Please do all work in the exam booklets provided.

You may use either Gaussian or MKSA units on this exam.

1. (10 pts.) Show that the charge induced in a small area $A$ on a grounded conducting plane by a point charge not in that plane is proportional to the solid angle subtended at the point charge by area $A$.

2. (20 pts.) A hollow dielectric sphere of dielectric constant $\epsilon = 3$ has inner radius one half its outer radius. When this sphere is placed in an initially uniform electric field $E_0$, what is the resulting electric field strength at the center of the sphere?

3. (30 pts.) Two circular wires of radii $a$ and $b$ have a common center, and are free to turn on an insulating axis which is a diameter of both. Find the torque about this diameter required to hold the two wire loops at rest when their planes are at right angles and they are carrying currents $I$ and $I'$, supposing that $b \ll a$. Give both the leading term, and the first correction in a power of the small ratio $b/a$.

Hint: This requires evaluating the first correction to both the axial and transverse magnetic field components near the center of the larger loop. Recall that the torque about a point is $\vec{\tau} = \vec{r} \times \vec{F}$ where force $\vec{F}$ is applied at distance $\vec{r}$. 
Solutions

1. Let charge $q$ be at perpendicular distance $a$ from the grounded conducting plane. The small area $A$ has its center at distance $r$ from the foot of the perpendicular to charge $q$. The charge $q'$ induced in the area $A$ is related by

\[ q' = \sigma A = \frac{EA}{4\pi}, \]

where $E$ is the electric field strength at the surface of the conducting plane.

We calculate $E$ using the image method, supposing that charge $-q$ is located at distance $a$ on the other side of the conducting plane from charge $q$. Then,

\[ E = -\frac{2q}{R^2} \frac{a}{R} = -2q \frac{\cos \theta}{R^2}, \]

where $R = \sqrt{a^2 + r^2}$ is the distance from charge $q$ to area $A$, and $\theta$ is the angle between vector $\mathbf{R}$ and the perpendicular from $q$ to the plane.

Combining eqs. (1) and (2), we have

\[ q' = -\frac{2qA \cos \theta}{4\pi R^2} = -\frac{q\Omega}{2\pi}, \]

where $\Omega = A \cos \theta / R^2$ is the solid angle subtended by area $A$ at charge $q$. For the whole plane, $\Omega = 2\pi$ and $q' = -q$.

2. This problem is closely related to that of a dielectric sphere in an otherwise uniform electric field. We choose the $z$ axis antiparallel to the initial field $E_0$, with the origin at the center of the dielectric sphere, where the potential is taken to be zero.

The potential of the initial field is then

\[ \phi_0 = E_0 z = E_0 r \cos \theta = E_0 r P_1(\theta), \]

where $\theta$ is the polar angle with respect to the $z$ axis and $P_1$ is the Legendre polynomial of order 1.

We recall from the case of a uniform dielectric sphere that the potential contains terms only in $P_1$, and we expect the same here.

Writing the inner radius of the sphere as $a$ and the outer radius as $b$, we expect that the potential will have the form

\[ \phi_1 = E_0 r P_1 + A \frac{r}{a} P_1, \quad (0 < r < a) \]

\[ \phi_2 = E_0 r P_1 + B \frac{r}{a} P_1 + C \frac{b^2}{r^2} P_1, \quad (a < r < b) \]

\[ \phi_3 = E_0 r P_1 + D \frac{b^2}{r^2} P_1, \quad (a < r < b) \]

since the perturbation to field $E_0$ must be finite at $r = 0$ and $\infty$. 
The potential is continuous at \( r = a \) and \( b \), so that

\[
A = B + C \frac{b^2}{a^2}, \tag{8}
\]

\[
B \frac{b}{a} + C = D. \tag{9}
\]

Also, the normal component of the electric displacement \( \mathbf{D} = \varepsilon \varepsilon \mathbf{E} \) is continuous at the boundaries, since \( \nabla \cdot \mathbf{D} = 0 \). Hence,

\[
\frac{\partial \phi_1(a)}{\partial r} = \varepsilon \frac{\partial \phi_2(a)}{\partial r}, \tag{10}
\]

and

\[
\varepsilon \frac{\partial \phi_2(b)}{\partial r} = \frac{\partial \phi_3(b)}{\partial r}, \tag{11}
\]

which yields

\[
E_0 + \frac{A}{a} = \varepsilon E_0 + \frac{\varepsilon B}{a} - 2\varepsilon \frac{C b^2}{a^3}, \tag{12}
\]

and

\[
\varepsilon E_0 + \frac{\varepsilon B}{a} - 2\varepsilon \frac{C}{b} = E_0 - 2 \frac{D}{b}. \tag{13}
\]

Inserting eq. (8) in (12), we get

\[
\frac{\varepsilon - 1}{a} B - (2\varepsilon + 1) \frac{b^2}{a^3} C = (1 - \varepsilon) E_0, \tag{14}
\]

while using eq. (9) in (13) gives

\[
\frac{\varepsilon + 2}{a} B - \frac{2(\varepsilon - 1)}{b} C = (1 - \varepsilon) E_0. \tag{15}
\]

These could be solved in general for \( A, B \) and \( C \), but here we consider the particular case that \( a = 1, b = 2 \) and \( \varepsilon = 3 \), for which eqs. (14) and (15) become

\[
B - 14 C = -E_0, \tag{16}
\]

and

\[
5 B - 2 C = -2 E_0. \tag{17}
\]

We quickly find that

\[
B = -\frac{13}{34} E_0, \quad C = \frac{3}{68} E_0, \tag{18}
\]

and from eq. (11),

\[
A = B + 4 C = -\frac{7}{34} E_0. \tag{19}
\]

The electric field strength at the center of the dielectric sphere is

\[
E(0) = E_0 + A = \frac{27}{34} E_0. \tag{20}
\]

A dielectric sphere is not as effective as a conducting sphere in shielding its interior from an external electric field.
3. (Problem 12, p. 448 of *The Mathematical Theory of Electricity and Magnetism* by J. Jeans.)

The leading term of the torque is given by $\vec{\mu} \times \mathbf{B}(0)$, where

$$\mu = \frac{\pi I' b^2}{c}$$  \hspace{1cm} (21)

is the magnetic moment of the small loop of radius $b$ that carries current $I'$, and

$$B(0) = \frac{1}{c} \int \frac{\mathbf{I} \times d\mathbf{l}}{r^2} = \frac{2\pi a I}{ca^2} = \frac{2\pi I}{ca}$$  \hspace{1cm} (22)

is the magnetic field at the center of the loops due to the current $I$ in the loop of radius $a$. When the two loops are at right angles, the vectors $\vec{\mu}$ and $\mathbf{B}(0)$ are also at right angles, so the magnitude of the leading term of the torque is

$$\tau = \frac{2\pi II'b^2}{c^2a}$$  \hspace{1cm} (23)

To evaluate the torque in greater detail, we consider the variation of the magnetic field over the small loop, and use the basic torque equation

$$\vec{\tau} = \int \mathbf{r} \times d\mathbf{F} = \frac{1}{c} \int \mathbf{r} \times [I'dl' \times \mathbf{B} \text{(due to I)}].$$  \hspace{1cm} (24)

We use a coordinate system in which the centers of the loops are at the origin, with the axis of loop $a$ is along the $z$ axis. We take the sign of current $I$ to be such that the resulting magnetic field at the origin is in the $+z$ direction. The axis of loop $b$ is defined to be the $y$ axis, and the sign of current $I'$ is such that the magnetic moment $\vec{\mu}$ is along the $+y$ axis. Then, we desire the $x$ component of the torque $\tau_x$ about the origin:

$$\tau_x = \frac{1}{c} \int b\hat{r} \times [I'b\hat{\phi}d\phi \times (B_z\hat{z} + B_\rho\hat{\rho})]|_{x}$$

$$= \frac{b^2I'}{c} \int_0^{2\pi} d\phi \cos \phi (\cos \phi B_z + \sin \phi B_\rho),$$  \hspace{1cm} (25)

where angle $\phi$ is measured in the $x$-$z$ plane with respect to the $z$ axis, such that for a point on loop $b$, $\rho = b \sin \phi$ and $z = b \cos \phi$.

If we don’t recall the results of problem 7, set 4, the magnitude of $B_\rho$ can be estimated quickly using the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ and a “pillbox” surface of radius $\rho$ and thickness $dz$ whose axis is along the $z$ axis:

$$0 = \int \nabla \cdot \mathbf{B} d\text{Vol} = \int \mathbf{B} \cdot d\mathbf{S}$$

$$\approx \pi \rho^2 (B_z(0, z + dz) - B_z(0, z)) + 2\pi \rho dz B_\rho(\rho, z).$$

$$\approx \pi \rho^2 dz \frac{\partial B_z(0, z)}{\partial z} + 2\pi \rho dz B_\rho(\rho, z).$$  \hspace{1cm} (26)
Hence,
\[ B_\rho(\rho, z) \approx -\frac{\rho}{2} \frac{\partial B_z(0, z)}{\partial z}. \]  
(27)

Then, near the center of loop \( a \) its magnetic field obeys \( \nabla \times \mathbf{B} = 0 \), and in particular
\[ \frac{\partial B_z(\rho, z)}{\partial \rho} = \frac{\partial B_z(\rho, z)}{\partial z} \approx -\frac{\rho}{2} \frac{\partial^2 B_z(0, z)}{\partial z^2}, \]  
(28)

using eq. (27). We can integrate this to find
\[ B_z(\rho, z) \approx B_z(0, z) - \frac{\rho^2}{4} \frac{\partial^2 B_z(0, z)}{\partial z^2}, \]  
(29)
in agreement with the results of Problem 7, Set 4.

For points along the \( z \) axis the magnetic field due to loop \( a \) is
\[ B_z(0, z) = \frac{1}{c} \int \frac{I \times d\mathbf{l}}{r^2} \bigg|_{z} = \frac{2\pi a^2 I}{c(a^2 + z^2)^{3/2}} \approx \frac{2\pi I}{ca} \left(1 - \frac{3z^2}{2a^2}\right), \]  
(30)
where the approximation can be used when we evaluate the field on loop \( b \) for which \(|z| \leq b \ll a\). Thus,
\[ \frac{\partial B_z(0, z)}{\partial z} = -\frac{6\pi a^2 z I}{c(a^2 + z^2)^{5/2}} \approx -\frac{6\pi z I}{ca^3}, \]  
(31)
and
\[ \frac{\partial^2 B_z(0, z)}{\partial z^2} = -\frac{6\pi a^2 I(a^2 - 4z^2)}{c(a^2 + z^2)^{7/2}} \approx -\frac{6\pi I}{ca^3}, \]  
(32)

Using eqs. (27) and (31), the transverse magnetic field at a point on loop \( b \) is
\[ B_\rho(\rho, z) \approx \frac{3\pi I \rho z}{ca^3} = \frac{3\pi b^2 I \cos \phi \sin \phi}{ca^3}, \]  
(33)
and eqs. (29), (30) and (32) give the axial field as
\[ B_z(\rho, z) \approx \frac{2\pi I}{ca} \left(1 - \frac{3z^2}{2a^2}\right) + \frac{3\pi I \rho^2}{2ca^3} = \frac{2\pi I}{ca} \left(1 - \frac{3b^2 \cos^2 \phi}{2a^2}\right) + \frac{3\pi b^2 I \sin^2 \phi}{2ca^3}. \]  
(34)

Combining eqs. (25), (33) and (34) we find
\[ \tau_x \approx \frac{\pi b^2 II'}{c^2 a} \int_0^{2\pi} d\phi \left(2 \cos^2 \phi - \frac{3b^2 \cos^4 \phi}{a^2} + \frac{3b^2 \cos^2 \phi \sin^2 \phi}{2a^2} + \frac{3b^2 \cos^2 \phi \sin^2 \phi}{a^2}\right) \]
\[ = \frac{\pi b^2 II'}{c^2 a} \int_0^{2\pi} d\phi \left(2 \cos^2 \phi - \frac{3b^2 \cos^2 \phi}{a^2} + \frac{15b^2 \sin^2 2\phi}{8a^2}\right) \]
\[ = \frac{2\pi^2 b^2 II'}{c^2 a} \left(1 - \frac{9b^2}{16a^2}\right). \]  
(35)

[The answer in MKSA units is obtained on setting \( c = 1 \) in the magnetic force equation, and replacing \( 1/c \) by \( \mu_0/4\pi \) in the Biot-Savart law, so \( 2\pi^2/c^2 \to \pi\mu_0/2 \).]