Pl. 406 Final Exam Solutions

1. 3 Nucleon Charges: +, - , 0

\[ A = N_+ + N_0 + N_- \quad \Rightarrow \quad Z = N_+ - N_- = \text{Total Charge} \]

a) Mass Formula: \( M = A \left( 1 - \frac{A}{3} \right) \)

\[ + b \frac{A^{2/3}}{A_0} \]

Loss of binding energy per nucleon in extended nuclear medium

\[ + \frac{A^{2/3}}{A_0} \]

Loss of binding energy at surface, whose area varies as \( A^{2/3} \)

\[ + \frac{2}{A_0} \]

Electrostatic energy of charge \( Z \) is

\[ \text{Sphere of radius } r \approx A^{1/3} \]

\[ \frac{Z}{A} \]

Symmetric energy

\[ \text{In Fermi-gas model of nucleus} \]

\[ \left( + \frac{Z}{A} \right) \]

[Total energy, which we ignore here]

Of these, only the symmetric energy might not be 'obvious', compared to the mass formula in our universe. (In fact, we find a slightly different form below...)

You may wish to recall the Fermi gas argument:

For each of the 3 nucleon types \( N_i \sim V_n E_{F_i} \sim \frac{A}{3} \)

or \( E_{F_i} \sim \left( \frac{N_i}{A} \right)^{1/3} = \text{Fermi energy of type } i \)

\[ U_i \equiv \frac{N_i}{A} E_{F_i} \sim \frac{N_i}{A} \frac{A}{3} = \text{Energy (mass) of type } i \text{ in the nuclear potential well} \]

Now \( N_i \sim A^{1/3} \) so write \( N_i \sim A^{1/3} + \left( N_i - A^{1/3} \right) = \frac{A}{3} \left( 1 + \epsilon_i \right) \)

\( \epsilon_i = \frac{3N_i - A}{A} \)

\[ U_i \sim N_i \frac{\epsilon_i^{1/3}}{A^{1/3}} \sim \frac{A^{1/3}}{A^{1/3}} \left( 1 + \epsilon_i \right)^{1/3} \sim A \left( 1 + \frac{\epsilon_i}{3} + \frac{1}{6} \left( \frac{\epsilon_i}{3} \right)^2 \epsilon_i^2 + \ldots \right) \]

\[ \sum_i U_i \sim A + \frac{\epsilon_i}{A} (3N_i - A)^2 \]

Notation: \( \sum_i \epsilon_i = 0 \)

From the definitions of \( A \)

\[ N_+ = \frac{A + Z - N_0}{2} \]

\[ N_- = \frac{A - Z - N_0}{2} \]

\[ 3N_+ - A = \frac{A}{2} + \frac{3Z}{2} - \frac{3N_0}{2} = \frac{(A - 3N_0) + 3Z}{2} \]

\[ 3N_- - A = \frac{A}{2} - \frac{3Z}{2} - \frac{3N_0}{2} = \frac{(A - 3N_0) - 3Z}{2} \]

\[ \sum_i (3N_i - A)^2 = \frac{3}{4} (A - 3N_0)^2 + 2\frac{Z}{A} \frac{Z}{A} + (3N_0 - A)^2 \]

So the symmetric term can be written as \( \sum_i \frac{Z^2 + (A - 3N_0)^2}{A} \)
1. \( b) \) At fixed \( A \), \( M(z, No) \approx M_0 + M_1 z^2 + M_2 (A - 3N_0)^2 \)

The minimum is clearly at \( z = 0 \); \( N_0 \approx A/3 \Rightarrow N_+ = N_- N_0 = A/3 \)

19. \( ^{2} \text{STABLE NUCLEI HAVE EQUAL NUMBERS OF ALL } z \text{ TYPES OF NUCLEI, AND ZERO (OR VERY SMALL) NET CHARGE} \)

\( 2 \approx 0 \Rightarrow \text{NO COULOMB BARRIER AGAINST FISSION, OR AGAINST} \)

\( ^{141} \text{DECAY WHERE} \ \ ^{141} = (2N_+)(2N_-)(2N_0) \)

**BUT ARE THESE DECAYS ALLOWED BY ENERGY CONSERVATION?**

For \( z < 0 \), \( N_0 \approx A/3 \), \( M(z) = A(1-a) + b A^{3/2} \)

**FISSION:** \( A \rightarrow A_+ + A_- \quad M(A_1) = \frac{A}{2} (1-a) + b A^{3/2} \)

\[ M(A) - M(A_1) = b A^{3/2} \left( 1 - \frac{2}{3} \frac{A}{A_0} \right) \]

**IF THE \( ^{141} \text{ WAS GROWN} \text{ ENERGY GIVEN Away} \quad M(^{141}) = b (1-a) + b b^{2/3} \)**

If the \( ^{141} \text{ HAS FINITE ENERGY GIVEN Away} \quad M(^{141}) = b (1-a) + b b^{2/3} \)

**AND** \( M(A) - M(A-C) \approx A/3 (1-a) + b (A-6)^{3/2} \)

\[ A/3 (1-a) + b (A-6)^{3/2} = A/3 (1 - \frac{6}{3} \frac{A}{A_0}) \]

\[ = (6A)^{3/2} (1-a) + b A^{3/2} - \frac{4b}{A^{3/2}} \]

\[ M(A) - M(^{141}) = b \left( \frac{4}{A^{3/2}} - 6 b^{3/2} \right) \]

**HIGH A NUCLEI ARE STABLE IN THIS UNIVERSE!**

**IN FACT IF TWO NUCLEI COLLIDE, IT IS ENERGETICALLY FAVORABLE THAT THEY COALESCE.**

**\( ^{141} \text{ STARS} \)** **CONSIST OF A SINGLE GIANT NUCLEUS, WITH \( z \approx 0 \)**

**THEY DON'T GLOW DUE TO ANY NUCLEAR PROCESSES (BUT LIKE A NEUTRON STAR.)**

So Life Is Bad In This Universe For Various Reasons:

- No STARS
- No Chemistry as \( z \approx 2 \) ALWAYS
- No Life...
2) a) For a rotational excitation, \( E = \frac{1}{2} \sum w_i^2 \) where \( \sum w_i^2 \) 

so \( E = \frac{\sum w_i^2}{2I} = \frac{3(3+1)}{2I} \) not the eigenvalues of \( \sum w_i^2 \)

Now a rotating sphere (or even ellipsoid) is symmetric under the parity transformation \( P \rightarrow -P \Rightarrow P = + \) only. But \( P \neq (-1)^I \) for reflections, so have \( \text{even } I \) only.

Hence \( I^+ = 0^+, 2^+, \ldots, (2J)^+ \) is expected sequence of states.

Also, expect \( I = 0, 2, 4, 6, 8 \)

\[ 2I \cdot E = 3(3+1) \]

\[ 0 \quad 6 \quad 20 \quad 42 \quad 72 \]

This sequence of energies is fairly well matched to the data, with \( 2I \cdot 100 \approx 6 \text{ (keV)}^{-1} \) or \( I \approx 0.03 \text{ (keV)}^{-1} \).

Compare \( I \approx \frac{2}{5} M R^2 \) for \( A = 170 \)

\[ M = 170 \cdot 930 \text{ MeV} \]

\[ R = 1.1 \text{ A}^{1/3} \text{ fermi} \approx 6 \text{ fm} \]

\[ I = (0.4) (170)(933) 36 \text{ MeV \cdot fermi}^2 \]

\[ \text{Now } 1 \text{ fermi}^2 \approx \frac{1}{200 \text{ MeV}} \]

So \( I = (0.4) (170)(933)(36) \approx 5000 \text{ MeV} \cdot \text{fermi}^2 \approx 0.05 \text{ (keV)}^{-1} \)

Thus \( I \) model \( < I \) model suggests the whole nucleus is not rotating as a rigid body...

Equivalently \( I \) model \( \approx \frac{5000}{1 \text{ MeV}} = 10^4 \text{ fermi}^2 \)

\[ I \text{ data} \approx 5 \times 10^3 \text{ fermi}^2 \]
In the shell model the ground state has neutrons filling the $1s_{1/2}$, $1p_{3/2}$, and $1p_{1/2}$ shells.

Protons fill the $1s_{1/2}$ and $1p_{3/2}$ shells, but one vacancy in the $1p_{1/2}$.

The unpaired proton determines the ground state to be $\frac{1}{2}^-$.

1st excited state is $\frac{1}{2}^+$, consistent with the unpaired proton excited to the $1d_{5/2}$ shell.

2nd excited state is $\frac{1}{2}^+$: now the unpaired proton is in the next higher shell, $2s_{1/2}$.

3rd excited state is $\frac{3}{2}^-$; if the unpaired proton were in the next higher shell, $1d_{3/2}$, would expect $\frac{3}{2}^+$!

Instead, could be that a proton from the $1p_{3/2}$ shell is excited into the $1p_{1/2}$ filling the latter. Then the unpaired $1p_{3/2}$ proton determines the state as $\frac{3}{2}^-$. 

4th excited state is $\frac{3}{2}^+$ could be that a neutron from the filled $1p_{1/2}$ shell is excited to the $1d_{5/2}$

Turn the unpaired proton & neutron in the $1p_{1/2}$ together, give $3p_{3/2}$, and the excited neutron determines the state to be $\frac{3}{2}^+$. 

5th excited state is $\frac{3}{2}^+$. The unpaired proton in the $1p_{3/2}$ is excited to the $1d_{3/2}$...

\[ q'(3685) \]
\[ e^+ \rightarrow \gamma \] \[ e^- \] \[ \gamma \text{ has } J=0 \text{ by hypothesis} \]
\[ \gamma \text{ has } J^z=\pm 1 \text{ only}, \text{ as is real photon} \]
\[ \Rightarrow \text{ total } J_2' = \pm 1 \text{ only} \]

So angular distribution is given by projection $J_2 = \pm 1$ onto $J_2' = \pm 1$.

\[ \Delta_2(\theta) \sim |D_{11}|^2 + |D_{1-1}|^2 + |D_{-1,1}|^2 + |D_{-1,-1}|^2 \]
\[ \sim \left( \frac{1 + \cos \theta}{2} \right)^2 + \left( \frac{1 - \cos \theta}{2} \right)^2 \sim 1 + \cos \theta \]
USE ISOSPIN ANALYSIS. BUT WITH SOME CAUTION, ISOSPIN IS NOT CONSERVED IN THE WEAK INTERACTION $c\bar{u} \rightarrow s\bar{d}$!

Indeed $c\bar{u}$ has $I_3 = -\frac{1}{2}$ (since $c\bar{u}$ has $I_3 = -\frac{1}{2}$)

while $s\bar{d}$ has $I_3 = \frac{1}{2}$

But ISOSPIN IS CONSERVED IN THE STRONG INTERACTION $s\bar{d} \rightarrow s\bar{q}_R q_d$

so we can write $(s\bar{d}) \rightarrow c_i (s \bar{u})_i (u \bar{d})_i + C_2 (s \bar{d}) \rightarrow (u \bar{d})_i (u \bar{d})_i$.

$K^- \rightarrow \pi^+$

or $\frac{1}{2}, \frac{1}{2} \rightarrow \left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \end{array}\right)$

using the CLEBSCH-GORDAN CRIG SHEET for $c_i \bar{d}$.

Thus $\Gamma(D^0 \rightarrow K^- \pi^+) = \frac{|\langle \frac{1}{2}, \frac{1}{2} | \cdot \frac{1}{2}, \frac{1}{2} \rangle|^2}{(\frac{1}{2})^2} = 4$

And another CABIBB - FAUROED DIAGRAM is $D^0 \rightarrow K^- \pi^+$

And also $D^0 \rightarrow K^- \pi^+$

These MM WELL HAVE COMPARABLE STRENGTHS, AND SO TEND TO REDUCE THE RATIO OF BRANCHING RATIOS TO LESS THAN 4....
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Cases 1 & 4 have $J_z = J_z'$ both 0, so expect isotropic angular distribution.

Case 2 has $J_z = -1$, $J_z' = -1$

Case 3 has $J_z = 1$, $J_z' = 1$

So expect angular dist $\sim \frac{1}{12} \left( \frac{1}{r^2} \right)$

Integrating over angle $\int_0^{2\pi} 120 \frac{1}{r^2} = \frac{1}{4} \left( \frac{2 + \frac{2}{3}}{\frac{1}{3}} \right) = \frac{2}{3}$

Compared to $\int_0^{2\pi} 120 = 2$.

I.e., cases 2 & 3 have $\frac{1}{3}$ the cross section of 1 & 4.

To get the vertex factors, consider the diagram:

$$\text{AMPLI} = \frac{g}{c^2 \theta W} \left( I_3 - \frac{1}{2} \theta W Q \right) V_{e_L} \left( \frac{1}{M^2} \right) \left( I_3 - \frac{1}{2} \theta W Q \right) e_R$$

Propagator:

$$\sim \frac{G}{\sqrt{2}} \left( \frac{1}{2} \right) \left\{ \begin{array}{l} -\frac{1}{2} - (\theta W Q)(-1) \quad \text{for} \ e_L \\ 0 - (\theta W Q)(-1) \quad \text{for} \ e_R \end{array} \right.$$  

Using $I_3 (\nu \nu \bar{\nu}) = \frac{1}{2}$

$I_3 (e_L) = -\frac{1}{2}$

$I_3 (e_R) = 0$

Weak Isospin.

$\angle \sim \alpha (\text{AMPLI}) \cdot (\text{ANGULAR FACTOR}) \cdot (\text{CM ENERGY})^2$  

By dimensional analysis, $\angle \sim \frac{G}{\sqrt{2}} (\text{AMPLI}) \sim G$.

So $\angle \sim G \sqrt{c_m} \left\{ \left( \frac{1}{2} - (\theta W Q) \right)^2 + \frac{1}{3} (\theta W Q)^2 \quad \text{for} \ nu \ e_L \right.$

$\left. \left( \frac{1}{3} \left( \frac{1}{2} - (\theta W Q) \right)^2 + (\theta W Q)^2 \quad \text{for} \ nu \ e_R \right. \right.$

$\uparrow \quad \uparrow$

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I.e., there is an overall factor of $\frac{4}{11}$ missing from the quick estimate.