PRINCETON UNIVERSITY
Ph304 Problem Set 1
Electrodynamics

(Due 5 pm, Tuesday Feb. 11, 2003 in Sullivan’s mailbox, Jadwin atrium)

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Problem sessions: Sundays, 7 pm, Jadwin 303

Text: Introduction to Electrodynamics, 3rd ed.
Errata at http://academic.reed.edu/physics/faculty/griffiths.html
Reading: Griffiths chap.1 as needed, secs. 2.1-2.3.

1. Griffiths’ prob. 1.60.

2. Griffiths’ prob. 1.62. In part a), comment on whether \( \nabla \cdot r^n \hat{r} \) is meaningful at the origin, by using the divergence integral theorem for a sphere of radius \( a \), and for a spherical shell of inner radius \( b \) and outer radius \( a \).

3. Variant of Griffiths’ prob. 2.7. After working Griffiths’ prob. 2.7 you are meant to be impressed at how effective Gauss’ law (2.13-14) is for problems of high symmetry. But since you already know Gauss’ law, it may be more instructive to work a variant: Find the electric potential \( V(z) \) relative to infinity everywhere along the axis of symmetry (the \( z \) axis) of a HEMISPHERICAL shell of radius \( R \) with uniform charge density \( \sigma \). In particular, what is \( V(z = 0) \) at the center of curvature of the shell. Then use eq. (2.23) to find the electric field \( E_z(z) \). Show that the value \( E_z(z) - E_z(-z) \) based on a hemispherical shell corresponds to the field \( E_z \) at a distance \( z \) from the center of a uniform spherical shell of charge.

4. Griffiths’ prob. 2.18.

5. Griffiths’ prob. 2.47.

The following digression is not part of the curriculum of Ph304, but you might find it interesting.

Electric potential problems in two dimensions can often be usefully related to functions of a complex variable, \( z = x + iy \). In particular, any analytic function \( f(z) = u + iv \) obeys

\[
\begin{align*}
 i \frac{\partial f}{\partial x} &= if' \frac{\partial z}{\partial x} = if' = i \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}, \\
 \frac{\partial f}{\partial y} &= f' \frac{\partial z}{\partial y} = if' = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}.
\end{align*}
\]

Since both of the above lines are equal to \( if' \), we can equate their real and imaginary parts to find the so-called Cauchy-Reimann relations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

Taking second derivatives, we also find

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.
\]

Thus, both functions \( u(x, y) \) and \( v(x, y) \) obeys Laplace’s equation, \( \nabla^2 V = 0 \) for the electric potential in a charge-free region in two dimensions.

Hence, any analytic function of a complex variable provides us with not one but two solutions to electrostatics problems. Mathematics hands us the solutions; the game is to figure out what the problem is.....
From the functions $u$ and $v$ we can, of course, deduce the associated electric fields $\mathbf{E}_u = -\nabla u$ and $\mathbf{E}_v = -\nabla v$. In general, lines of electric field are orthogonal to their corresponding equipotential surfaces.

Note that the Cauchy-Riemann equations imply that the lines of the field $\mathbf{E}_u$ are orthogonal to the lines of the field $\mathbf{E}_v$. Hence the equipotentials of field $\mathbf{E}_u \ (= \text{lines of constant } u)$ lie along the lines of field $\mathbf{E}_v$, and vice versa.

So the use of complex functions for two dimensional problems can give us quick prescriptions for both equipotentials and field lines.

Example: The function defined by the inverse relation $z = f + e^f$ describes the equipotentials and electric fields of a (semi-infinite) parallel plate capacitor. Can you show that the plates are at $-\infty < x < -1$ and $y = \pm \pi$?

From sec. 202 of A Treatise on Electricity and Magnetism by J.C. Maxwell.

Example: The function $f = -2\lambda \ln(z - z_0)$ describes the potential and field due to a line charge $\lambda$ located at $(x_0, y_0)$, where, of course, $z_0 = x_0 + iy_0$. From this, we see that the situation of Griffiths’ prob. 2.47 is described by the function

$$f(z) = -2\lambda \ln \frac{z - a}{z + a}.$$  

Then, $Re(f) = V(x, y)$ can be used to show that the equipotentials are circles, AND by considering $Im(f) = \text{constant}$, you can show that the electric fields lines are also circles (which always include the wires at $(\pm a, 0)$). See the figure on p. 3.

It turns out the equipotentials of two wires carrying opposite line charges have the same form as the magnetic field lines of two wires carrying opposite currents:

From sec. 61.2 of *Electromagnetic Fields and Interactions* by R. Becker.

Griffiths’ prob. 2.52 is not assigned, but summarizes a famous bit of lore. The derivation of eq. (2.57) given in the book of Smythe is elegantly algebraic. For a highly geometric derivation due to Lord Kelvin, see http://puhep1.princeton.edu/~mcdonald/examples/ellipsoid.pdf