1. Deduce the nonrelativistic form factors,

\[ F(q^2) = \int \rho(r) e^{i q r} d^3r, \quad (1) \]

for the spherically symmetric charge densities with characteristic radius \( R \),

\[ \rho(r) = \begin{cases} 
3Q/4\pi R^3 & (r < R), \\
0 & (r > R), 
\end{cases} \quad (2) \]

\[ \rho(r) = \frac{Q}{4\pi R^2} \delta(r - R), \quad (3) \]

and

\[ \rho(r) = \frac{Q}{2\pi \sqrt{2\pi} R^3} e^{-r^2/2R^2}, \quad (4) \]

all of which have total charge \( Q \). Expand these form factors to order \((qR)^2\).

A neutral particle might have charge distributions \( \rho_+ \) and \( \rho_- \) with the above forms, but with different values of the characteristic radii \( R_+ \) and \( R_- \).

The data are often fit to the form,\(^1\)

\[ F_n(q^2) = \frac{Q}{[1 + (qR)^2]^n}; \quad (5) \]

with \( n = 2 \). What are the corresponding forms of the charge distributions \( \rho_n(r) \) for \( n = 1, 2 \) and \( 3 \)?

2. **Arbitrary 2 \times 2 Unitary Matrices and Pauli Spin Matrices**

This problem concerns operators that act on 2-component spinors. Such operators can be expressed as \( 2 \times 2 \) matrices. Operators that preserve the normalization of a state are called **unitary**.

Two of the simplest unitary operators on 2-component spinors are the identify matrix \( I_2 = I \), and the spin-flip operator \( X \) (called the **NOT** operator in quantum computation),

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (6) \]

\(^1\)For a review of nucleon form factors, see C.F. Perdrisat et al., *Nucleon electromagnetic form factors*, Prog. Part. Nucl. Phys. 59, 694 (2007),

An arbitrary 2 × 2 unitary matrix \( U \) can be written as
\[
U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
(7)
where \( a, b, c \) and \( d \) are complex numbers such that \( UU^\dagger = I \). The decomposition (7) is somewhat trivial. Express the general unitary matrix \( U \) as the sum of four unitary matrices, times complex coefficients, of which two are the classical unitary matrices \( I \) and \( X \) given above. Denote the “partner” of \( I \) by \( Z \) and the “partner” of \( X \) by \( Y \) such that
\[
XY = iZ, \quad YZ = iX, \quad ZX = iY.
\]
(8)
You have, of course, rediscovered the so-called Pauli spin matrices,\(^2\)\(^3\)
\[
\sigma_x (= \sigma_1) = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y (= \sigma_2) = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z (= \sigma_3) = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
(9)
As usual, we define the Pauli “vector” \( \sigma \) as the triplet of matrices
\[
\sigma = (\sigma_x, \sigma_y, \sigma_z).
\]
(10)
Show that for ordinary 3-vectors \( a \) and \( b \),
\[
(a \cdot \sigma)(b \cdot \sigma) = (a \cdot b) \, I + i \, \sigma \cdot a \times b.
\]
(11)
With this, show that a general 2 × 2 unitary matrix can be written as
\[
U = e^{i\delta} \left( \cos \frac{\theta}{2} \, I + i \sin \frac{\theta}{2} \, \hat{u} \cdot \sigma \right) = e^{i\delta} e^{i\frac{\theta}{2} \hat{u} \cdot \sigma},
\]
(12)
where \( \delta \) and \( \theta \) are real numbers and \( \hat{u} \) is a real unit vector.\(^4\) By the exponential \( e^O \) of an operator \( O \) we mean the Taylor series \( \sum_n O^n/n! \) where \( O^0 = I \).

What is the determinant of the matrix representation of \( U \)? The subset of 2 × 2 unitary matrices with unit determinant is called the special unitary group \( SU(2) \). What is the version of eq. (12) that describes 2 × 2 special unitary operators?

You may wish to convince yourself of a factoid related to eq. (12), namely that if \( A \) is a square matrix of any order such that \( A^2 = I \), then \( e^{i\theta A} = \cos \theta \, I + i \sin \theta \, A \), provided that \( \theta \) is a real number. It follows that \( A \) can also be written in the exponential form
\[
A = e^{i\pi/2} e^{-i\pi/2} = e^{-i\pi/2} e^{i\pi/2}.
\]
(13)
\(^2\)W. Pauli, Zur Quantenmechanik des magnetischen Elektrons, Z. Phys. 43, 601 (1927),
\(^3\)The Pauli spin matrices (and the unit matrix \( I \)) are not only unitary, they are also hermitian, meaning that they are identical to their adjoints: \( \sigma_j^\dagger = \sigma_j \).
\(^4\)Note that if make the replacements \( \theta \rightarrow -\theta \) and \( \hat{u} \rightarrow -\hat{u} \) we obtain another valid representation of \( U \), since the physical operation of a rotation by angle \( \theta \) about an axis \( \hat{u} \) is identical to a rotation by \(-\theta \) about the axis \(-\hat{u}\).
There are several unitary operators of interest, such as the Pauli matrices, that are their own inverse. If we call such an operator $V$, then its exponential representation of $V$ can be written in multiple ways,

$$V = e^{i\delta} e^{i\hat{2} \cdot \sigma} = V^{-1} = e^{-i\delta} e^{-i\hat{2} \cdot \sigma}. \quad (14)$$

3. Give the explicit $4 \times 4$ matrix form of the four Dirac matrices $\gamma_\mu$,\(^5\) as well as that for $\gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3$, in their representation via the $2 \times 2$ Pauli matrices $I$ and $\sigma_i$, $i = 1, 2, 3$,

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad (15)$$

It should be then evident that $\text{tr}(\gamma_\mu) = 0 = \text{tr}(\gamma_5)$, where $\text{tr}$ is the trace operator. Then, it immediately follows that $\text{tr}(\phi) = 0$, where $\phi \equiv a^\mu \gamma_\mu$ and $a_\mu$ is an arbitrary 4-vector.

Show that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I_4, \quad (16)$$

where $\eta_{\mu\nu}$ has diagonal elements $1, -1, -1, -1$ and $I_4$ is the $4 \times 4$ unit matrix,\(^6\) and hence that

$$\text{tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu}, \quad \text{and} \quad \text{tr}(\phi \phi) = 4a_\mu b^\mu \equiv 4ab. \quad (17)$$

Show also that

$$\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \quad (18)$$

and hence that

$$\text{tr}(\phi \phi \phi \phi) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]. \quad (19)$$

A factoid which you need not demonstrate is that the Dirac equivalent of eq. (11) is

$$\phi \phi = abI_4 + \frac{a^\mu b^\nu}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \quad (20)$$

If you think that matrix manipulation is the key to physics, then you might enjoy my course, Physics of Quantum Computation,


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\(^5\)The matrices $\gamma_\mu$ were introduced by Dirac in the form used here, but with his $\gamma_4$ being our $\gamma_0$, in sec. 3 of The Quantum Theory of the Electron, Proc. Roy. Soc. London A 117, 610 (1928),

\(^6\)The matrix $I_4$ is typically denoted by 1.