1. The form

\[ U = e^{i \delta} \left( \cos \frac{\theta}{2} \mathbf{I} + i \sin \frac{\theta}{2} \mathbf{u} \cdot \mathbf{\sigma} \right) = e^{i \delta} e^{i \frac{\theta}{2} \mathbf{u} \cdot \mathbf{\sigma}}, \tag{1} \]

of a general 2 \( \times \) 2 unitary matrix \([\text{(Set 2, eq. (12)}]\ suggests that these matrices have something to do with rotations. Certainly, a matrix that describes the rotation of a vector is a unitary transformation.

A general 2-component (spinor) state \( |\psi\rangle = \psi_+ |+\rangle + \psi_- |-\rangle \), where \( |\psi_+|^2 + |\psi_-|^2 = 1 \), can also be written as

\[ |\psi\rangle = e^{i \delta} \left( \cos \theta |+\rangle + e^{i \phi} \sin \theta |-\rangle \right). \tag{2} \]

The overall phase \( \delta \) has no meaning to a measurement of \( |\psi\rangle \). So, it is tempting to interpret parameters \( \theta \) and \( \phi \) as angles describing the orientation in a spherical coordinate system \( (r, \theta, \phi) \) of a unit 3-vector that is associated with the state \( |\psi\rangle \). The state \( |+\rangle \) might then correspond to the unit 3-vector \( \mathbf{z} \) that points up along the \( z \)-axis, while \( |-\rangle \leftrightarrow -\mathbf{z} \).

However, this doesn’t work! The suggestion is that the state \( |+\rangle \) corresponds to angles \( \theta = 0, \phi = 0 \) and state \( |-\rangle \) to angles \( \theta = \pi, \phi = 0 \). With this hypothesis, eq. (2) gives a satisfactory representation of a spin-up state as \( |+\rangle \), but it implies that the spin-down state would be \( |-+\rangle = e^{i \pi} \) times the spin-up state, which is not really distinct from the spin-up state.

We fix up things be writing

\[ |\psi\rangle = e^{i \delta} \left[ \cos \frac{\theta}{2} |+\rangle + e^{i \phi} \sin \frac{\theta}{2} |-\rangle \right], \tag{3} \]

and identifying angles \( \theta \) and \( \phi \) with the polar and azimuthal angles of a unit 3-vector in an abstract 3-space (sometimes called the Bloch sphere). That is, we associate the state \( |\psi\rangle \) with the unit 3-vector whose components are \( \psi_x = \sin \theta \cos \phi, \psi_y = \sin \theta \sin \phi \) and \( \psi_z = \cos \theta \). Now, the associations

\[ \text{spin up} \leftrightarrow (\theta = 0, \phi = 0) \leftrightarrow |+\rangle, \quad \text{spin down} \leftrightarrow (\theta = \pi, \phi = 0) \leftrightarrow |-\rangle, \tag{4} \]

given by eq. (3) are satisfactory.
We then infer from eq. (3) that the spin-up and spin-down states in the direction \((\theta, \phi)\) are, to within an overall phase factor,

\[
|+(\theta, \phi)\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad |-(\theta, \phi)\rangle = |+(\pi - \theta, \phi + \pi)\rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{i\phi} \end{pmatrix}. \tag{5}
\]

The Problem: Deduce the up and down 2-component spinor states along direction \((\theta, \phi)\) in a spherical coordinate system via rotation matrices (where first a rotation is made by angle \(\theta\) and then by angle \(\phi\)).

Rotation Matrices

A general rotation in 3-space is characterized by 3 angles. We follow Euler in naming these angles as in the figure above.\(^1\) The rotation takes the axis \((x, y, z)\) into the axes \((x', y', z')\) in 3 steps:

(a) A rotation by angle \(\alpha\) about the \(z\)-axis, which brings the \(y\)-axis to the \(y_1\) axis and the

(b) A rotation by angle \(\beta\) about the \(y_1\)-axis, which brings the \(z\)-axis to the \(z'\)-axis.

(c) A rotation by angle \(\gamma\) about the \(z'\)-axis, which brings the \(y_1\)-axis to the \(y'\)-axis (and the \(x\)-axis to the \(x'\)-axis).

The 2 \(\times\) 2 unitary matrix that corresponds to this rotation (of coordinate axes) is

\[
R(\alpha, \beta, \gamma) = \begin{pmatrix} \cos \frac{\alpha}{2} e^{i(\alpha+\gamma)/2} & \sin \frac{\alpha}{2} e^{i(-\alpha+\gamma)/2} \\ -\sin \frac{\alpha}{2} e^{i(\alpha-\gamma)/2} & \cos \frac{\alpha}{2} e^{-i(\alpha+\gamma)/2} \end{pmatrix}
\]

\[
= \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos \frac{\beta}{2} & \sin \frac{\beta}{2} \\ -\sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}
\]

\[
= R_{z'}(\gamma)R_{y_1}(\beta)R_{z}(\alpha), \tag{6}
\]

\(^1\)From sec. 58 of Landau and Lifshitz, *Quantum Mechanics*. 

2
where the decomposition into the product of 3 rotation matrices follows from the particular rules

\[
R_x(\phi) = \begin{pmatrix}
\cos \frac{\phi}{2} & i\sin \frac{\phi}{2} \\
 i\sin \frac{\phi}{2} & \cos \frac{\phi}{2}
\end{pmatrix},
\]

\[
R_y(\phi) = \begin{pmatrix}
\cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\
 -\sin \frac{\phi}{2} & \cos \frac{\phi}{2}
\end{pmatrix},
\]

\[
R_z(\phi) = \begin{pmatrix}
e^{i\phi/2} & 0 \\
 0 & e^{-i\phi/2}
\end{pmatrix}.
\]

Convince yourself that the combined rotation (6) could also be achieved if first a rotation is made by angle \(\gamma\) about the \(z\) axis, then a rotation is made by angle \(\beta\) about the original \(y\) axis, and finally a rotation is made by angle \(\alpha\) about the original \(z\) axis.

There is unfortunately little consistency among various authors as to the conventions used to describe rotations. I follow the notation of Barenco et al., who appear to write eq. (6) simply as

\[
R(\alpha, \beta, \gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha).
\]

Occasionally one needs to remember that in eq. (10) the axes of the second and third rotations are the results of the previous rotation(s).

Note that according to eqs. (7)-(9),

\[
\sigma_x = \sigma_1 = -iR_x(180^\circ), \quad \sigma_y = \sigma_2 = -iR_y(180^\circ), \quad \sigma_z = \sigma_3 = -iR_z(180^\circ),
\]

and also

\[
\sigma_x = iR_x(-180^\circ), \quad \sigma_y = iR_y(-180^\circ), \quad \sigma_z = iR_z(-180^\circ),
\]

so that the Pauli spin matrices are equivalent to the formal matrices for \(180^\circ\) rotations only up to a phase factor \(i\).

Show that a more systematic relation between the Pauli spin matrices and the rotation matrices is that eqs. (7)-(9) can be written as

\[
R_u(\phi) = e^{i\phi/2 \hat{u} \cdot \sigma},
\]

which describes a rotation of the coordinate axes in Bloch space by angle \(\phi\) about the \(\hat{u}\) axis (in a right-handed convention).

Rather than rotating the coordinate axes, we may wish to rotate vectors in Bloch space by an angle \(\phi\) about a given axis \(\hat{u}\), while leaving the coordinate axes fixed. The operator

\[
R_u(-\phi) = e^{-i\phi/2 \hat{u} \cdot \sigma}
\]

\footnote{The order of operations is that the rightmost rotation in eq. (6) is to be performed first.}

\footnote{http://physics.princeton.edu/~mcdonald/examples/QM/barenco_pra_52_3457_95.pdf}
performs this type of rotation. With this in mind, you can finally do the main problem posed on p. 2.


General (spin-1/2) particle and antiparticle 4-spinors for plane-wave states with rest mass \( m \), 3-momentum \( \mathbf{p} \), energy \( E = \sqrt{p^2 + m^2} \) and spacetime waveform \( e^{-ipx} = e^{-ip_{\mu}x^\mu} \) can be written as

\[
\begin{align*}
    u &= \sqrt{E + m} \begin{pmatrix} \chi \\ \frac{p \cdot \sigma}{E + m} \chi \end{pmatrix} = \left( \begin{pmatrix} \sqrt{E + m} \chi \\ \sqrt{E + m} \hat{p} \cdot \sigma \chi \end{pmatrix}, (15) \right. \\
    v &= \sqrt{E - m} \begin{pmatrix} \frac{p \cdot \sigma}{E - m} \chi \\ \chi \end{pmatrix} = \left( \begin{pmatrix} \sqrt{E - m} \chi \\ \sqrt{E + m} \hat{p} \cdot \sigma \chi \end{pmatrix} \right), (16) 
\end{align*}
\]

where \( \chi \) is a 2-spinor, recalling pp. 86 and 88 of Lecture 6 of the Notes. These 4-spinors obey the Dirac equations \( \not{p}u = mu \) and \( -\not{p}v = mv \). In the high-energy limit, these 4-spinors simplify to

\[
\begin{align*}
    u \rightarrow \begin{pmatrix} \chi \\ \hat{p} \cdot \sigma \chi \end{pmatrix}, \\
    v \rightarrow \begin{pmatrix} \hat{p} \cdot \sigma \chi \\ \chi \end{pmatrix}, (17)
\end{align*}
\]

where in this problem we (largely) ignore the normalization factor of the 4-spinors.

The positive and negative helicity spinor states for a particle with 3-momentum \( \mathbf{p} \) in direction \( (\theta, \phi) \) have \( \chi_+ = |+(\theta, \phi)\rangle \) and \( \chi_- = |-(\theta, \phi)\rangle \), respectively, recalling eq. (5), while the helicity states of an antiparticle have \( \chi_+ = |-(\theta, \phi)\rangle \) and \( \chi_- = |+(\theta, \phi)\rangle \), respectively.

Give the explicit forms of the helicity spinors \( u_+(0), u_-(0), v_+(0) \) and \( v_-(0) \), and \( u_+(\theta, \phi), u_-(\theta, \phi), v_+(\theta, \phi) \) and \( v_-(\theta, \phi) \) for high-energy (anti)particles moving and at angles 0 and \((\theta, \phi)\) to the \( +z \)-axis.

If these are pointlike particles of charge \( e \), their electromagnetic interaction is described by the 4-current \( j_\mu = e \gamma_\mu \). Verify that the matrix elements \( \langle \bar{u}_-(\theta)|\gamma_\mu|u_+(0)\rangle \) vanish for \( \mu = 0, 1, 2, 3 \), and similarly that \( \langle \bar{v}_+(\theta)|\gamma_\mu|u_+(0)\rangle = 0 \). Remember that \( \bar{v} = v^\dagger \gamma_0 \), etc.

3. The cross section for inelastic scattering of electrons off some target can be expressed in terms of two generalized structure functions \( W_{1,2}(q^2, \nu) \) where \( q = p_{\text{ei}} - p_{\text{ef}} \) and \( \nu = q_0 = E_i - E_f \), as on p. 131, Lecture 8 of the Notes. If the inelastic scattering is due to the interaction of the virtual photon emitted by the incident electron with a spin-1/2, charge \( Q \), mass \( m \) constituent of the target, such that the rest of the target is a “spectator” to this interaction, then the cross section is that given on p. 99, Lecture
6 of the Notes, and we infer that

\[ W_1(q \ast 2, \nu) = \frac{-q^2}{4m^2} Q^2 \delta \left( \nu + \frac{q^2}{2m} \right), \quad W_2(q^2, \nu) = Q^2 \delta \left( \nu + \frac{q^2}{2m} \right). \]  

(18)

An argument of Bjorken\(^5\) is that the lab-frame energy difference between the initial and final electron can be written as

\[ E_i - E_f = \nu = q_0 = \frac{qP}{M}, \]  

(19)

where \( P \) is the energy-momentum 4-vector of the target (of rest mass \( M \)), which is just \( P = (M, 0, 0, 0) \) in the lab frame. Then, in a frame in which the target has very high momentum, the 4-vector \( p \) of a constituent which carries (scalar) fraction \( x \) of the target’s 3-momentum can be written approximately as \( p \approx xP \). A consequence of this approximation is that the constituent mass \( m \) is related by \( m^2 = p^2 \approx x^2 P^2 = x^2 M^2 \), \( i.e., \) that \( m \approx xM \) (as appropriate for consideration of very high-energy scattering).

This permits us to rewrite eq. (18) as\(^6\)

\[ W_1 = \frac{-q^2}{4M^2 x^2} Q^2 \delta \left( \nu + \frac{q^2}{2Mx} \right), \quad W_2 = Q^2 \delta \left( \nu + \frac{q^2}{2Mx} \right). \]  

(20)

Supposing the constituents are distributed with the target (as viewed from a frame in which the target has high speed) with probability \( f(x) \, dx \), give expressions for the generalized structure functions \( W_1 \) and \( W_2 \) in terms of a single variable \( x \).


\(^6\)A different version of this argument is given on p. 139, Lecture 8 of the Notes, where a Breit frame is used.
Solutions

1. Rotation and Pauli Spin Matrices

The rotations (7)-(9) are readily seen to be exponentials of the Pauli matrices,

\[
R_x(\phi) = \begin{pmatrix}
\cos \frac{\phi}{2} & i \sin \frac{\phi}{2} \\
0 & 0
\end{pmatrix} = \cos \frac{\phi}{2} I + i \sin \frac{\phi}{2} \sigma_x = e^{i \frac{\phi}{2} \sigma_x},
\]

(21)

\[
R_y(\phi) = \begin{pmatrix}
\cos \frac{\phi}{2} & -i \sin \frac{\phi}{2} \\
0 & 0
\end{pmatrix} = \cos \frac{\phi}{2} I + i \sin \frac{\phi}{2} \sigma_y = e^{i \frac{\phi}{2} \sigma_y},
\]

(22)

\[
R_z(\phi) = \begin{pmatrix}
0 & 0 \\
e^{i \phi/2} & e^{-i \phi/2}
\end{pmatrix} = \cos \frac{\phi}{2} I + i \sin \frac{\phi}{2} \sigma_z = e^{i \frac{\phi}{2} \sigma_z},
\]

(23)

recalling eq. (1).

Digression: Pauli Spin Matrices and Rotations

The NOT operation, \( X = \sigma_x \), that “flips” a bit can be interpreted as a rotation by 180° of the Bloch-sphere state vector about the \( x \)-axis. Thus,

\[
\sigma_x \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
e^{i \beta} \sin \frac{\theta}{2} \\
\cos \frac{\theta}{2}
\end{pmatrix},
\]

(24)

while a rotation \( R_x(180^\circ) \) by 180° about the \( x \)-axis in our abstract spherical coordinate system takes \( \alpha \) to \( \pi - \alpha \) and \( \beta \) to \( -\beta \),

\[
R_x(180^\circ) \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\pi - \alpha}{2} \\
e^{-i \beta} \sin \frac{\pi - \alpha}{2}
\end{pmatrix} = e^{-i \beta} \begin{pmatrix}
e^{i \beta} \sin \frac{\theta}{2} \\
\cos \frac{\theta}{2}
\end{pmatrix}.
\]

(25)

Since the overall phase of a state does not affect its meaning, our prescription can be considered satisfactory thus far.

Can we interpret the operation \( \sigma_y \) as a rotation by 180° about the \( y \)-axis? On one hand,

\[
\sigma_y \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
i e^{i \beta} \sin \frac{\theta}{2} \\
\cos \frac{\theta}{2}
\end{pmatrix},
\]

(26)

while a rotation \( R_y(180^\circ) \) by 180° about the \( y \)-axis in our abstract spherical coordinate system takes \( \alpha \) to \( \pi - \alpha \) and \( \beta \) to \( -\beta \),

\[
R_y(180^\circ) \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\pi - \alpha}{2} \\
e^{i (\pi - \beta)} \sin \frac{\pi - \alpha}{2}
\end{pmatrix} = i e^{-i \beta} \begin{pmatrix}
i e^{i \beta} \sin \frac{\theta}{2} \\
\cos \frac{\theta}{2}
\end{pmatrix}.
\]

(27)

Similarly, we interpret the operation \( \sigma_z \) as a rotation by 180° about the \( z \)-axis:

\[
\sigma_z \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
\cos \frac{\theta}{2} \\
e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta}{2} \\
-e^{i \beta} \sin \frac{\theta}{2}
\end{pmatrix},
\]

(28)
while a rotation $R_z(180^\circ)$ by $180^\circ$ about the $z$-axis in our abstract spherical coordinate system takes $\alpha$ to $\alpha$ and $\beta$ to $\pi + \beta$,

$$R_z(180^\circ) \left( \begin{array}{c} \cos \frac{\alpha}{2} \\ e^{i\beta} \sin \frac{\alpha}{2} \end{array} \right) = \left( \begin{array}{c} \cos \frac{\alpha}{2} \\ e^{i(\pi+\beta)} \sin \frac{\alpha}{2} \end{array} \right) = \left( \begin{array}{c} \cos \frac{\alpha}{2} \\ -e^{i\beta} \sin \frac{\alpha}{2} \end{array} \right). \quad (29)$$

**Solution to the Main Problem**

We desire to rotate states in Bloch space, rather than the coordinate axes thereof, so we must heed eq. (14).

Referring to the figure on p. 3 (and reproduced below), rotation of the $z$-axis to the direction $(\theta, \phi)$, keeping the $x$-axis in the original $x$-$y$ plane, could be accomplished with rotation angles $\alpha = 0$, $\beta = \theta$, $\gamma = \phi$ in the general rotation (6). Hence, rotation of a Bloch vector along the $z$-axis to one along the direction $(\theta, \phi)$ can be accomplished by the rotation operator,

$$R(0, -\theta, -\phi) = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} & -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} & \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}. \quad (30)$$

Then,

$$| + (\theta, \phi) \rangle = R(0, -\theta, -\phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \quad (31)$$

$$| - (\theta, \phi) \rangle = R(0, -\theta, -\phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \quad (32)$$

as on p. 113, Lecture 7 of the Notes, but which differ from eq. (5) by phase factors.
2. From Prob. 1, the helicity 2-spinors for a particle are

\[
\chi_+ = | \pm (\theta, \phi) \rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \quad \chi_- = | - (\theta, \phi) \rangle = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}. \tag{33}
\]

The operator \( \hat{\mathbf{p}}(\theta, \phi) \cdot \mathbf{\sigma} \) has the form

\[
\hat{\mathbf{p}}(\theta, \phi) \cdot \mathbf{\sigma} = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}, \quad \text{such that} \quad \hat{\mathbf{p}}(\theta, \phi) \cdot \mathbf{\sigma} \chi_\pm = \pm \chi_\pm. \tag{34}
\]

**Digression:** If a general 2-spinor is written as \( \chi = a_+ \chi_+ + a_- \chi_- \), in terms of helicity spinors for the \((\theta, \phi)\) direction, then \([I \pm \hat{\mathbf{p}}(\theta, \phi) \cdot \mathbf{\sigma}] \chi = 2a_\pm \chi_\pm \). Hence,

\[
\frac{I \pm \hat{\mathbf{p}}(\theta, \phi) \cdot \mathbf{\sigma}}{2}
\]

are 2-spinor helicity projection operators for the direction \((\theta, \phi)\).

The particle and antiparticle helicity 4-spinors are, recalling eqs. (15)-(16) and defining \( \sqrt{E_+} = \sqrt{E + m} \) and \( \sqrt{E_-} = \sqrt{E - m} = p/\sqrt{E + m} \),

\[
u_+(\theta, \phi) = \begin{pmatrix} \sqrt{E + m} \chi_+ \\ \frac{p}{\sqrt{E + m}} \hat{\mathbf{p}} \cdot \mathbf{\sigma} \chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+} \chi_+ \\ \sqrt{E_+} \chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sqrt{E_+} \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \tag{36}
\]

\[
u_- (\theta, \phi) = \begin{pmatrix} \sqrt{E + m} \chi_- \\ \frac{p}{\sqrt{E + m}} \hat{\mathbf{p}} \cdot \mathbf{\sigma} \chi_- \end{pmatrix} = \begin{pmatrix} \sqrt{E_-} \chi_- \\ -\sqrt{E_-} \chi_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-} \sin \frac{\theta}{2} e^{-i\phi/2} \\ \sqrt{E_-} \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \tag{37}
\]

\[
u_+(\theta, \phi) = \begin{pmatrix} \sqrt{E + m} \chi_- \\ \frac{p}{\sqrt{E + m}} \hat{\mathbf{p}} \cdot \mathbf{\sigma} \chi_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-} \chi_- \\ \sqrt{E_-} \chi_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-} \sin \frac{\theta}{2} e^{-i\phi/2} \\ \sqrt{E_-} \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \tag{38}
\]

\[
u_- (\theta, \phi) = \begin{pmatrix} \sqrt{E + m} \chi_+ \\ \frac{p}{\sqrt{E + m}} \hat{\mathbf{p}} \cdot \mathbf{\sigma} \chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+} \chi_+ \\ \sqrt{E_+} \chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sqrt{E_+} \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}. \tag{39}
\]
In case of high-speed motion, for which $E_+ \approx E_- \approx E$, along the $+z$-axis the 4-spinors are

$$u_+(0) \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_-(0) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_+(0) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_-(0) \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (40)$$

Recalling that

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (41)$$

we have that

$$\gamma_0 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \gamma_1 u_+(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\gamma_2 u_+(0) = \begin{pmatrix} 0 \\ i \\ 0 \\ -i \end{pmatrix}, \quad \gamma_3 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (42)$$

To evaluate matrix elements such as $\langle \bar{u}_f | \gamma_\mu | u_i \rangle$ we recall that this equals $u_+^\dagger \gamma_0 \gamma_\mu u_i$, so we multiply eq. (42) by $\gamma_0$ to obtain

$$\gamma_0 \gamma_0 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \gamma_0 \gamma_1 u_+(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad (42)$$
\[
\gamma_0 \gamma_2 u_+(0) = \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix}, \quad \gamma_0 \gamma_3 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\]

Now, we can use eqs. (37)-(39) to see that
\[
\langle \bar{u}^- (\theta, \phi) | \gamma_{\mu} | u_+(0) \rangle = u^\dagger (\theta, \phi) \gamma_0 \gamma_{\mu} u_+(0) = 0 = \langle \bar{v}^+ (\theta, \phi) | \gamma_{\mu} | u_+(0) \rangle.
\]

Hence, a high-energy pointlike spin-1/2 particle of a given helicity cannot couple a particle of the opposite helicity via the electromagnetic interaction, nor can it annihilate with an antiparticle of the same helicity. It is possible for a high-energy spin-1/2 particle of a given helicity to scatter into a particle of the same helicity, or annihilate with an antiparticle of opposite helicity, via single-photon emission.

Examples where helicity conservation in the high-energy limit is useful in providing a simplified understanding include \(e^+e^-\) annihilation into a pair of spin-0 or spin-1/2 particles, as well as elastic scattering of electrons off spin-0 and spin-1/2 particles, as discussed on p. 118 ff, Lecture 7 of the Notes.

**Digression:** If we label the four components of a general spinor \(\psi_i\), \(i = 1, 4\), then
\[
\phi^\dagger \psi = \phi_1^* \phi_1 + \phi_2^* \phi_2 + \phi_3^* \phi_3 + \phi_4^* \phi_4,
\]
while
\[
\bar{\phi} \psi = \phi^\dagger \gamma_0 \psi = \phi_1^* \phi_1 + \phi_2^* \phi_2 - \phi_3^* \phi_3 - \phi_4^* \phi_4.
\]

Then, from eqs. (36)-(39),
\[
\begin{align*}
\bar{u}_+ u_+ &= \bar{u}_- u_- = v_+ v_+ = v_- v_- = 2E, \\
u_+^\dagger u_+ &= u_+^\dagger v_+ = v_-^\dagger v_- = 0, \\
u_+^\dagger v_- &= 2p = -u_-^\dagger v_+,
\end{align*}
\]
while
\[
\begin{align*}
\bar{u}_+ u_+ &= \bar{u}_- u_- = \bar{v}_+ v_+ = \bar{v}_- v_- = 2m, \\
\bar{u}_+ v_- &= \bar{v}_+ v_+ = \bar{v}_- v_- = -\bar{u}_- v_+ = 0.
\end{align*}
\]

That is, the \(u\) and \(v\) spinors are orthogonal with respect to the scalar product \(\bar{\phi} \psi\), but not with respect to \(\phi^\dagger \psi\).

**Digression:** The generalization to 4-spinors of the 2-spinor helicity projection operators (35) is,
\[
\frac{1}{2} \begin{pmatrix} I \pm \hat{p}(\theta, \phi) \cdot \sigma & 0 \\ 0 & I \pm \hat{p}(\theta, \phi) \cdot \sigma \end{pmatrix} = \frac{1}{2} \gamma_0 \hat{p}(\theta, \phi) \cdot \gamma_5 \gamma^\dagger_5,
\]
\footnote{We now follow the usual convention in writing the unit 4 \times 4 matrix as 1.}
recalling from eq. (68) of Set 3 that
\[
\gamma_0 \gamma_i \gamma_5 = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.
\] (53)

That is, recalling eqs. (36)-(39),
\[
[1 \pm \gamma_0 \hat{p}(\theta, \phi) \cdot \gamma \gamma_5] u_\pm(\theta, \phi) = \pm 2 u_\pm(\theta, \phi),
\] (54)
\[
[1 \pm \gamma_0 \hat{p}(\theta, \phi) \cdot \gamma \gamma_5] v_\pm(\theta, \phi) = \pm 2 v_\pm(\theta, \phi).
\] (55)

On p. 116, Lecture 7 of the Notes, it was argued that in the high-energy limit we can consider a different form of the helicity projection operator, more properly called the chirality projection operator,
\[
\frac{1 \pm \gamma_5}{2} = \frac{1}{2} \begin{pmatrix} I & \pm I \\ \pm I & I \end{pmatrix},
\] (56)
which leads, for momentum \(p = p \hat{z}\) along the z-axis, to
\[
\frac{1 + \gamma_5}{\sqrt{E + m}} u_\pm = \frac{1 + \gamma_5}{\sqrt{E + m}} \begin{pmatrix} \chi_\pm \\ \pm \frac{p}{E + m} \chi_\pm \end{pmatrix} = \left(1 \pm \frac{p}{E + m}\right) \begin{pmatrix} \chi_\pm \\ \chi_\pm \end{pmatrix}
\] (57)
\[
\frac{1 - \gamma_5}{\sqrt{E + m}} u_\pm = \frac{1 - \gamma_5}{\sqrt{E + m}} \begin{pmatrix} \chi_\pm \\ \pm \frac{p}{E + m} \chi_\pm \end{pmatrix} = \left(1 \mp \frac{p}{E + m}\right) \begin{pmatrix} \chi_\pm \\ -\chi_\pm \end{pmatrix}
\] (58)
\[
\frac{1 + \gamma_5}{\sqrt{E + m}} v_\pm = \frac{1 + \gamma_5}{\sqrt{E + m}} \begin{pmatrix} \pm \frac{p}{E + m} \chi_\mp \\ \chi_\mp \end{pmatrix} = \left(1 \pm \frac{p}{E + m}\right) \begin{pmatrix} \chi_\mp \\ \chi_\mp \end{pmatrix}
\] (59)
\[
\frac{1 - \gamma_5}{\sqrt{E + m}} v_\pm = \frac{1 - \gamma_5}{\sqrt{E + m}} \begin{pmatrix} \pm \frac{p}{E + m} \chi_\mp \\ \chi_\mp \end{pmatrix} = \left(1 \mp \frac{p}{E + m}\right) \begin{pmatrix} -\chi_\mp \\ \chi_\mp \end{pmatrix}
\] (60)

Only in the high-energy limit do the chirality and the helicity projection operators produce that same results.

The result of the positive (negative) chirality operator on a particle 4-spinor is call a righthanded (lefthanded) spinor, and conversely for antiparticles,
\[
\frac{1 + \gamma_5}{2} u \equiv u_R, \quad \frac{1 - \gamma_5}{2} u \equiv u_L, \quad \frac{1 + \gamma_5}{2} v \equiv v_L, \quad \frac{1 - \gamma_5}{2} v \equiv v_R,
\] (61)

Then, eq. (58) reminds us the a lefthanded particle is not precisely a negative helicity state, etc. That is, \(u_L = (1 - \gamma_5)u/2\) contains a positive-helicity component of
amplitude \((E + m - p)/(E + m + p) \approx m/2E\) relative to the nominal negative helicity component. A famous application of this in the decays \(\pi \rightarrow \mu \nu\) vs. \(\pi \rightarrow e\nu\) is discussed on p. 293, Lecture 16 of the Notes. See also Set 9, Prob. 1b.

For general particle and antiparticle 4-spinors,

\[
\begin{align*}
    u &= \sqrt{E + m} \left( \frac{\chi}{\sigma \cdot p + E + m} \right), \\
    v &= \sqrt{E + m} \left( \frac{\sigma \cdot p \zeta}{E + m} \right),
\end{align*}
\]  

(62)

with 2-spinors \(\chi\) and \(\zeta\) that obey \(\chi^\dagger \chi = 1 = \zeta^\dagger \zeta\), we have that \(u^\dagger u = 2E = v^\dagger v\) and \(\bar{u}u = 2m = -\bar{v}v\). The the right- and lefthanded 4-spinors of eqs. (61) and (62) are then,

\[
\begin{align*}
    u_{R,L} &= \frac{\sqrt{E + m}}{2} \left( \frac{1 \pm \sigma \cdot p}{E + m} \right) \chi, \\
    v_{R,L} &= \frac{\sqrt{E + m}}{2} \left( \frac{\sigma \cdot p \pm 1}{E + m} \right) \zeta,
\end{align*}
\]  

(63)

These 4-spinors have normalizations,

\[
\begin{align*}
    u_{R}^\dagger u_{R} &= E + \chi^\dagger \sigma \cdot p \chi, \\
    u_{L}^\dagger u_{L} &= E - \chi^\dagger \sigma \cdot p \chi, \\
    v_{R}^\dagger v_{R} &= E - \zeta^\dagger \sigma \cdot p \zeta, \\
    v_{L}^\dagger v_{L} &= E + \zeta^\dagger \sigma \cdot p \zeta,
\end{align*}
\]  

(64), (65)

while

\[
\bar{u}_{R} u_{R} = \bar{u}_{L} u_{L} = \bar{v}_{R} v_{R} = \bar{v}_{L} v_{L} = 0.
\]  

(66)

They also satisfy the relations,

\[
\begin{align*}
    u_{R}^\dagger u_{R} &= u_{R}^\dagger v_{R} = u_{L}^\dagger v_{L} = 0, \\
    u_{R}^\dagger v_{L} &= \chi^\dagger (E + \sigma \cdot p) \zeta, \\
    u_{L}^\dagger v_{R} &= \chi^\dagger (E - \sigma \cdot p) \zeta,
\end{align*}
\]  

(67), (68)

while

\[
\bar{u}_{R} v_{L} = \bar{u}_{L} v_{R} = 0, \quad \bar{u}_{R} u_{L} = -\bar{v}_{R} v_{L} = m, \quad \bar{u}_{R} v_{R} = \bar{u}_{L} v_{L} = m \chi^\dagger \zeta.
\]  

(69)

3. Convoluting the generalized structure functions (18) with a longitudinal-momentum distribution \(f(x)\) of constituents of a target particle of mass \(M\), we have

\[
W_2(x) = \int_0^1 f(x') \, dx' \, Q^2 \delta \left( \nu + \frac{q^2}{2Mx'} \right) = \frac{Q^2 f(x)}{-q^2/2Mx^2} = \frac{Q^2 x f(x)}{\nu},
\]  

(70)

where we recall that

\[
\int f(x) \, dx \, \delta[y(x)] = \int f(x) \frac{dy}{g'(x)} \delta(y) = \frac{f(x)}{g'(x)}, \quad \text{for} \quad x = y^{-1}(0).
\]  

(71)

so that in the present case,

\[
\nu = -\frac{q^2}{2Mx}.
\]  

(72)
The result (70) is usually recast as
\[ \nu W_2(x) = Q^2 x f(x) \equiv F_2(x). \] (73)

Similarly,
\[ W_1(x) = \int_0^1 f(x') \frac{-q^2}{4M^2 x'^2} Q^2 \delta \left( \nu + \frac{q^2}{2M x'} \right) = \frac{-q^2}{4M^2 x'^2} \frac{Q^2 f(x)}{-q^2/2M x^2} = \frac{Q^2 f(x)}{2M}, \] (74)
which is usually recast as
\[ 2MW_1(x) = Q^2 f(x) \equiv 2F_1(x). \] (75)