Polarization Precession

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(January 14, 1970; updated May 7, 2015)

1 Problem

Deduce an expression for the rate of precession of the spin polarization unit vector $\hat{\mathbf{P}}$ of a relativistic charged particle in laboratory electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The polarization vector is to be measured in the so-called comoving rest frame of the particle obtained by boosting from the lab frame along the direction $\hat{\mathbf{l}}$ that is tangent the the particle's laboratory trajectory, and whose axes are defined by the (lab-frame) triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/dt$. However, express the time dependence of this polarization vector in terms of lab-frame quantities. Take into account the so-called Thomas precession [1, 2] of vectors in the rest frame of an accelerating particle when the (instantaneous) rest frame is defined by a boost along the instantaneous lab-frame velocity.

2 Solution

Many high-energy-physics experiments measure the spin polarization or other quantities depending on the polarization of a charged particle. If the particle passes through an electromagnetic field (such as that of a particle accelerator or a magnetic spectrometer) the spin may precess. We derive an expression for the spin precession for arbitrary fields, polarizations and particle velocities. Among the many previous discussions of the precession [2, 2, 3, 4, 5, 6, 7, 8], most restricted their examples to velocities either normal or parallel to the magnetic field.[1]

Rather than use the Dirac equation [5, 6, 7] (or some relativistic wave equation for higher spin) to calculate the precession, we first consider the (instantaneous) rest frame of the particle, where nonrelativistic wave equations apply [1, 2, 3, 4, 8]. The results may then be transformed to the lab with complete generality. However, if we construct a polarization 4-vector and transform it to the lab frame, the meaning of the various lab components of the 4-vector is unclear. One usually returns to the rest frame to understand the polarization vector. Our approach is to deal always with the physically meaningful rest-frame polarization vector, but to use the laboratory values of all other quantities.[2]

[1]The earliest discussion, that of Thomas [1, 2], was of full generality – although he did not use the comoving rest frame. However, it appears that the common application of the Thomas precession to atomic systems (the emphasis of the Nature article [1]) resulted in it being nearly forgotten that his results also apply to free electrons in macroscopic electromagnetic fields.

[2]Let us illustrate that the rest-frame polarization is indeed the physically meaningful quantity to a lab observer. Consider the decay of a polarized particle. Calculations of the correlation between the polarization and the decay angular distribution are always made in the rest frame of the particle. As another example, consider the two-body scattering involving a spin-1/2 particle whose polarization is to be measured. The only parity-conserving operator inthe scattering amplitude involving spin is $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, where $\boldsymbol{\sigma}$ is the Pauli spin matrix vector and $\hat{\mathbf{n}}$ is the normal to the scattering plane. Thus, only the component of the spin transverse

If the particle is not accelerated then its rest frame is a single inertial frame, and the polarization 3-vector $\hat{\mathbf{P}}$ obeys

$$\left.\frac{d\hat{\mathbf{P}}}{d\tau}\right|_{\text{nonaccel.}} = g\frac{e}{2m}\hat{\mathbf{P}} \times \mathbf{B}^\star, \tag{1}$$

where

$$\begin{aligned} \tau &= \text{proper time}, \\ g &= \text{gyromagnetic ratio } (= 2(2.79) \text{ for protons}), \\ e &= \text{electric charge}, \\ m &= \text{mass}, \\ \mathbf{B}^\star &= \text{magnetic field strength in the rest frame}, \end{aligned}$$

and we work in units where the speed of light is unity. This equation follows from classical electromagnetism using the correspondence principle.[3]

Before turning to the important issue of accelerated motion, we first express τ and $\mathbf{B}^\star$ in terms of lab-frame quantities. Let the instantaneous 3-velocity of the particle be $\boldsymbol{\beta}$ with respect to the lab frame, and let

$$\beta = |\boldsymbol{\beta}|\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{and} \qquad \hat{\mathbf{l}} = \frac{\boldsymbol{\beta}}{\beta}\,, \tag{5}$$

such that $\hat{\mathbf{l}}$ is the unit tangent to the particle's trajectory. Then, the Lorentz transformation from the (instantaneous) rest frame to the lab frame gives

$$d\tau = \frac{dt}{\gamma}\,, \tag{6}$$

where t is the time in the lab frame. The transformation of the magnetic field is

$$\mathbf{B}^\star = \mathbf{B}_\parallel + \gamma(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}) = \gamma\mathbf{B} + (\gamma - 1)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \gamma\beta\,\hat{\mathbf{l}} \times \mathbf{E}, \tag{7}$$

to the particle's direction is significant. This component is the same in the rest, lab and center of mass frames, so that calculation of the spin direction in the particle's rest frame is sufficient.

[3]We illustrate this for spin-1/2 particles, for which $\mathbf{P} = \langle\boldsymbol{\sigma}\rangle$. Now

$$\frac{d\langle\boldsymbol{\sigma}\rangle}{dt} = \left\langle\frac{d\boldsymbol{\sigma}}{dt}\right\rangle = \frac{i}{\hbar}\langle[H, \boldsymbol{\sigma}]\rangle\,, \tag{2}$$

where H is the Hamiltonian. Since we work in the rest frame, we may use the nonrelativistic Hamiltonian. The only part of the Hamiltonian for a free particle in an electromagnetic field that does not commute with $\boldsymbol{\sigma}$ is

$$H' = -\frac{g}{2}\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}^\star. \tag{3}$$

Hence,

$$\frac{d\langle\boldsymbol{\sigma}\rangle}{dt} = -i\frac{g}{2}\frac{e}{2m}[\boldsymbol{\sigma}\cdot\mathbf{B}^\star, \boldsymbol{\sigma}] = g\frac{e}{2m}\boldsymbol{\sigma}\times\mathbf{B}^\star, \tag{4}$$

whose expectation value is eq. (1).

where $B_{\|} = (\mathbf{B} \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}}$ is the longitudinal part of the lab magnetic field, $\mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{\|}$, and $\mathbf{E}$ is the lab electric field.

Using eqs. (6) and (7) in (1), the polarization in the rest frame (still assuming the particle is not accelerated) obeys the equation of motion

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{nonaccel.}} = \hat{\mathbf{P}} \times \frac{e}{m}\frac{g}{2}\left[\mathbf{B} - \left(1 - \frac{1}{\gamma}\right)(\mathbf{B} \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}} - \beta\,\hat{\mathbf{l}} \times \mathbf{E}\right]. \tag{8}$$

In general, if the particle is subject to fields that can precess the polarization, its trajectory is also deflected. Then, the rest frame of the particle is an accelerated frame, and care is required when using Lorentz transformations between the lab frame and the instantaneous rest frame. The latter frame is, by construction, an inertial frame, but leads to the concept of the rest frame as a sequence of inertial frames each with a different boost from the lab frame. The subtle consequence of this construction is that there is an additional precession, the Thomas precession [1, 2], of a rest-frame vector such as the polarization.

At this point we should distinguish between two different "rest" frames, both of whose origins are at the position of the particle. The first, to be called the rest frame, is the frame obtained by a boost from the lab frame along the particle's velocity. We have been using this rest frame so far.

The second frame, to be called the comoving frame, differs from the rest frame defined above by a rotation of the coordinate axes onto the lab-frame triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/dt$.

We report the precession of the polarization in the comoving frame, rather than that in the lab frame, since the geometry of the comoving frame is more closely related to the geometry of the lab trajectory of the particle.

In Appendix A it is shown that a vector $\mathbf{s}$ that has no explicit time dependence in the rest frame obeys

$$\left.\frac{d\mathbf{s}}{dt}\right|_{\text{comoving}} = \mathbf{s} \times \boldsymbol{\omega} \tag{9}$$

relative to the comoving rest frame, where

$$\boldsymbol{\omega} = \gamma\,\hat{\mathbf{l}} \times \frac{d\hat{\mathbf{l}}}{dt}. \tag{10}$$

That is, vector $\mathbf{s}$ precesses with frequency $\boldsymbol{\omega}$ with repsect to the triad form by $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\boldsymbol{\omega}$ in the comoving frame. This is the so-called Thomas precession [1, 2]. The equation of motion of the polarization in the comoving frame is therefore

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{comoving}} = \left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{nonaccel.}} + \hat{\mathbf{P}} \times \boldsymbol{\omega}, \tag{11}$$

where (8) is the equation of motion in the rest frame for nonaccelerated motion.

To calculate the vector $\boldsymbol{\omega}$ using eq. (10) we need $d\hat{\mathbf{l}}/dt$. For this, we use the relativistic form of the Lorentz force,

$$\frac{du}{dt} = \frac{e}{m} F \cdot u, \tag{12}$$

where $u = (\gamma, \gamma\boldsymbol{\beta})$ is the 4-velocity and F is the electromagnetic field tensor,

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{13}$$

The well-known resulting equations are

$$\frac{d\gamma}{dt} = \frac{e}{m}\beta\,\mathbf{E}\cdot\hat{\mathbf{l}}, \tag{14}$$

$$\frac{d(\gamma\beta\,\hat{\mathbf{l}})}{dt} = \frac{e}{m}(\mathbf{E} + \beta\,\hat{\mathbf{l}}\times\mathbf{B}). \tag{15}$$

Differentiating the identity $\gamma^2\beta^2 = \gamma^2 - 1$, we also have

$$\frac{d(\gamma\beta)}{dt} = \frac{1}{\beta}\frac{d\gamma}{dt} = \frac{e}{m}\mathbf{E}\cdot\hat{\mathbf{l}}. \tag{16}$$

Solving for $d\hat{\mathbf{l}}/dt$ we find[4]

$$\frac{d\hat{\mathbf{l}}}{dt} = \frac{1}{\gamma\beta}\frac{d(\gamma\beta\hat{\mathbf{l}})}{dt} - \frac{\hat{\mathbf{l}}}{\gamma\beta}\frac{d(\gamma\beta)}{dt} = \frac{e}{m}\left[\frac{\mathbf{E} - (\mathbf{E}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}}{\gamma\beta} + \frac{\hat{\mathbf{l}}\times\mathbf{B}}{\gamma}\right]. \tag{18}$$

Then, from eq. (10) we have

$$\omega = \gamma\,\hat{\mathbf{l}}\times\frac{d\hat{\mathbf{l}}}{dt} = \frac{e}{m}\left[\frac{\hat{\mathbf{l}}\times\mathbf{E}}{\beta} - \mathbf{B} - (\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}\right], \tag{19}$$

Inserting eq. (19) in (11) we have

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{comoving}} = \hat{\mathbf{P}}\times\frac{e}{m}\left\{\left(\frac{g}{2}-1\right)[\mathbf{B} - (\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}] + \frac{g}{2\gamma}(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \left(\frac{g}{2} - 1 - \frac{g}{2\gamma^2}\right)\frac{\hat{\mathbf{l}}\times\mathbf{E}}{\beta}\right\}. \tag{20}$$

This is the main result for the precession of the polarization vector with respect to the comoving frame.[5]

If $d\hat{\mathbf{P}}/dt = 0$ in eq. (20), then $\hat{\mathbf{P}}$ is constant with respect to the triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\boldsymbol{\omega}$.

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[4] Equation (18) is strictly true only in homogeneous field [9]. In an inhomogeneous magnetic field there is an additional force on the particle due to the magnetic moment interaction $\boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B})$ (plus other terms for higher multipole moments). However, this correction is extremely small compared to the usual magnetic force $e\boldsymbol{\beta}\times\mathbf{B}$. For example, suppose $\boldsymbol{\mu}$ is parallel to $\mathbf{B}$ and normal to $\boldsymbol{\beta}$. Then, the ratio of these two terms is

$$\frac{F_{\text{mag dipole}}}{F_{\text{Lorentz}}} = \frac{\mu}{e\beta}\frac{\nabla B}{B} \approx \frac{\hbar}{m\beta}\frac{\nabla B}{B} = \frac{\lambda\!\!\!^{-}_C}{\beta}\frac{\nabla B}{B}\,, \tag{17}$$

where $\lambda\!\!\!^{-}_C = \hbar/m$ is the Compton wavelength of the particle. Even for a relativistic electron in a rapidly varying magnetic field with $\nabla B/B \approx 1/\text{cm}$, the ratio is 10^{-11}. Hence, the results of this problem are applicable to inhomgenous fields on any reasonable laboratory scale.

[5] Thomas [2] gave a very similar result [his eq. (4.121)], which reports the precession relative to a fixed direction in the lab. As shown in Appendix A, our formalism can be converted to this case by replacing the factor γ in the definition (10) of $\boldsymbol{\omega}$ by the factor $\gamma - 1$. The resulting expression for the polarization precession is also displayed in eq. (11.170) of J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).

2.1 Discussion

From now on we put $\mathbf{E} = 0$.[6]

2.1.1 $\hat{\mathbf{l}} \parallel \mathbf{B}$

In this case,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\mathrm{comoving}} = \frac{e}{m}\frac{g}{2}\frac{\hat{\mathbf{P}} \times \mathbf{B}}{\gamma}, \qquad (\hat{\mathbf{l}} \perp \mathbf{B}), \tag{21}$$

which is the nonrelativistic result (1) divided by the time-dilation factor γ. If $\mathbf{B}$ is constant, the particle's trajectory is a straight line parallel to $\mathbf{B}$, and the polarization precesses about this fixed direction. This effect has been utilized in measurements of the gyromagnetic ratio of the electron [10].

2.1.2 $\hat{\mathbf{l}} \perp \mathbf{B}$

Here,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\mathrm{comoving}} = \frac{e}{m}\left(\frac{g}{2} - 1\right)\hat{\mathbf{P}} \times \mathbf{B}, \qquad (\hat{\mathbf{l}} \parallel \mathbf{B}). \tag{22}$$

The polarization vector of a Dirac particle (for which $g = 2$) would not precess at all. From eq. (18), the particle's trajectory obeys

$$\frac{d\hat{\mathbf{l}}}{dt} = \frac{1}{\gamma\beta}\frac{d(\gamma\beta\hat{\mathbf{l}})}{dt} - \frac{\hat{\mathbf{l}}}{\gamma\beta}\frac{d(\gamma\beta)}{dt} = \frac{e}{m}\frac{\hat{\mathbf{l}} \times \mathbf{B}}{\gamma}. \tag{23}$$

Thus, the precession rate of $\hat{\mathbf{P}}$ is $\gamma(g/2 - 1)$ times that of $\hat{\mathbf{l}}$. Hence, if the trajectory bends through angle θ_l, the polarization vector precesses by angle

$$\theta_P = \gamma\left(\frac{g}{2} - 1\right)\theta_l. \tag{24}$$

This result is useful for quick calculations as to the effect of a bending magnet on polarization. For a slow proton, eq. (23) gives $\theta_P \approx 2\theta_l$. Hence, a 45° bend can interchange transverse and longitudinal polarization (if $\hat{\mathbf{P}} \perp \mathbf{B}$), while a 90° bend flips the polarization [11]. Of course, if $\hat{\mathbf{P}} \parallel \mathbf{B}$, the precession has no physical effect.

The precession frequency is

$$\Omega_P = \left(\frac{g}{2} - 1\right)\frac{eB}{m}, \tag{25}$$

which does not depend on γ. Hence, in a given magnetic field a fast muon will undergo more spin precession during its lifetime, which is proportional to γ, than will a slow muon. This allows a more precise determination of the muon gyromagnetic ratio at relativistic energies [12].

[6] Note added Dec. 2002. In the recent BNL experiment to measure $g - 2$ for the muon, http://phyppro1.phy.bnl.gov/g2muon/index.shtml, it was desired to have electrostatic quadrupole focusing elements in addition to the magnetic field $\mathbf{B} \perp \mathbf{l}$ that cause the muons to execute circular orbits. The precession due to the (spatially varying) electric field was eliminated, following eq. (20), by using muons with $\gamma = \sqrt{(g/2)/(g/2-1)} \approx \sqrt{2\pi/\alpha} \approx 29.3$, noting that $g/2 \approx 1 + \alpha/2\pi$, where $\alpha \approx 1/137$ is the fine structure constant.

2.1.3 $\hat{\mathbf{l}}$ neither $\perp$ nor $\parallel$ $\mathbf{B}$

In this case the particle's trajectory is helical. From eq. (20) we see that the polarization does not precess about $\mathbf{B}$, but rather about

$$\mathbf{B}' = \mathbf{B}_\perp + \frac{g}{\gamma(g-2)}\mathbf{B}_\parallel, \tag{26}$$

which lies in the plane of $\mathbf{B}$ and $\mathbf{l}$. If we designate the angle between $\mathbf{B}$ and $\mathbf{l}$ by θ, and the angle between $\mathbf{B}'$ and $\mathbf{l}$ by θ', then

$$\tan\theta' = \left(\frac{g}{2} - 1\right)\frac{2\gamma}{g}\tan\theta \ (= 0.64\gamma\tan\theta \text{ for protons}). \tag{27}$$

When $\gamma = 1.56$ for protons, $\tan\theta' = \tan\theta$ and $\mathbf{B}' = \mathbf{B}$, which occurs at a kinetic energy of 525 MeV.

As $\gamma \to \infty$, $\tan\theta' \to 1$, and $\hat{\mathbf{P}}$ precesses about the transverse part of $\mathbf{B}$. For applications to many bending magnets at finite γ, $\mathbf{l}$ will be nearly normal to $\mathbf{B}$ and the difference between $\mathbf{B}'$ and $\mathbf{B}$ will be small. A first correction to eq. (24) can be made via an "effective edge" approximation to the magnetic field. That is, replace the actual, spatially varying field with a constant field $\mathbf{B}_{\text{eff}}$ that produces the same total bend in the particle's trajectory. Then, $\mathbf{B}_{\text{eff}} \cdot \hat{\mathbf{l}}$ is constant, so that $\mathbf{B}'$ is constant is magnitude and precess about $\mathbf{B}$ at the same frequency as does $\hat{\mathbf{l}}$. The corrected form of eq. (24) is

$$\theta_P = \gamma\left(\frac{g}{2} - 1\right)\theta_l \frac{|\mathbf{B}'|}{|\mathbf{B}|} = \gamma\left(\frac{g}{2} - 1\right)\theta_l \sqrt{1 + \left(\frac{g^2}{\gamma^2(g-2)^2} - 1\right)\cos^2\theta}, \tag{28}$$

where θ is the angle between $\mathbf{B}$ and $\mathbf{l}$. Note that in eq. (24) both θ_P and θ_l are azimuthal angles measured with respect to $\mathbf{B}$, while in eq. 28) θ_P is with respect to $\mathbf{B}'$.

A Appendix: The Thomas Precession

We wish to find the time dependence in the "comoving" rest frame of a spacelike vector that has no explicit time dependence in the rest frame of an accelerated particle. That is, we seek an expression for the apparent time depedence of the vector that arises because the rest frame is constructed from a sequence of instantaneous inertial frames based on the time-dependent velocity $\boldsymbol{\beta}(t)$ of the particle.

The particle has lab frame 4-velocity u where

$$u = (\gamma, \gamma\boldsymbol{\beta}) = (\gamma, \gamma\beta\hat{\mathbf{l}}). \tag{29}$$

The "rest" frame is obtained from the lab frame by a boost, without rotation of the coordinate axes, along the direction $\hat{\mathbf{l}}$. The "comoving" frame is a rotating frame in which the particle is at rest and whose axes are always aligned with the (lab frame) triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$, and $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/dt$. The comoving frame differs from the rest frame by a time-dependent rotation.

Consider a spacelike 4-vector $s = (0, \mathbf{s}_R)$ in the rest frame, that satisfies $\partial \mathbf{s}_R/\partial\tau = 0$, *i.e.*, that has no explicit time dependence in the rest frame. In the rest frame the 4-velocity has components $u = (1, \mathbf{0})$, so the invariant product of s and u vanishes,

$$s \cdot u = 0, \tag{30}$$

which relation is therefore true in all (inertial) frames. The time derivative of eq. (30) also vanishes, so we have

$$\frac{ds}{d\tau} \cdot u = -s \cdot \frac{du}{d\tau}\,, \tag{31}$$

where τ is the proper time. This suggests that we can write

$$\frac{ds}{d\tau} = -\left(s \cdot \frac{du}{d\tau}\right) u \tag{32}$$

as the covariant form of the rest-frame condition $\partial \mathbf{s}_R/\partial\tau = 0$.

Equation (32) implies that $d\mathbf{s}_R/d\tau = 0$ if the rest frame is inertial. We extend the use of eq. (32) to the case of accelerated motion by first applying it in the lab frame to find $d\mathbf{s}_L/d\tau$, then transforming $\mathbf{s}_L$ back to the rest frame, followed by taking the time derivative of this transformation. This procedure is sufficient to reveal the Thomas precession.

In the lab frame (indeed, in any frame) where we write $s = (s_{L,0}, \mathbf{s}_L)$, the condition (30) plus the expression (29) tell us that

$$s_{L,0} = \mathbf{s}_L \cdot \boldsymbol{\beta} = \beta\, \mathbf{s}_L \cdot \hat{\mathbf{l}}. \tag{33}$$

From eq. (32) we find the lab-frame time dependence

$$\frac{d\mathbf{s}_L}{d\tau} = -\left(s_{L,0}\frac{d\gamma}{d\tau} - \mathbf{s}_L \cdot \frac{d\gamma\beta\,\hat{\mathbf{l}}}{d\tau}\right)\gamma\beta\,\hat{\mathbf{l}} = (\gamma^2 - 1)\left(\mathbf{s}_L \cdot \frac{d\hat{\mathbf{l}}}{d\tau}\right)\hat{\mathbf{l}} + \frac{1}{\gamma}\frac{d\gamma}{d\tau}(\mathbf{s}_L \cdot \hat{\mathbf{l}})\hat{\mathbf{l}}, \tag{34}$$

recalling eq. (16). Although $d\mathbf{s}_L/d\tau$ is not zero in the lab frame, we would not say that vector $\mathbf{s}_L$ is precessing there. That is, a lab-frame observer does not see the Thomas precession when considering only lab-frame quantities.

The Lorentz transformation between the lab frame L and the (instantaneous) rest frame R for the spatial part of 4-vector s is

$$\begin{aligned}
\mathbf{s}_L &= \mathbf{s}_R - (\mathbf{s}_R \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}} + \gamma[(\mathbf{s}_R \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}} + \boldsymbol{\beta} s_{R,0}] = \mathbf{s}_R + (\gamma - 1)(\mathbf{s}_R \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}}, && (35)\\
\mathbf{s}_R &= \mathbf{s}_L - (\mathbf{s}_L \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}} + \gamma[(\mathbf{s}_L \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}} - \boldsymbol{\beta} s_{L,0}] = \mathbf{s}_L + \left(\frac{1}{\gamma} - 1\right)(\mathbf{s}_L \cdot \hat{\mathbf{l}})\,\hat{\mathbf{l}}, && (36)
\end{aligned}$$

since $s_{R,0} = 0$ and $s_{L,0}$ is given by eq. (33).

Taking the time derivative of eq. (36) and using eq. (34) we find

$$\frac{d\mathbf{s}_R}{d\tau} = (\gamma - 1)\left(\mathbf{s}_L \cdot \frac{d\hat{\mathbf{l}}}{d\tau}\right)\hat{\mathbf{l}} + \left(\frac{1}{\gamma} - 1\right)(\mathbf{s}_L \cdot \hat{\mathbf{l}})\frac{d\hat{\mathbf{l}}}{d\tau}\,. \tag{37}$$

Inserting eq. (35) in this we find

$$\frac{d\mathbf{s}_R}{d\tau} = (\gamma - 1)\left[\left(\mathbf{s}_R \cdot \frac{d\hat{\mathbf{l}}}{d\tau}\right)\hat{\mathbf{l}} - (\mathbf{s}_R \cdot \hat{\mathbf{l}})\frac{d\hat{\mathbf{l}}}{d\tau}\right] = \mathbf{s}_R \times (\gamma - 1)\left(\hat{\mathbf{l}} \times \frac{d\hat{\mathbf{l}}}{d\tau}\right). \tag{38}$$

This is the Thomas precession of the vector $\mathbf{s}_R$ in the rest frame. It is a "relativistic" effect in that $\gamma - 1 \approx v^2/2c^2$, which is negligible for $v \ll c$.

Since $\hat{\mathbf{l}} \cdot d\hat{\mathbf{l}}/d\tau = 0$, we obtain the identity

$$\frac{d\hat{\mathbf{l}}}{d\tau} = \left(\hat{\mathbf{l}} \times \frac{d\hat{\mathbf{l}}}{d\tau}\right) \times \hat{\mathbf{l}} \tag{39}$$

in the lab frame. Hence, $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/d\tau$ is the instantaneous angular velocity of the particle's trajectory. Since $d\hat{\mathbf{l}}/d\tau$ is transverse to $\hat{\mathbf{l}}$, eq. (39) holds in the rest frame also. Thus, the part of $d\mathbf{s}_R/d\tau$ coming from the term -1 in the factor $\gamma - 1$ of eq. (38) is due to the rotation of the trajectory in the rest frame. Hence, the precession of $\mathbf{s}$ relative to the precession of the trajectory can be written

$$\frac{d\mathbf{s}_C}{d\tau} = \mathbf{s}_C \times \gamma\left(\hat{\mathbf{l}} \times \frac{d\hat{\mathbf{l}}}{d\tau}\right), \tag{40}$$

where we use the subscript C to indicate that this expression holds in the comoving rest frame whose axes are aligned with the triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/dt$.

The covariant description of the precession used in this Appendix can be extended to include the spin precession caused by electromagnetic fields. This is the approach of Bargmann, Michel and Telegdi [8], which is expanded upon by Hagedorn [13]. The specialization to the comoving frame is straightforward, but they do not do it. The result of such analysis is eq. (20).

B Appendix: Precession of an Intrinsic Electric Dipole Moment

Suppose the charged particle had an intrinsic electric dipole moment $\mathbf{p}$. A famous argument (first given a bit indirectly in [14]) is that such a moment would have to be aligned with the spin of the particle, as this the only vector associated with an elementary particle at rest.[7] If we accept this argument, the preceding discussion for polarization precession can be readily extended to include a possible intrinsic electric dipole moment.

If the (spin-1/2) particle is not accelerated then its rest frame is a single inertial frame, and the electric dipole moment 3-vector $\mathbf{p} = p\hat{\mathbf{P}}$ obeys the torque equation

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}^\star = \left.\frac{d\mathbf{L}}{d\tau}\right|_{\text{nonaccel.}} = \left.\frac{\hbar}{2}\frac{d\hat{\mathbf{P}}}{d\tau}\right|_{\text{nonaccel.}} \tag{41}$$

[7]Such an electric dipole moment violates the symmetries of space inversion (parity) and time reversal (see, for example, [15].

which adds a contribution to eq. (1) for the polarization vector $\mathbf{P}$, which should now read

$$\left.\frac{d\hat{\mathbf{P}}}{d\tau}\right|_{\rm nonaccel.} = g\frac{e}{2m}\hat{\mathbf{P}} \times \mathbf{B}^\star + \frac{2p}{\hbar}\hat{\mathbf{P}} \times \mathbf{E}^\star, \tag{42}$$

The equivalent of eq. (8) including the electric dipole moment is

$$\begin{aligned}\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\rm nonaccel.} &= \hat{\mathbf{P}} \times \frac{e}{m}\frac{g}{2}\left[\mathbf{B} - \left(1 - \frac{1}{\gamma}\right)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \beta\,\hat{\mathbf{l}} \times \mathbf{E}\right] \\ &\quad + \hat{\mathbf{P}} \times \frac{e}{m}p\lambda\!\!\!^{-}_C\left[\mathbf{E} - \left(1 - \frac{1}{\gamma}\right)(\mathbf{E}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \beta\,\hat{\mathbf{l}} \times \mathbf{B}\right]. \end{aligned} \tag{43}$$

where $\lambda\!\!\!^{-}_C = e\hbar/m$ is the Compton wavelength of the particle.

The motion of the charge in laboratory $\mathbf{E}$ and $\mathbf{B}$ fields is again given by eqs. (14)-(17), if we again ignore the tiny force on the dipole moment due to field inhomogeneity. Then, the extention of eq. (20) to include the electric dipole moment is

$$\begin{aligned}\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\rm comoving} &= \hat{\mathbf{P}} \times \frac{e}{m}\left\{\left(\frac{g}{2} - 1\right)[\mathbf{B} - (\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}] + \frac{g}{2\gamma}(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \left(\frac{g}{2} - 1 - \frac{g}{2\gamma^2}\right)\frac{\hat{\mathbf{l}} \times \mathbf{E}}{\beta}\right. \\ &\quad \left. + p\lambda\!\!\!^{-}_C\left[\mathbf{E} - \left(1 - \frac{1}{\gamma}\right)(\mathbf{E}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \beta\,\hat{\mathbf{l}} \times \mathbf{B}\right]\right\}. \end{aligned} \tag{44}$$

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