector [13, 16],^{12,13}

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}. \quad (6)$$

The form (5) was also used by Poincaré in 1900 [36], following Lorentz' convention [26] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector be $(c/4\pi) \mathbf{D} \times \mathbf{H}$. In 1903 Abraham [38] argued for

$$\mathbf{p}_{\text{EM}}^{(\text{Abraham})} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}, \quad (7)$$

⁹The result (4) was anticipated by Heaviside in eqs. (6a) and (7a) of [18] (1886).

¹⁰In the same paper [23], Heaviside gave the first correct derivation of the electromagnetic fields of a uniformly moving charge with any velocity $v < c$.

¹¹Variants of this argument were given by Heaviside in 1891, sec. 45 of [25], and much later in sec. 18-4 of [61], where it is noted that Faraday's law, $\nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B} / \partial t$, combined with the Maxwell equation for \mathbf{H} implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted in sec. 265 of [33].

¹²The idea that an energy-flux vector is the product of energy density and energy-flow velocity seems to be due to Umov [10], based on Euler's continuity equation [1] for mass flow, $\nabla \cdot (\rho \mathbf{v}) = -\partial \rho / \partial t$.

¹³Thomson argued, in effect, that the field-momentum density (5) is related by $\mathbf{p}_{\text{EM}} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [24, 29]. See also eq. (19), p. 79 of [25], and p. 6 of [42]. It turns out that the energy-flow velocity defined by $\mathbf{v} = \mathbf{S}/u$ can exceed c (see, for example, sec. 2.1.4 of [105] and sec. 4.3 of [108]).

and in 1908 Minkowski [40] advocated the form^{14,15}

$$\mathbf{p}_{\text{EM}}^{(\text{Minkowski})} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}. \quad (8)$$

The forms (5)-(8) all involve either $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ or $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, where \mathbf{P} and \mathbf{M} are the densities of electric and magnetic dipoles due to electric charges and currents. As such, they involve momentum of the “mechanical” dipoles as well as the of the electromagnetic fields \mathbf{E} and \mathbf{B} , which has led to extensive debate as to the physical interpretation of these various forms [116]. The author’s view that it is often best to avoid this debate, and restrict discussion of “field” momentum to the “electromagnetic-field-only” momentum,

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}. \quad (9)$$

The integral of the electromagnetic-field-momentum density (9) can be written for static fields in four equivalent forms,

$$\mathbf{P}_{\text{EM}} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \frac{V \mathbf{J}}{c^2} d\text{Vol} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} \quad (\text{statics}), \quad (10)$$

where ρ is the (total) electric charge density, $\mathbf{A} = \int \mathbf{J} d\text{Vol}/cr$ is the magnetic vector potential (where $\nabla \cdot \mathbf{A} = 0$ for static fields in both the Coulomb and Lorenz gauges), \mathbf{J} is the electric current density due to electric charges, $\mathbf{E} = \int \rho \hat{\mathbf{r}} d\text{Vol}/r^2$ is the electric field, $\mathbf{B} = \int \mathbf{J} \times \hat{\mathbf{r}} d\text{Vol}/cr^2$ is the magnetic field, and $V = \int \rho d\text{Vol}/r$ is the electric scalar potential.¹⁶ The third form was introduced by Furry [69],¹⁷ and the fourth form is due to Aharonov *et al.* [92].

2.2 Field Angular Momentum

In retrospect, one can recognize that electromagnetic field momentum was first considered Darboux (1878) [12] as a vector constant of the motion in the interaction of an electric charge with a (hypothetical) magnetic pole. This constant vector was further considered by Poincaré in 1896 [35], but its relation to electromagnetic field momentum was only realized much later. For a review, see [118].

The concept of electromagnetic field momentum was first explicitly mentioned in 1904 by J.J. Thomson [39], who considered an electric charge outside a small solenoid magnet, and then an electric charge q plus a single magnetic pole p . For the latter case, he computed the field angular momentum as,¹⁸

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \mathbf{p}_{\text{EM}} d\text{Vol} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \frac{pq}{c} \hat{\mathbf{z}}, \quad (11)$$

¹⁴Minkowski, like Poynting [13], Heaviside [16] and Abraham [38], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [40]. Heaviside wrote the momentum density in the Minkowski form (8) on p. 108 of [25].

¹⁵For some remarks on the “perpetual” Abraham-Minkowski debate see [111].

¹⁶For a review, see [102]. For discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence, see, for example [109].

¹⁷The density $V \mathbf{J}/c^2$ is the static limit of an expression for the field-momentum density proposed by Livens (1926, sec. 3.3 below), bottom of p. 263 of [46]. This density is also the static limit of \mathbf{S}/c^2 for several of the alternative forms of the Poynting vector proposed by Slepian (1942, sec. 3.5 below) [55].

¹⁸See also sec. 2.2 of [114].

independent of the distance between p and q , where $\hat{\mathbf{z}}$ points from the electric charge q to the magnetic pole p .

The vector (11) is the constant vector considered by Darboux and Poincaré, but this went unnoticed for many years.

In 1904 the notion of quantizing angular momentum was still years away, and the provocative result (11), that the angular momentum of a magnetic pole plus electric charge is independent of their separation, went unremarked until 1931 when Dirac [49] argued that $pq/c = \hbar/2$. See also sec. 6.12 of [99].

3 Some History of Poynting-Vector Alternatives

3.1 Birkeland

In 1894, Birkeland [32] argued (in German) that if the Poynting vector is to be a function only of the electromagnetic fields, and not of their derivatives, then the only possible form is the original version of Poynting, our eq. (2).

A version of Birkeland’s argument has been given in English at the end of [78].

3.2 Macdonald

The earliest definite proposal for an alternative to the Poynting vector seems to be that by Macdonald (1902), p. 72 of [37],

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} + \mathbf{s}, \quad \text{with} \quad \mathbf{s} = \frac{1}{8\pi} \frac{\partial}{\partial t} (\mathbf{A} \times \mathbf{H}) \quad (\text{Macdonald}). \quad (12)$$

Here, $\nabla \cdot \mathbf{s} \neq 0$, but Macdonald also proposed an alternative form for the energy density u such that the second of eq. (2) is still satisfied.

Macdonald’s expressions for \mathbf{S} and u are not gauge invariant,¹⁹ so this author considers them not to be valid for a physical description of Nature.^{20,21}

¹⁹Maxwell was aware of the arbitrariness of the electromagnetic potentials Ψ and \mathbf{A} , sec. 98, pp. 499-500 of [7]. He understood that his preference for $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge) is a choice, not a physical requirement, and that one can always enforce this condition by what is now called a gauge transformation, eqs. (74) and (77) of [7] on rewriting φ as Ψ' .

²⁰Of course, with a particular choice of gauge, valid computations can be made using the electromagnetic potentials.

²¹One of the first to question the physical significance of the electromagnetic potentials was Heaviside (1888) in the *Postscript* to [21], pp. 47-50, titled *On the Metaphysical Nature of the Propagation of the Potentials*. On p. 47, he wrote: *We make acquaintance, experimentally, not with potentials, but with forces, and we formulate observed facts with the least amount of hypothesis, in terms of the electric force \mathbf{E} and magnetic force \mathbf{H} . In Maxwell’s development of Faraday’s views, \mathbf{E} and \mathbf{H} actually represent the state of the medium anywhere.*

Such doubt was a topic of debate at the 1888 Bath Meeting of the British Association, where FitzGerald spoke of the “murder of Ψ ”, p. 624 of [20]. See also [22] and chap. 7 of [95].

Heaviside indicated an awareness as to the arbitrariness of the electromagnetic potentials Ψ and \mathbf{A} on p. 48 of [21], where he stated: *Thus we have Ψ , \mathbf{A} , and $\dot{\mathbf{A}}$ required, involving seven scalar specifications to find the six in \mathbf{E} and \mathbf{H} .* However, he seems to have followed Maxwell in taking the electromagnetic potentials to

3.2.1 Schott

Macdonald's alternatives were noted by Schott (1912) on pp. 5-6 of [41], who thereafter used Poynting's standard form.

3.2.2 Tralli

Macdonald's vector (12) was presented without attribution in sec. 9-13 of [60] (1963).

3.3 Livens

In 1917, Livens discussed Poynting's derivation of eq. (2) in sec. 2 of [43], and then in sec. 3 he reviewed Macdonald's derivation of eq. (12). In sec. 4, he proposed a different alternative to the Poynting vector,

$$\mathbf{S} = V\mathbf{J}_{\text{total}} \quad (\text{Livens}), \quad (13)$$

where V is the electrical scalar potential, and $\mathbf{J}_{\text{total}} = \mathbf{J} + \partial\mathbf{D}/\partial t$ is the total current density as first considered by Maxwell in eq. (112), p. 19, of [6]. Like Maxwell, Livens tacitly used the Coulomb gauge.

The form (13) has the appeal that in case of a steady conduction current the flow of energy follows the flow of electric current, in contrast to Poynting's vision (footnote 4 above).

Livens also made this argument in secs. 627-628, pp. 555-556 of his textbook [44], and in sec. 229, pp. 242-244 of the 2nd edition [46].

Livens briefly mentioned electromagnetic field momentum in the Abraham form (7) on p. 598 of [44], and gave greater discussion in secs. 239-242, pp. 258-264 of [46]. In particular, he proposed an alternative to electromagnetic-field-momentum density on p. 263 of [46] as,

$$\mathbf{p}_{\text{EM}} = \frac{V\mathbf{J}_{\text{total}}}{c^2} + \left(-\frac{1}{c} \frac{\partial\mathbf{A}}{\partial t} \right) \times \frac{\mathbf{B}}{4\pi c} \quad (\text{Livens}). \quad (14)$$

Livens' forms are subject to the general objection that physical quantities must be gauge invariant.

It is, however, noteworthy that in case of static fields (where $\mathbf{J}_{\text{total}} = \mathbf{J}$), Livens' form (14) for the field-momentum density reduces to the form of eq. (10), as later advocated by Furry [69].

3.3.1 Carpenter

Livens' vector (13) was advocated by Carpenter [96] in 1989.

be in the Coulomb gauge. At the top of p. 49 of [21], Heaviside noted what Maxwell had missed regarding the Coulomb gauge: *Thus we have instantaneous propagation of Ψ to infinity.* Then he added: *I prefer, however, to say that this is only a mathematical fiction.*

3.4 Bateman

In 1922, Bateman [45] proposed a variant of the Poynting vector as part of an effort to explain the quantum behavior of atoms via classical electromagnetism,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} - \frac{\psi}{2\pi} \frac{\partial}{\partial t} \nabla \psi, \quad \text{where} \quad \psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{\rho(\mathbf{r}', t')}{\gamma' |\mathbf{r} - \mathbf{r}'|} d\text{Vol}' \quad (\text{Bateman}), \quad (15)$$

$t' = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time, and $\gamma' = 1/\sqrt{1 - v^2(\mathbf{r}', t')/c^2}$ is the Lorentz factor of the moving source charges at the retarded time.

This peculiar suggestion depends on a potential ψ , and is not gauge invariant.

3.5 Slepian

In 1942, Slepian (who had a delightful sense of humor²²) discussed Poynting's theorem [54] and offered 8 alternatives to standard Poynting vector [55]. One of these, eq. (13) of [55], involves the magnetization density \mathbf{M} which is not typically considered explicitly in the context of the Poynting vector, and the other seven involve the electromagnetic potentials as well as the fields.²³

Slepian's third form, eq. (19) of [55] is,²⁴

$$\mathbf{S}_3 = V \mathbf{J}_{\text{total}} + \frac{c}{4\pi} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \times \mathbf{H} \quad (\text{Slepian}), \quad (17)$$

where the electric current density $\mathbf{J}_{\text{total}}$ includes the displacement current (density) $\partial \mathbf{D}/\partial t$.²⁵

Slepian remarked after his eq. (19): *The first term $V \mathbf{J}_{\text{total}}$ is the formula for energy flow used extensively by electrical engineers of power systems, and where displacement currents are only a small part of the total current, may be determined directly by wattmeters.*^{26,27}

The seven alternatives of Slepian which involve the electromagnetic potentials are unphysical, in that they are not gauge invariant.

²²For an example of Slepian's wry humor regarding the Poynting vector, see [56, 57, 106].

²³The variant involving the magnetization \mathbf{M} is a precursor to considerations by the author [113] that the standard Poynting vector can be rewritten in 729 different ways using different combinations of auxiliary fields. These variants ultimately contain the same physics as the standard Poynting vector, but the notation used makes them appear to be very different in many cases.

²⁴

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} = \left(-\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \times \frac{c\mathbf{H}}{4\pi} = V \nabla \times \frac{c\mathbf{H}}{4\pi} - \nabla \times V \frac{c\mathbf{H}}{4\pi} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \times \frac{c\mathbf{H}}{4\pi} \\ &= V \mathbf{J}_{\text{total}} + \frac{c}{4\pi} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \times \mathbf{H} - \nabla \times V \frac{c\mathbf{H}}{4\pi} = \mathbf{S}_3 - \nabla \times \frac{cV\mathbf{H}}{4\pi}, \end{aligned} \quad (16)$$

such that \mathbf{S} and \mathbf{S}_3 differ only by the vector $\mathbf{s} = \nabla \times cV\mathbf{H}/4\pi$, whose divergence is zero.

²⁵Although this differs from Livens' form (13), note that \mathbf{S}_3/c^2 is the same as Livens' expression (14) for the field-momentum density \mathbf{p}_{EM} , if Livens' \mathbf{B} were changed to \mathbf{H} .

²⁶For more extended comments by Slepian on the use of $V \mathbf{J}_{\text{total}}$, see [54].

²⁷In the electric-power industry there is interest in scalar measures of electric power, such as the product VI of voltage and current. The merits of a vector measure of energy flow, such as the standard Poynting vector \mathbf{S} , or even the alternative $V \mathbf{J}_{\text{total}}$, are not always obvious. Some of the ongoing debate on this theme is at [101, 104, 107, 115].

3.5.1 Carter

Slepian's third form, eq. (19) of [55], was called the "Slepian vector" on p. 321 of [66] (and was later reinvented by Lai [74], sec. 3.8 below).

3.6 Hines

In 1951, Hines [58] reviewed the alternatives of Macdonald, eq. (12) and Livens, eq. (13), and then proposed the hybrid variant,

$$\mathbf{S} = \frac{1}{8\pi} \left(\mathbf{A} \times \frac{\partial \mathbf{H}}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \right) + V \mathbf{J}_{\text{total}} \quad (\text{Hines}), \quad (18)$$

which differs slightly from Slepian's sixth form, eq. (26) of [55].

In 1979, Wallace and McConnell [72] argued that use of Hines' form (18) does not lead to the correct rate of radiation by an accelerated, nonrelativistic charge. But, see also [81, 87].

3.7 Butler

In 1969, Butler [70] argued that the momentum density $\mathbf{p} = \mathbf{S}/c^2$ should be part of an energy-momentum-density 4-vector, which (he claimed) could be arranged with the form

$$\mathbf{S} = \frac{E^2 - B^2}{8\pi} \gamma^2 \mathbf{v}, \quad (\text{Butler}), \quad (19)$$

where \mathbf{v} is the velocity of the electric charge in a Universe that contains only a single charge, and $\gamma = 1/\sqrt{1 - v^2/c^2}$. He did not explain what happens in a Universe with more than one charge.

3.8 Lai

In 1980, Lai [74] proposed a variant on the form (18) of Hines,

$$\mathbf{S} = -\frac{1}{4\pi} \frac{\partial A}{\partial t} \times \mathbf{H} + V \mathbf{J}_{\text{total}} \quad (\text{Lai}). \quad (20)$$

3.8.1 Kobe

In 1981, Kobe [79] pointed out that Lai's form (20) is the same as Slepian's third form, eq. (19) of [55], our eq. (17), as well as objecting to all forms that are not gauge invariant.

3.8.2 Peters

In 1981, Peters [77] also objected to Lai's form (20) on the grounds that it is not gauge invariant, and added a second objection that, with this form, \mathbf{S}/c^2 is not a reasonable expression for the field-momentum density.

3.8.3 Romer

In 1981, Romer [78] seconded the objections by Kobe and Peters to Lai's form (20), and added a comment about Birkeland's analysis (mentioned in sec. 3.1 above).

3.8.4 Puthoff

Lai's form (20) was advocated by Puthoff (2010) [110].

3.9 Other Commentaries

3.9.1 Jeans

In the 5th edition of his text, p. 519 of [47], Jeans (1927) added cautionary remarks about the Poynting vector similar to those of Heaviside (p. 93 of [31]): *The integral of the Poynting Flux over a closed surface gives the total flow of energy into or out of a surface, but it has not been proved, and we are not entitled to assume, that there is an actual flow of energy at every point equal to the Poynting Flux. For instance if an electrified sphere is placed near to a bar magnet, this latter assumption would require a perpetual flow of energy at every point in the field except the special points at which the electric and magnetic lines of force are tangential to one another. It is difficult to believe that this predicted circulation of energy can have any physical reality. On the other hand it is to be noticed that such a circulation of energy is almost meaningless. The circulation of a fluid is a definite conception because it is possible to identify the different particles of a fluid; we can say for instance whether or not the particles entering a small element of volume are identical or not with an equal number of particles coming out, but the same is not true of energy.*

3.9.2 Mason and Weaver

Mason and Weaver (1929), sec. 54 of [48], reviewed Poynting's theorem, and on p. 268 remarked that its integral form *suggests the interpretation given, rather than demands it. This is, of course, recognized by every careful reader of electrodynamics.*

3.9.3 Sumpner

In 1934, Sumpner, an enthusiast of Heaviside [50], published an article [51] which cast doubt on all the work of Poynting because an argument in Poynting's paper [14] supposedly implied that the speed of the flow of electromagnetic energy is $\approx 10^8$ times the speed of light.

See also sec. 3.3 of [59].

3.9.4 O'Rahilly

Although he offered no alternatives to the Poynting vector, O'Rahilly (1938) gave, on pp. 275-323 of [52], a very extensive criticism of perceived inconsistencies in the use and interpretation of the Poynting vector, and of the electromagnetic-field energy and momentum densities.

3.9.5 Stratton

In sec. 2.19 of [53], Stratton (1941) noted: *The classical interpretation of Poynting's theorem appears to rest to a considerable degree on hypothesis. Various alternative forms of the theorem have been offered from time to time, but none of these has the advantage of greater plausibility or greater simplicity to recommend it, and it is significant that thus far no other interpretation has contributed anything of value to the theory. The hypothesis of an energy density in the electromagnetic field and a flow of intensity $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ has, on the other hand, proved extraordinarily fruitful. A theory is not an absolute truth but a self-consistent analytical formulation of the relations governing a group of natural phenomena. By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision.*

3.9.6 Hammond

In 1958, Hammond [59] reviewed the physics of the Poynting vector and commented on several proposed alternatives.²⁸

3.9.7 Feynman

In addition to the comments on the Poynting vector mentioned in sec. 1 above, Feynman posed the now-famous disk paradox related to field angular momentum in sec. 17-4 of [61]. This paradox was perhaps inspired by a comment of J.J. Thomson, p. 348 of [39], and has led to extensive additional commentary, including [63, 64, 65, 73, 76, 80, 82, 84, 85, 86, 88, 89, 90, 91, 93, 94, 97, 98].

At the end of sec. 27-6, Feynman gave a verbal resolution of the paradox: *Do you remember the paradox we described in Section 17-4 about a solenoid and some charges mounted on a disc? It seemed that when the current turned off, the whole disc should start to turn. The puzzle was: Where did the angular momentum come from? The answer is that if you have a magnetic field and some charges, there will be some angular momentum in the field. It must have been put there when the field was built up. When the field is turned off, the angular momentum is given back. So the disc in the paradox would start rotating. This mystic circulating flow of energy, which at first seemed so ridiculous, is absolutely necessary. There is really a momentum flow. It is needed to maintain the conservation of angular momentum in the whole world.*

3.9.8 Romer

In 1966, Romer [63, 65] commented on field angular momentum in another paradox [62, 103].

²⁸Hammond seems to have been misled by Slepian's subtle humor into supposing, sec. 3.5 of [59], that the latter favored an alternative form involving the electromagnetic potentials.

3.9.9 Shockley

In 1968, Shockley [68] argued that certain thought experiments involving pulsed electric currents “prove” that the standard form (2) of the Poynting vector is the only valid one.²⁹

3.9.10 Lorrain

In 1982, Lorrain [75] made a brief comment, in reference to [74], that some examples of static electric and magnetic fields contain angular momentum, and that a valid expression for the energy-flux vector \mathbf{S} should be consistent with the density of field angular momentum being,

$$\mathbf{l}_{\text{EM}} = \mathbf{r} \times \mathbf{p}_{\text{EM}} = \mathbf{r} \times \frac{\mathbf{S}}{c^2}. \quad (21)$$

This criterion would exclude all alternatives in secs. 4-8 above, if we suppose that the vector potential is independent of time for static electric and magnetic fields.³⁰

3.9.11 Backhaus and Schäfer

In 1984, Backhaus and Schäfer [87] noted that some arguments against alternatives to the Poynting vector are not as decisive as claimed, without commenting on the key issue of gauge invariance.

3.9.12 Bossavit

A recent (2018) paper by Bossavit [119] argued that with use of differential forms one can “prove” the uniqueness of the standard version of the Poynting vector.

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²⁹Shockley is also notable for popularizing the term “hidden momentum” [67], inspired by examples (first given in [39]) in which the static electromagnetic fields of a system “at rest” have nonzero net field momentum. As alternative forms of the Poynting vector have played little role in discussions of “hidden momentum,” we do not consider it further here. A review by the author on this topic is at [112].

³⁰In the Gibbs gauge [34, 117] the scalar potential V is identically zero, and the vector potential for static fields is given by $\mathbf{A} = -c \mathbf{E} t$.

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