1. Deduce the nonrelativistic form factors,

\[ F(q^2) = \int \rho(r) e^{iqr} d^3r, \quad (1) \]

for the spherically symmetric charge densities with characteristic radius \( R \),

\[ \rho(r) = \begin{cases} 
3Q/4\pi R^3 & (r < R), \\
0 & (r > R), 
\end{cases} \quad (2) \]

\[ \rho(r) = \frac{Q}{4\pi R^2} \delta(r - R), \quad (3) \]

and

\[ \rho(r) = \frac{Q}{2\sqrt{2\pi} R^3} e^{-r^2/2R^2}, \quad (4) \]

all of which have total charge \( Q \). Expand these form factors to order \( (qR)^2 \).

A neutral particle might have charge distributions \( \rho_+ \) and \( \rho_- \) with the above forms, but with different values of the characteristic radii \( R_+ \) and \( R_- \).

The data are often fit to the form,\(^1\)

\[ F_n(q^2) = \frac{Q}{[1 + (qR)^2]^n}, \quad (5) \]

with \( n = 2 \). What are the corresponding forms of the charge distributions \( \rho_n(r) \) for \( n = 1, 2 \) and 3?

2. **Arbitrary 2 × 2 Unitary Matrices and Pauli Spin Matrices**

This problem concerns operators that act on 2-component spinors. Such operators can be expressed as 2 × 2 matrices. Operators that preserve the normalization of a state are called unitary.

Two of the simplest unitary operators on 2-component spinors are the identify matrix \( I_2 = I \), and the spin-flip operator \( X \) (called the NOT operator in quantum computation),

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (6) \]

\(^1\)For a review of nucleon form factors, see C.F. Perdrisat et al., *Nucleon electromagnetic form factors*, Prog. Part. Nucl. Phys. 59, 694 (2007),

An arbitrary $2 \times 2$ unitary matrix $U$ can be written as
\[
U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
(7)
where $a$, $b$, $c$ and $d$ are complex numbers such that $UU^\dagger = I$. The decomposition (7) is somewhat trivial. Express the general unitary matrix $U$ as the sum of four unitary matrices, times complex coefficients, of which two are the classical unitary matrices $I$ and $X$ given above. Denote the “partner” of $I$ by $Z$ and the “partner” of $X$ by $Y$ such that
\[
XY = iZ, \quad YZ = iX, \quad ZX = iY.
\]
(8)
You have, of course, rediscovered the so-called Pauli spin matrices,\(^2,3\)
\[
\sigma_x (= \sigma_1) = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y (= \sigma_2) = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z (= \sigma_3) = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
(9)
As usual, we define the Pauli “vector” $\sigma$ as the triplet of matrices
\[
\sigma = (\sigma_x, \sigma_y, \sigma_z).
\]
(10)
Show that for ordinary 3-vectors $a$ and $b$,
\[
(a \cdot \sigma)(b \cdot \sigma) = (a \cdot b) I + i \sigma \cdot a \times b.
\]
(11)
With this, show that a general $2 \times 2$ unitary matrix can be written as
\[
U = e^{i\delta} \left( \cos \frac{\theta}{2} I + i \sin \frac{\theta}{2} \mathbf{u} \cdot \sigma \right) = e^{i\delta} e^{i\frac{\pi}{2} \mathbf{u} \cdot \sigma},
\]
(12)
where $\delta$ and $\theta$ are real numbers and $\mathbf{u}$ is a real unit vector.\(^4\) By the exponential $e^O$ of an operator $O$ we mean the Taylor series $\sum_n O^n/n!$ where $O^0 = I$.

What is the determinant of the matrix representation of $U$? The subset of $2 \times 2$ unitary matrices with unit determinant is called the special unitary group SU(2). What is the version of eq. (12) that describes $2 \times 2$ special unitary operators?

You may wish to convince yourself of a factoid related to eq. (12), namely that if $A$ is a square matrix of any order such that $A^2 = I$, then $e^{i\theta A} = \cos \theta I + i \sin \theta A$, provided that $\theta$ is a real number. It follows that $A$ can also be written in the exponential form
\[
A = e^{i\pi/2} e^{-i\pi/2} A = e^{-i\pi/2} e^{i\pi/2} A.
\]
(13)
\(^3\)The Pauli spin matrices (and the unit matrix $I$) are not only unitary, they are also hermitian, meaning that they are identical to their adjoints: $\sigma_j^\dagger = \sigma_j$.
\(^4\)Note that if make the replacements $\theta \to -\theta$ and $\mathbf{u} \to -\mathbf{u}$ we obtain another valid representation of $U$, since the physical operation of a rotation by angle $\theta$ about an axis $\mathbf{u}$ is identical to a rotation by $-\theta$ about the axis $-\mathbf{u}$. 

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There are several unitary operators of interest, such as the Pauli matrices, that are their own inverse. If we call such an operator $V$, then its exponential representation of $V$ can be written in multiple ways,

$$V = e^{i\delta} e^{i\frac{\vec{v} \cdot \sigma}{2}} = V^{-1} = e^{-i\delta} e^{-i\frac{\vec{v} \cdot \sigma}{2}}.$$  

(14)

3. Give the explicit $4 \times 4$ matrix form of the four Dirac matrices $\gamma_\mu$\(^5\) as well as that for $\gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3$, in their representation via the $2 \times 2$ Pauli matrices $I$ and $\sigma_i$, $i = 1, 2, 3$,

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$  

(15)

It should be then evident that $\text{tr}(\gamma_\mu) = 0 = \text{tr}(\gamma_5)$, where $\text{tr}$ is the trace operator. Then, it immediately follows that $\text{tr}(\phi) = 0$, where $\phi \equiv a_\mu \gamma_\mu$ and $a_\mu$ is an arbitrary 4-vector.

Show that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} I_4,$$  

(16)

where $\delta_{\mu\nu}$ has diagonal elements 1, $-1$, $-1$, $-1$ and $I_4$ is the $4 \times 4$ unit matrix,\(^6\) and hence that

$$\text{tr}(\gamma_\mu \gamma_\nu) = 4\delta_{\mu\nu}, \quad \text{and} \quad \text{tr}(\phi \phi) = 4a_\mu b_\mu \equiv 4ab.$$  

(17)

Show also that

$$\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}),$$  

(18)

and hence that

$$\text{tr}(\phi \phi \phi) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)].$$  

(19)

A factoid which you need not demonstrate is that the Dirac equivalent of eq. (11) is

$$\phi \phi = abI_4 + \frac{a^\mu b^\nu}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).$$  

(20)

If you think that matrix manipulation is the key to physics, then you might enjoy my course, Physics of Quantum Computation,


\(^5\)The matrices $\gamma_\mu$ were introduced by Dirac in the form used here, but with his $\gamma_4$ being our $\gamma_0$, in sec. 3 of The Quantum Theory of the Electron, Proc. Roy. Soc. London A 117, 610 (1928), http://physics.princeton.edu/~mcdonald/examples/QED/dirac_prsla_117_610_28.pdf.

\(^6\)The matrix $I_4$ is typically denoted by 1.