“Hidden” Momentum in a Current Loop
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(June 30, 2012; updated April 30, 2014)

1 Problem

Discuss the momentum of a loop of electrical current \(I\) that is subject to a uniform external electric field \(E\) in the plane of the loop.\(^1\)

Does the current contain hidden momentum, \(P\)\(\text{hidden}\), defined for a subsystem by\(^2\)

\[
P\text{hidden} \equiv P - Mv_{\text{cm}} - \oint_{\text{boundary}} (x - x_{\text{cm}})(p - \rho v_b) \cdot d\text{Area}, \tag{1}
\]

where \(P\) is the total momentum of the subsystem, \(M = U/c^2\) is its total “mass,” \(c\) is the speed of light in vacuum, \(U\) is its total energy, \(x_{\text{cm}}\) is its center of mass/energy, \(v_{\text{cm}} = dx_{\text{cm}}/dt\), \(p\) is its momentum density, \(\rho = u/c^2\) is its “mass” density, \(u\) is its energy density, and \(v_b\) is the velocity (field) of its boundary?

This problem was first considered by J.J. Thomson in 1904 [3, 4, 5, 6]. The present version first appeared on p. 215 of [7]. See also [8, 9, 10, 11, 12, 13, 14].

2 Solution

If a charge \(e\) of rest mass \(m\) is moving with (variable) velocity \(v\) in (variable) electric potential \(V\) its total energy has the constant value\(^3\)

\[
U = \gamma mc^2 + eV, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{2}
\]

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\(^1\)The current is not “shielded” from the external electric field, and does not correspond to the current in either a resistive or superconducting wire.

\(^2\)The definition (1) was suggested by Daniel Vanzella [1]. See also [2].

\(^3\)The expression (2) is not gauge invariant. However, we will only be concerned with comparison of energies of charges at two different potentials, \(\Delta KE = mc^2 \Delta \gamma = -e \Delta \Phi\). For additional remarks on eq. (2), see [15].
In the present example, the change in a charge’s mechanical energy when moving distance \( w \) “upwards” through this field is
\[
\Delta U_{\text{mech}} = m(\gamma_t - \gamma_b)c^2 = eEw = e\Delta V, \tag{3}
\]
where \( t \) and \( b \) refer to the top and bottom of the loop, \( w \) is its height, and \( \Delta V \) is the difference in the external electric potential between the bottom and the top of the loop.\(^4\)

We now consider the system to contain three subsystems, the circulating charges, the electromagnetic fields (which include both the external electric field and the fields of the charges), and other mechanical apparatus at rest in the lab frame, including the sources of the external electric field and a nonconducting tube that constrains the charges, without friction, to move in the rectangular circuit.

The electrical current of the circulating charges in the bottom and top segments of the loop is given by
\[
I = en_bv_b = en_tv_t, \tag{4}
\]
where \( n \) is the number of moving charges per unit length. The total momentum of the charges can now be written
\[
P_{\text{charges}} = (n_t lm\gamma_t v_t - n_b lm\gamma_b v_b) \hat{x} = \frac{II\Delta U_{\text{mech}}}{c^2e} \hat{x} = \frac{IlwE}{c^2} \hat{x} = \frac{m \times E}{c}, \tag{5}
\]
noting that the magnetic moment \( m \) of the loop of circulating charge is given (in Gaussian units) by
\[
m = \frac{I\text{Area}}{c} = -\frac{Ilw}{c} \hat{z}, \tag{6}
\]
where \( \hat{z} \) is out of the paper.

The center-of-mass velocity of the charges is given by
\[
M_{\text{charges}} = (n_t lm\gamma_t v_t - n_b lm\gamma_b v_b) \hat{x} = P_{\text{charges}}, \tag{7}
\]
where \( M_{\text{charges}} \) is the sum of the masses of all charges. This result is peculiar in that the tube that constrains the charges is at rest, and the charge distribution within that tube is stationary, from a macroscopic perspective. Equation (7) corresponds to a microscopic view in which charges enter and leave each segment of the loop, such that the center of mass of the charges in, say, the top segment moves to the right at speed \( v_t \) until a charge leaves on the right and another enters from the left; at that moment the center of mass instantaneously moves to the left by distance \( d_t/n_t l \), where \( d_t \) is the spacing between charges in the top segment. That is, the motion of the microscopic center of mass is a Zitterbewegung.

It is more appropriate to provide a macroscopic description, in which we can speak of a steady current \( I \), rather than a series of delta functions of current spaced at small time intervals. The macroscopic description is obtained by averaging over distances larger than the gaps \( d \) between the moving charges. In the macroscopic description the Zitterbewegung

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\(^4\)The increase in the mechanical energy of the charges is compensated by a reduction in the energy of the electric field. In particular, the interference energy between the external field and the field of the energized charges is negative, as discussed in [16].
is ignored, and we set the macroscopic center-of-mass velocity \( \bar{v}_{\text{cm, charges}} \) of the charges to zero,

\[ \bar{v}_{\text{cm, charges}} = 0. \]  

(8)

The subsystem of charges is then said to contain “hidden” momentum according to the definition (1),

\[ P_{\text{hidden, charges}} = P_{\text{charges}} - M_{\text{charges}} \bar{v}_{\text{cm, charges}} = P_{\text{charges}} = \frac{m \times E}{c}. \]  

(9)

The subsystem of the electromagnetic fields has nonzero momentum,

\[ P_{\text{EM}} = \int \frac{E \times B}{4\pi c} \, d\text{Vol} = \int \frac{J V}{c^2} \, d\text{Vol} = \frac{E \times m}{c} = -P_{\text{charges}}, \]  

(10)

where the third and fourth forms of eq. (10) are due to Furry [8].\(^5\) Thus, the total momentum of the system is zero, whereas the total momentum of the system of circulating rockets was nonzero.

The center of energy of the electromagnetic fields is at rest in the lab frame, so \( v_{\text{cm, EM}} = 0 \). Hence, according to definition (1), the electromagnetic subsystem has nonzero “hidden” momentum,

\[ P_{\text{hidden, EM}} = P_{\text{EM}} - M_{\text{EM}} v_{\text{cm, EM}} = P_{\text{EM}} = \frac{E \times m}{c} = -P_{\text{charges}} = -P_{\text{hidden, charges}}, \]  

(11)

where \( M_{\text{EM}} \equiv \frac{U_{\text{EM}}}{c^2} \).

The key result of this problem is the clarification that the center-of-mass velocity to be used in application of definition (2) is the macroscopic average velocity, in the same sense that the electrical current \( I \) is a macroscopic average. Then, mechanical momentum which is explicitly noted in a microscopic description can, in some cases, be characterized as “hidden” in the macroscopic description.

### 3 Transient Effects

In the preceding the current and external electric field were assume to be steady. We now consider several possibilities of transient behavior, in which the current varies, or in which the electric field varies.\(^6,7\)

\(^5\) The result (10) appears on pp. 347-348 of [4] and in sec. 285 of [5], but Thomson appears not to have noted the paradox that a system “at rest” has nonzero field momentum. For discussion of these early results, see [6].

\(^6\) The example of an electric charge plus magnetic dipole is a variant on the configuration of the Aharonov-Bohm effect [17], in which an electric charge moves outside a long solenoid magnet where the external field \( B_{\text{solenoid}} \) is negligible but the external vector potential \( A_{\text{solenoid}} \) is nonzero. For discussion by the author of classical aspects of the Aharonov-Bohm effect, see [18].

\(^7\) An exotic possibility, that the entire system collapses to a black hole, has been considered in [19].
3.1 The Current is Excited/De-excited

Suppose the external electric field is static, but the current in the loop is created or destroyed by a transient, external magnetic flux though the loop. The transient electric field around the loop modifies the electrical current in the loop, doing work to create/destroy the magnetic field energy of the loop (and the small kinetic energy of the circulating charges). In this process, equal and opposite amounts of mechanical and electromagnetic momentum are given/taken away from the system of loop plus external field, so no net force is exerted on that system during the transient phase.

3.2 The External Field is Due to a Single, Moving Charge

Suppose the external electric field is due to a single electric charge \( q \) (with no magnetic moment) that has velocity \( \mathbf{v} \) with \( v \ll c \), while the magnetic moment \( \mathbf{m} \) remains constant. This charge is subject to the Lorentz force of the magnetic dipole field of the current loop, which force does no work on the charge, such that the speed \( v \), the relativistic mass \( m_q/\sqrt{1 - v^2/c^2} \), and the kinetic energy of the charge remain constant. However, the direction of the velocity \( \mathbf{v} \) changes with time, so the charge is accelerated and radiates energy and momentum. An issue raised by [19] is whether the total radiated momentum is significant compared to \( P_{EM} = qm \sin \theta/cr^3 \), where \( \theta \) is the angle between \( \mathbf{m} \) and the line joining \( q \) and the location of the magnetic dipole \( \mathbf{m} \), and \( r \) is the length of that line.

In the following we will only consider cases where \( \theta = 90^\circ \); that is, the charge moves in the plane perpendicular to the constant magnetic moment \( \mathbf{m} \). We also suppose that the mass of the current loop is much larger than the (rest) mass \( m_q \) of the charge, so that it is a good approximation to consider the current loop to be at rest, and at the origin.

3.2.1 The Charge Moves Radially from \( r_0 \) to \( \infty \)

Initially, the charge is at radius \( r_0 \) with respect to the magnetic moment \( \mathbf{m} \), and its small velocity is radially outwards, \( \mathbf{v} = v \hat{\mathbf{r}} \). We also suppose that \( v \) is small enough that the trajectory of the charge is nearly radial at all times.

The Lorentz force on the moving charge \( q \) is

\[
F_q = \frac{dP_q}{dt} = q \frac{\mathbf{v}}{c} \times \mathbf{B}_m = q \frac{\mathbf{v}}{c} \times \left( \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}}{r^3} \right) = q \frac{\mathbf{m} \times \mathbf{v}}{cr^3},
\]

(12)

for motion of \( q \) in the plane perpendicular to \( \mathbf{m} \). Hence, if \( \mathbf{v} \approx v \hat{\mathbf{r}} \), then the azimuthal component of eq. (12) is

\[
\frac{dP_\phi}{dt} \approx \frac{qmv}{cr^3}.
\]

(13)

The radial component of the momentum of the charge is \( P_r \approx m_q v \), so the rate of change of the angle \( \phi \) of the charge’s trajectory with respect to the radial direction is

\[
\frac{d\phi}{dt} \approx \frac{1}{P_r} \frac{dP_\phi}{dt} \approx \frac{qm}{cm_q r^3}.
\]

(14)
For nearly radial motion, $dr \approx v \, dt$, so that the total change $\Delta \phi$ in the azimuthal orientation of the trajectory of the charge as it moves from $r_0$ to $\infty$ is related by

$$\frac{d\phi}{dr} \approx \frac{1}{v} \frac{d\phi}{dt} \approx \frac{q m}{v m_q c^2 r^3}, \quad \Delta \phi = \int_{r_0}^{\infty} \frac{d\phi}{dr} \, dr = \int_{r_0}^{\infty} \frac{c}{v} \frac{q m}{m_q c^2 r^3} \, dr = \frac{c}{v} \frac{q m}{2 m_q c^2 r_0^2}.$$  

(15)

Hence, the condition for quasiradial outward motion is

$$\frac{q m}{m_q c^2 r_0^2} \ll \frac{v}{c} \ll 1 \quad \text{(quasiradial outward motion).}$$  

(16)

As discussed in sec. 2.2.6 of [20], rate of radiation of momentum by the accelerated charge is

$$\frac{dP_{\text{rad}}}{dt} = \frac{dU_{\text{rad}}}{dt} \frac{v}{c^2} = \frac{2q^2 F_q^2}{3m_q c^6} \approx \frac{2q^4 m^2 v^3}{3m_q c^7 r_0^6},$$  

(17)

recalling the Larmor formula for $v \ll c$ that $dU_{\text{rad}}/dt = 2q^2 a^2/3c^3 = 2q^2 F_q^2/3m_q c^3$. The total radiated momentum $P_{\text{rad}}$ in the radial direction, follows as

$$\frac{dP_{\text{rad}}}{dr} \approx \frac{1}{v} \frac{dP_{\text{rad}}}{dt} \approx \frac{2q^4 m^2 v^2}{3m_q c^7 r_0^6}, \quad P_{\text{rad}} = \int_{r_0}^{\infty} \frac{dP_{\text{rad}}}{dr} \, dr \approx \frac{2q^4 m^2 v^2}{15m_q c^7 r_0^6}.$$  

(18)

The initial electromagnetic momentum of the system follows from eq. (10) as

$$P_{\text{EM}} = \frac{q m}{c r_0^2},$$  

(19)

so that the ratio of the radiated momentum to the initial electromagnetic momentum is

$$\frac{P_{\text{rad}}}{P_{\text{EM}}} \approx \frac{q^3 m v^2}{m_q c^5 r_0^3} = \frac{v^2}{c^2} \frac{q m}{m_q c^2 r_0^2} \frac{q^2}{m_q c^2} \ll \frac{v^3}{c^3} \frac{q^2}{m_q c^2} \frac{r_0}{r_0}.$$  

(20)

For $r_0$ large compared to the classical charge radius $q^2/m_q c^2$, the ratio (20) is negligible.

In sum, as the charge moves outwards quasiradially at constant speed, the electromagnetic field momentum and the net mechanical momentum of the electric current drop to zero, remaining equal and opposite at all times in the quasistatic approximation.

3.2.2 The Charge Orbits the Magnetic Dipole

For magnetic moment $m = m \hat{z}$, a positive electric charge $q$ can move counterclockwise in a circular orbit of radius $r$ with speed $v$ given by the Lorentz force as

$$\frac{v}{c} = \frac{q m}{m_q c^2 r^2},$$  

(21)

when $v/c \ll 1$. We suppose that the initial radius $r_0$ is large enough that $v/c \ll 1$.

As noted, for example, in sec. 2.5.5 of [20], the accelerated charge radiates total power

$$\frac{dU_{\text{rad}}}{dt} = \frac{2q^4 (v/c \times B)^2}{3m_q^2 c^3} = \frac{2q^4 m^2 v^2}{3m_q^2 c^5 r^6} = \frac{2q^6 m^4}{3m_q^4 c^7 r^{10}}.$$  

(22)
The radiation produces a back reaction on the charge which can be characterized by the radiation reaction force $F_{\text{react}} = -Kv$, such that the power delivered by this force is the negative of the radiated power,

$$F_{\text{react}} \cdot v = -Kv^2 = -\frac{dU_{\text{rad}}}{dt}, \quad (23)$$

and hence

$$F_{\text{react}} = -Kv \quad \text{with} \quad K = \frac{2q^4m^2}{3m_q^2c^5r^6}. \quad (24)$$

As a consequence, the charged particle slows down according to

$$m_q \frac{dv}{dt} = F_{\text{react}} = -Kv, \quad v = v_0 e^{-Kt/m_q}, \quad r = \sqrt{\frac{qm}{m_qcv}} = r_0 e^{Kt/2m_q}, \quad (25)$$

and its trajectory spirals slowly out to infinite radius where the asymptotic velocity is zero. During this process the electromagnetic field momentum and the net mechanical momentum go to zero, and the initial kinetic energy of the charge is transformed into electromagnetic radiation. Momentum is radiated by the accelerating charge, but the net radiated momentum over the large number of turns in the spiral trajectory is negligible, so there is no significant “kick” imparted to the magnetic dipole.\(^9\)

### 3.2.3 General Motion of the Charge

An analytic solution for the general motion of the charge is not possible, but electromagnetic radiation is generated as the charge accelerates, which reduces the kinetic energy and the speed of the charge. The charge either comes to rest at some finite distance from the magnetic moment, or moves to infinity with asymptotic speed either zero or nonzero. In the first case the final electromagnetic field momentum and final net momentum of the current are nonzero, equal and opposite, and smaller than their initial values. In the second case the final electromagnetic field momentum and final net momentum of the current are zero. In no case does anything striking take place.

### 3.2.4 The Charge Collides with the Magnetic Dipole

A special case is that the electric charge $q$ has initial velocity $v_0 \perp m$ such that it eventually collides (totally inelastically) with the magnetic dipole $m$. Just before this collision the charge has velocity $v_f (\perp m)$. Suppose the magnetic dipole has initial velocity $-v_f m_q/m$, such that the final state of charge + moment has zero (mechanical) velocity.

In the initial state there is electromagnetic momentum (and electromagnetic energy) associated with the separate motions of the electric charge and of the magnetic dipole.

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\(^8\)Equation (24) also follows from the well-known form $F_{\text{react}} = (2q^2/3c^3) \dot{\mathbf{a}} = (2q^2/3c^3) \mathbf{\omega} \times \mathbf{a} = (2q^2/3c^3) \left(-v \hat{z}/r \times (-v^2/r) \mathbf{\hat{r}} \right) = -(2q^2v^2/3c^3r^2) \mathbf{v} = -(2q^4m^2/3m_q^2c^3r^6) \mathbf{v}$.

\(^9\)The radiation fields transfer net energy and momentum to the magnetic dipole (including net work done by magnetic field [21]), but this effect is negligible in the present case.
We consider that these momenta (and energies) are “renormalized” into the “mechanical” momenta (and energies) of the charge and dipole. In addition, there is a nonzero interaction field momentum in the initial state, which is well approximated for low velocities by eq. (10), $P_{EM, int} \approx E_q \times m/c$. And to the same approximation, the initial currents of the magnetic dipole contain hidden mechanical momentum, $P_{hidden, mech} \approx -P_{EM, int} \approx -E_q \times m/c$.

As the electric charge approaches the magnetic dipole the electric field experienced by the latter grows large, so the interaction field momentum and the hidden mechanical momentum grow large, and are not necessarily well approximated by the forms in the preceding paragraph. However, in the final state, where we assume that the electric charge comes to rest at the center of the current loop of the magnetic dipole, the final interaction field momentum and the final hidden mechanical momentum are zero.

As the charge moves towards the magnetic dipole they both accelerate, emitting electromagnetic energy and momentum. The electric charge radiates much more than does the magnetic dipole, assuming that the latter’s mass is much larger. Then, the final state contains net radiated momentum, whose direction is roughly along the line from the initial position of the charge to the dipole. Conservation of momentum tells us that the final state of the charge + dipole is not at rest, but has constant velocity, and mechanical momentum equal and opposite to the net radiated momentum.

An observer who does not consider the radiated momentum would say that the final state of charge + dipole has an unexpected “kick.” While consideration of electromagnetic radiation shows that this “kick” has nothing to do with “hidden momentum” and everything to do with radiation, the naïve observer might associate it with “hidden momentum.”

A Appendix: An All-Mechanical Current Loop

As suggested by J. Franklin (private communication), we can imagine the figure on p. 1 corresponds to a set of runners on a racetrack. On the left, vertical leg of the track the runners convert some of their internal energy into increased kinetic energy, ideally in a reversible manner (such that on the right, vertical leg the runners convert some of their kinetic energy back into internal energy). In this idealization the total energy $m \gamma c^2$ of each runner is constant in time; the rest mass $m$ decreases when the kinetic energy increases, and vice versa. So, if the runners are distributed such that the “current” is constant around the track,

$$I = n_b v_b = n_t v_t,$$

then in the frame of the track the total momentum of the runners is

$$P_{runners} = (n_t l m_t \gamma_t v_t - n_b l m_b \gamma_b v_b) \hat{x} = 0.$$

Hence, there is no “hidden” momentum in this all-mechanical example.

As remarked in footnote 7 of [2], it appears that “hidden” momentum exists only in examples where (moving) “matter” interacts with a “field.”
References


http://arxiv.org/abs/1303.7456
