“Hidden” Momentum in a Current Loop
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(June 30, 2012; updated October 11, 2017)

1 Problem

Discuss the momentum of a loop of electrical current $I$ that is subject to a uniform external electric field $E$ in the plane of the loop.\(^1\)

\[ P_{\text{hidden}} \equiv P - M \mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) \cdot (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\text{Area}, \]

(1)

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass,” $c$ is the speed of light in vacuum, $U$ is its total energy, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, $u$ is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary.

This problem was first considered by J.J. Thomson in 1904 [3, 4, 5, 6]. The present version first appeared on p. 215 of [7]. See also [8, 9, 10, 11, 12, 13, 14].

2 Solution

If an electric charge $q$ of rest mass $m_q$ is moving with (variable) velocity $\mathbf{v}$ in an external electric field $\mathbf{E}$, its kinetic energy changes according to the work done on charge $q$,

\[ q \int_{i}^{f} \mathbf{E} \cdot d\mathbf{l} = \Delta \text{KE} = \Delta U_{\text{mech}} = m_q(\gamma_f - \gamma_i) \quad \text{where} \quad \gamma = 1/\sqrt{1 - v^2/c^2}, \]

(2)

independent of the possible presence of an external magnetic field $\mathbf{B}$. In the present example, let $\hat{x}$ point to the right in the figure above, with $\hat{y}$ upwards, such that the electric field is

---

\(^1\)The current is not “shielded” from the external electric field, and does not correspond to the current in either a resistive or superconducting wire.

\(^2\)The definition (1) was suggested by Daniel Vanzella [1]. See also [2].
\( E = E \hat{y} \). Then, the change in a charge’s mechanical energy when moving distance \( w \) upwards through this field is

\[
\Delta U_{\text{mech}} = m_q (\gamma_t - \gamma_b)c^2 = qEw, \tag{3}
\]

where \( t \) and \( b \) refer to the top and bottom of the loop, \( w \) is its height.\(^3\),\(^4\)

We now consider the system to contain three subsystems, the circulating charges, the electromagnetic fields (which include both the external electric field and the fields of the charges), and other mechanical apparatus at rest in the lab frame, including the sources of the external electric field and a nonconducting tube that constrains the charges, without friction, to move in the rectangular circuit.

The electrical current of the circulating charges in the bottom and top segments of the loop is given by

\[
I = qn_b v_b = qn_t v_t, \tag{4}
\]

where \( n \) is the number of moving charges per unit length. The total momentum of the charges can now be written

\[
P_{\text{charges}} = (n_t l m_q \gamma_t v_t - n_b l m_q \gamma_b v_b) \hat{x} = \frac{I I \Delta U_{\text{mech}}}{c^2 q} \hat{x} = \frac{I l w E}{c^2} \hat{x} = \frac{m \times E}{c}, \tag{5}
\]

noting that the magnetic moment \( m \) of the loop of circulating charge is given (in Gaussian units) by

\[
m = \int \frac{r \times J}{2c} d\text{Vol} = \frac{I \text{Area}}{c} = -\frac{I l w}{c} \hat{z}, \tag{6}
\]

where \( \hat{z} \) is out of the paper.

The center-of-mass velocity of the charges is given by

\[
M_{\text{charges}} \mathbf{v}_{\text{cm,charges}} = (n_t l m_q \gamma_t v_t - n_b l m_q \gamma_b v_b) \hat{x} = P_{\text{charges}, \hat{x}}, \tag{7}
\]

where \( M_{\text{charges}} \) is the sum of the masses of all charges. This result is peculiar in that the tube that constrains the charges is at rest, and the charge distribution within that tube is stationary, from a macroscopic perspective. Equation (7) corresponds to a microscopic view in which charges enter and leave each segment of the loop, such that the center of mass of the charges in, say, the top segment moves to the right at speed \( v_t \) until a charge leaves on the right and another enters from the left; at that moment the center of mass instantaneously moves to the left by distance \( d_t / n_t l \), where \( d_t \) is the spacing between charges in the top segment. That is, the motion of the microscopic center of mass is a Zitterbewegung.

\(^3\)The use of this argument in cases of time-dependent electric fields is considered in sec. 3 below for transient fields, and in sec. 4 below for wave fields.

\(^4\)It could be that the electric field is related only to a scalar potential \( V \), but this is not required in the present argument. In any case, the increase in the mechanical energy of the charges is compensated by a reduction in the energy of the electric field.

If the electric field is static, with \( E = -\nabla V \), then the interference energy between the external field and the field of the energized charges is readily shown to be negative, as discussed in [15].

Also, as noted in footnote 9 of [16], in case of an electrostatic field \( E \), the total energy of electric charge \( q \) has the constant value \( U = \gamma m_q e^2 + qV \). Then, the change in a charge’s mechanical energy when moving distance \( w \) upwards through this field is \( \Delta U_{\text{mech}} = m_q (\gamma_t - \gamma_b)c^2 = q\Delta V = qEw \), as in eq. (3).

For additional remarks on the relation \( U = \gamma m_q e^2 + qV \), see [17].
It is more appropriate to provide a macroscopic description, in which we can speak of a steady current $I$, rather than a series of delta functions of current spaced at small time intervals. The macroscopic description is obtained by averaging over distances larger than the gaps $d$ between the moving charges. In the macroscopic description the Zitterbewegung is ignored, and we set the macroscopic center-of-mass velocity $\vec{v}_{cm,\text{charges}}$ of the charges to zero,

$$\vec{v}_{cm,\text{charges}} = 0. \tag{8}$$

The subsystem of charges is then said to contain “hidden” momentum according to the definition (1),

$$P_{\text{hidden,charges}} = P_{\text{charges}} - M_{\text{charges}} \vec{v}_{cm,\text{charges}} = P_{\text{charges}} = \frac{m \times E}{c}. \tag{9}$$

The subsystem of the electromagnetic fields has nonzero momentum,

$$P_{\text{EM}} = \int \frac{E \times B}{4\pi c} d\text{Vol} = \int \frac{JV}{c^2} d\text{Vol} = \frac{E \times m}{c} = -P_{\text{charges}}, \tag{10}$$

where the third and fourth forms of eq. (10) are due to Furry [8]. Thus, the total momentum of the system is zero, whereas the total momentum of the system of circulating rockets was nonzero.

The center of energy of the electromagnetic fields is at rest in the lab frame, so $v_{cm,\text{EM}} = 0$. Hence, according to definition (1), the electromagnetic subsystem has nonzero “hidden” momentum,

$$P_{\text{hidden,EM}} = P_{\text{EM}} - M_{\text{EM}} \vec{v}_{cm,\text{EM}} = P_{\text{EM}} = \frac{E \times m}{c} = -P_{\text{charges}} = -P_{\text{hidden,charges}}, \tag{11}$$

where $M_{\text{EM}} \equiv U_{\text{EM}}/c^2$.

The key result of this problem is the clarification that the center-of-mass velocity to be used in application of definition (2) is the macroscopic average velocity, in the same sense that the electrical current $I$ is a macroscopic average. Then, mechanical momentum which is explicitly noted in a microscopic description can, in some cases, be characterized as “hidden” in the macroscopic description.

## 3 Transient Effects

In the preceding the current and external electric field were assume to be steady. We now consider several possibilities of transient behavior, in which the current varies, or in which the electric field varies.$^6$,$^7$

---

$^5$The result (10) appears on pp. 347-348 of [4] and in sec. 285 of [5], but Thomson appears not to have noted the paradox that a system “at rest” has nonzero field momentum. For discussion of these early results, see [6].

$^6$The example of an electric charge plus magnetic dipole is a variant on the configuration of the Aharonov-Bohm effect [18], in which an electric charge moves outside a long solenoid magnet where the external field $B_{\text{solenoid}}$ is negligible but the external vector potential $A_{\text{solenoid}}$ is nonzero. For discussion by the author of classical aspects of the Aharonov-Bohm effect, see [19].

$^7$An exotic possibility, that the entire system collapses to a black hole, has been considered in [20].
3.1 The Current is Excited/De-excited

Suppose the external electric field is static, but the current in the loop is created or destroyed by a transient, external magnetic flux though the loop. The transient electric field around the loop modifies the electrical current in the loop, doing work to create/destroy the magnetic field energy of the loop (and the small kinetic energy of the circulating charges). In this process, equal and opposite amounts of mechanical and electromagnetic momentum are given/taken away from the system of loop plus external field, so no net force is exerted on that system during the transient phase.

3.2 The External Field is Due to a Single, Moving Charge

Suppose the external electric field is due to a single electric charge \( Q \) (with no magnetic moment) that has velocity \( \mathbf{v} \) with \( v \ll c \), while the magnetic moment \( \mathbf{m} \) remains constant. This charge is subject to the Lorentz force of the magnetic dipole field of the current loop, which force does no work on the charge, such that the speed \( v \), the relativistic mass \( m_Q / \sqrt{1 - v^2 / c^2} \), and the kinetic energy of the charge remain constant. However, the direction of the velocity \( \mathbf{v} \) changes with time, so the charge is accelerated and radiates energy and momentum. An issue raised by [20] is whether the total radiated momentum is significant compared to \( P_{EM} = Q m \sin \theta / cr^3 \), where \( \theta \) is the angle between \( \mathbf{m} \) and the line joining \( Q \) and the location of the magnetic dipole \( \mathbf{m} \), and \( r \) is the length of that line.

In the following we will only consider cases where \( \theta = 90^\circ \); that is, the charge moves in the plane perpendicular to the constant magnetic moment \( \mathbf{m} \). We also suppose that the mass of the current loop is much larger than the (rest) mass \( m_q \) of the charge, so that it is a good approximation to consider the current loop to be at rest, and at the origin.

3.2.1 The Charge Moves Radially from \( r_0 \) to \( \infty \)

Initially, the charge is at radius \( r_0 \) with respect to the magnetic moment \( \mathbf{m} \), and its small velocity is radially outwards, \( \mathbf{v} = v \hat{r} \). We also suppose that \( v \) is small enough that the trajectory of the charge is nearly radial at all times.

The Lorentz force on the moving charge \( Q \) is

\[
F_Q = \frac{dP_Q}{dt} = Q \frac{\mathbf{v}}{c} \times \mathbf{B}_m = Q \frac{\mathbf{v}}{c} \times \left( \frac{3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m}}{r^3} \right) = Q \frac{\mathbf{m} \times \mathbf{v}}{cr^3},
\]

for motion of \( Q \) in the plane perpendicular to \( \mathbf{m} \). Hence, if \( \mathbf{v} \approx v \hat{r} \), then the azimuthal component of eq. (12) is

\[
\frac{dP_\phi}{dt} \approx \frac{Qmv}{cr^3}.
\]

The radial component of the momentum of the charge is \( P_r \approx m_Qv \), so the rate of change of the angle \( \phi \) of the charge’s trajectory with respect to the radial direction is

\[
\frac{d\phi}{dt} \approx \frac{1}{P_r} \frac{dP_\phi}{dt} \approx \frac{qm}{cm_qr^3}.
\]
For nearly radial motion, \( dr \approx v \, dt \), so that the total change \( \Delta \phi \) in the azimuthal orientation of the trajectory of the charge as it moves from \( r_0 \) to \( \infty \) is related by

\[
\frac{d\phi}{dr} \approx \frac{1}{v} \frac{d\phi}{dt} \approx \frac{c}{v} \frac{Qm}{m_q c^2 r^3} ; \quad \Delta \phi = \int_{r_0}^{\infty} \frac{d\phi}{dr} \, dr = \int_{r_0}^{\infty} \frac{c}{v} \frac{Qm}{m_Q c^2 r^3} \, dr = \frac{c}{v} \frac{Qm}{v 2m_Q c^2 r_0^3}.
\]  

Hence, the condition for quasiradial outward motion is

\[
\frac{Qm}{m_Q c^2 r_0^3} \ll \frac{v}{c} \ll 1 \quad \text{(quasiradial outward motion).}
\]  

As discussed in sec. 2.2.6 of [21], rate of radiation of momentum by the accelerated charge is

\[
\frac{dP_{rad}}{dt} = \frac{dU_{rad}}{dt} \frac{v}{c^2} = \frac{2Q^2 F_q^2}{3m_Q c^3} \frac{v}{c} \approx \frac{2Q^4 m^2 v^3}{3m_Q c^5 r^6},
\]

recalling the Larmor formula for \( v \ll c \) that \( dU_{rad}/dt = 2Q^2 a^2 / 3c^3 = 2Q^2 F_q^2 / 3m_Q c^3 \). The total radiated momentum \( P_{rad} \) in the radial direction, follows as

\[
\frac{dP_{rad}}{dr} \approx \frac{1}{v} \frac{dP_{rad}}{dt} \approx \frac{2Q^4 m^2 v^2}{3Q^3 c^5 r^6}, \quad P_{rad} = \int_{r_0}^{\infty} \frac{dP_{rad}}{dr} \, dr \approx \frac{2Q^4 m^2 v^2}{15m_Q c^5 r_0^3}.
\]  

The initial electromagnetic momentum of the system follows from eq. (10) as

\[
P_{EM} = \frac{Qm}{c^2 r_0^2},
\]

so that the ratio of the radiated momentum to the initial electromagnetic momentum is

\[
\frac{P_{rad}}{P_{EM}} \approx \frac{Q^3 m v^2}{m_Q c^5 r_0^3} = \frac{v^2}{c^2} \frac{Qm}{m_q c^2 r_0^2} \frac{Q^2}{m_Q c^6} \ll \frac{v^3 q^2 / m_Q c^2}{c^3 / r_0^3}.
\]

For \( r_0 \) large compared to the classical charge radius \( Q^2 / m_Q c^2 \), the ratio (20) is negligible.

In sum, as the charge moves outwards quasiradially at constant speed, the electromagnetic field momentum and the net mechanical momentum of the electric current drop to zero, remaining equal and opposite at all times in the quasistatic approximation.

### 3.2.2 The Charge Orbits the Magnetic Dipole

For magnetic moment \( \mathbf{m} = m \hat{z} \), a positive electric charge \( Q \) can move counterclockwise in a circular orbit of radius \( r \) with speed \( v \) given by the Lorentz force as

\[
\frac{v}{c} = \frac{Qm}{Q q c^2 r^2},
\]

when \( v/c \ll 1 \). We suppose that the initial radius \( r_0 \) is large enough that \( v/c \ll 1 \).

As noted, for example, in sec. 2.5.5 of [21], the accelerated charge radiates total power

\[
\frac{dU_{rad}}{dt} = \frac{2Q^4 (v/c \times B)^2}{3m_Q c^3} = \frac{2Q^4 m^2 v^2}{3m_Q c^5 r^6} = \frac{2Q^6 m^4}{3m_Q^4 c^7 r^{10}}.
\]
The radiation produces a back reaction on the charge which can be characterized by the radiation reaction force \( \mathbf{F}_{\text{react}} = -K \mathbf{v} \), such that the power delivered by this force is the negative of the radiated power,

\[
\mathbf{F}_{\text{react}} \cdot \mathbf{v} = -K v^2 = -\frac{dU_{\text{rad}}}{dt},
\]

and hence \(^8\)

\[
\mathbf{F}_{\text{react}} = -K \mathbf{v} \quad \text{with} \quad K = \frac{2Q^4m^2}{3m_Q^2c^5r^6}.
\]

As a consequence, the charged particle slows down according to

\[
m_Q \frac{dv}{dt} = F_{\text{react}} = -Kv, \quad v = v_0 e^{-Kt/m_Q}, \quad r = \sqrt{\frac{Qm}{m_Qcv}} = r_0 e^{Kt/2m_Q},
\]

and its trajectory spirals slowly out to infinite radius where the asymptotic velocity is zero. During this process the electromagnetic field momentum and the net mechanical momentum go to zero, and the initial kinetic energy of the charge is transformed into electromagnetic radiation. Momentum is radiated by the accelerating charge, but the net radiated momentum over the large number of turns in the spiral trajectory is negligible, so there is no significant “kick” imparted to the magnetic dipole.\(^9\)

### 3.2.3 General Motion of the Charge

An analytic solution for the general motion of the charge is not possible, but electromagnetic radiation is generated as the charge accelerates, which reduces the kinetic energy and the speed of the charge. The charge either comes to rest at some finite distance from the magnetic moment, or moves to infinity with asymptotic speed either zero or nonzero. In the first case the final electromagnetic field momentum and final net momentum of the current are nonzero, equal and opposite, and smaller than their initial values. In the second case the final electromagnetic field momentum and final net momentum of the current are zero. In no case does anything striking take place.

### 3.2.4 The Charge Collides with the Magnetic Dipole

A special case is that the electric charge \( q \) has initial velocity \( \mathbf{v}_0 \perp \mathbf{m} \) such that it eventually collides (totally inelastically) with the magnetic dipole \( \mathbf{m} \). Just before this collision the charge has velocity \( \mathbf{v}_f (\perp \mathbf{m}) \). Suppose the magnetic dipole has initial velocity \( -\mathbf{v}_f m_Q/m_m \), such that the final state of charge + moment has zero (mechanical) velocity.

In the initial state there is electromagnetic momentum (and electromagnetic energy) associated with the separate motions of the electric charge and of the magnetic dipole.

\(^8\)Equation (24) also follows from the well-known form \( \mathbf{F}_{\text{react}} = (2Q^2/3c^3) \dot{\mathbf{a}} = (2Q^2/3c^3) \mathbf{\omega} \times \mathbf{a} = (2Q^2/3c^3)(-v\hat{z}/r) \times (-v^2/r) \hat{r} = -(2Q^2v^2/3c^3r^2) \mathbf{v} = -(2Q^4m^2/3m_Q^2c^5r^6) \mathbf{v}.

\(^9\)The radiation fields transfer net energy and momentum to the magnetic dipole (including net work done by magnetic field [22]), but this effect is negligible in the present case.
We consider that these momenta (and energies) are “renormalized” into the “mechanical” momenta (and energies) of the charge and dipole. In addition, there is a nonzero interaction field momentum in the initial state, which is well approximated for low velocities by eq. (10), $P_{\text{EM, int}} \approx E_Q \times m/c$. And to the same approximation, the initial currents of the magnetic dipole contain hidden mechanical momentum, $P_{\text{hidden, mech}} \approx -P_{\text{EM, int}} \approx -E_Q \times m/c$.

As the electric charge approaches the magnetic dipole the electric field experienced by the latter grows large, so the interaction field momentum and the hidden mechanical momentum grow large, and are not necessarily well approximated by the forms in the preceding paragraph. However, in the final state, where we assume that the electric charge comes to rest at the center of the current loop of the magnetic dipole, the final interaction field momentum and the final hidden mechanical momentum are zero.

As the charge moves towards the magnetic dipole they both accelerate, emitting electromagnetic energy and momentum. The electric charge radiates much more than does the magnetic dipole, assuming that the latter’s mass is much larger. Then, the final state contains net radiated momentum, whose direction is roughly along the line from the initial position of the charge to the dipole. Conservation of momentum tells us that the final state of the charge + dipole is not at rest, but has constant velocity, and mechanical momentum equal and opposite to the net radiated momentum.

An observer who does not consider the radiated momentum would say that the final state of charge + dipole has an unexpected “kick.” While consideration of electromagnetic radiation shows that this “kick” has nothing to do with “hidden momentum” and everything to do with radiation, the naïve observer might associate it with “hidden momentum.”

4 Wave Fields

We now consider the possible validity of the argument of sec. 2 above for the case of a wave field, of wavelength $\lambda$ and angular frequency $\omega = 2\pi c/\lambda$, i.e., for a wave in a medium with unit (relative) permittivity and permeability.

One premise of the argument of sec. 2 is that the electric field $E$ be uniform over the loop, which implies that in the case of a wave field, the wavelength $\lambda$ is large compared to the sides $h$ and $w$ of the loop.

In addition, for the instantaneous “hidden” mechanical momentum to be $m \times E(t)/c$, it must be the period of the circulation of charges in the current loop be less than the period of the external wave field. That is, the angular velocity of the circulating charges must be larger than the angular velocity of the wave field.

We consider the implications of this requirement for the cases of current loops associated with permanent magnetism, and with diamagnetism.

4.1 Permanent Magnetism

In the case of permanent magnetism the magnetic moments are (almost entirely) associated with the intrinsic magnetic moments of electron, which have magnitude $e\hbar/2mc$. Although there is no satisfactory classical model for these intrinsic moments, we make a plausibility argument based on the model that an electron is a shell of charge of radius $\lambda_C = h/mc,$
the reduced Compton wavelength, which shell rotates with equatorial velocity $c$. In this model, the angular velocity of the moving charge is $c/\lambda_C$, so it is plausible that the relation $\mathbf{m} \times \mathbf{E}(t)/c$ describes “hidden” mechanical momentum of permanent magnetism in electromagnetic fields of (reduced) wavelength $\lambda$ larger than $\lambda_C$.

This is the same requirement that the electric field be uniform over the current distribution of the magnetic dipole.

This requirement is well satisfied by electromagnetic waves of optical frequencies.

### 4.2 Diamagnetism

Diamagnetism is associated with magnetic momentum to due orbital motion of atomic electrons that is induced by the external electromagnetic field.

For an order-of-magnitude estimate, we take the effective radius of the magnetic moment to be the Bohr radius $r_B = \lambda_C/\alpha$, where $\alpha = e^2/\hbar c$ is the fine-structure constant and $e$ is the charge of an electron.

Then, the requirement that the external field be uniform over the magnetic moment is that the wavelength of the external field be large compared to the Bohr radius, which is well satisfied by optical fields.

The angular frequency of the induced orbital motion of electrons is less than $v_{\text{max}}/r_B$, where

$$v_{\text{max}} \approx a_{\text{max}}t_{\text{wave}} \approx \frac{eE_0}{m} \frac{1}{\omega},$$

where $E_0$ is the peak electric field of the wave. For $\mathbf{m} \times \mathbf{E}(t)/c$ to describe the “hidden” mechanical momentum associated with diamagnetism, we must also have

$$\omega_{\text{orbital}} \approx \frac{v_{\text{max}}}{r_B} \approx \frac{eE_0}{m \omega r_B} = \frac{\alpha e E_0}{m \omega \lambda_C} \gg \omega, \quad E_0 \gg \frac{m \omega^2 \lambda_C}{\alpha e} = \frac{\hbar^2 \omega^2}{e^3} = \frac{E_{\text{crit}}}{\alpha} \left(\frac{\lambda_C}{\lambda}\right)^2,$$

where $E_{\text{crit}} = m^2 c^3/\epsilon h = 1.6 \times 10^{16}$ V/cm is the so-called QED critical field strength (above which a static electric field would spontaneously produce electron-positron pairs [23]). For optical waves, this requirement is that $E_0 \gg 10^4$ V/cm, which is a reasonably strong field, but which is readily achieved in laser beams.

### A Appendix: An All-Mechanical Current Loop

As suggested by J. Franklin (private communication), we can imagine the figure on p. 1 corresponds to a set of runners on a racetrack. On the left, vertical leg of the track the runners convert some of their internal energy into increased kinetic energy, ideally in a reversible manner (such that on the right, vertical leg the runners convert some of their kinetic energy back into internal energy). In this idealization the total energy $m \gamma c^2$ of each runner is constant in time; the rest mass $m$ decreases when the kinetic energy increases, and vice versa. So, if the runners are distributed such that the “current” is constant around the track,

$$I = n_b v_b = n_t v_t,$$
then in the frame of the track the total momentum of the runners is
\[ P_{\text{runners}} = (n_t lm_t \gamma_t v_t - n_b lm_b \gamma_b v_b) \hat{x} = 0. \] (29)

Hence, there is no “hidden” momentum in this all-mechanical example.

As remarked in footnote 7 of [2], it appears that “hidden” mechanical momentum exists only in examples where (moving) “matter” interacts with a “field.”

**B Appendix: Magnetic-Current Loop in a Magnetic Field**

If magnetic charges (monopoles) existed, we could consider the case of a loop of magnetic current in a uniform magnetic field \( B = B \hat{y} \), with geometry as in the figure on p. 1.

If we label a magnetic charge by \( q_m \) (with rest mass \( m_q \)), the force on it in an external magnetic field \( B \) is \( F = q_m B \). Then, all the discussion in sec. 2 holds with the substitutions \( q \rightarrow q_m, E \rightarrow B, \) and \( I \rightarrow I_m \) (for the current), except for the last equality in eq. (5). That is, the total momentum of the charges can now be written
\[ P_{\text{charges}} = (n_t lm_q \gamma_t v_t - n_b lm_q \gamma_b v_b) \hat{x} = \frac{I_m \Delta U_{\text{mech}}}{c^2 q} \hat{x} = \frac{I_m w B}{c^2} \hat{x}. \] (30)

Here, we note that a loop of circulating magnetic charge has an electric dipole moment \( p_m \) given (in Gaussian units) by
\[ p_m = -\int \frac{r \times J_m}{2c} \, d\text{Vol} \rightarrow - \frac{I_m \text{Area}}{c} = \frac{I_m l w}{c} \hat{z}, \] (31)
where \( \hat{z} \) is out of the paper. Thus, the total mechanical momentum of the moving magnetic charges is
\[ P_{\text{magnetic charges}} = -\frac{p_m \times B}{c}. \] (32)

**B.1 B due to Magnetic Charges**

For eq. (32) to be the only mechanical momentum in the example, we suppose that the external magnetic field is due to magnetic charges at rest, \( B = B_m \), rather than electric charges in motion. Then,
\[ P_{\text{hidden}} = P_{\text{magnetic charges}} = -\frac{p_m \times B_m}{c}. \] (33)

As for the case of a loop of circulating electric charges (with external electric field due to other electric charges), the velocity of the center of mass of the moving magnetic charges is zero, so the momentum (32) is a “hidden” mechanical momentum, to which the electromagnetic field momentum is equal and opposite, so that the total momentum is zero in this example.

---

10 For a review of classical electrodynamics with both electric and magnetic charges, see, for example, [24].
11 For a discussion of the first equality in eq. (31), see Appendix B of [25].
B.2 B due to Electric Currents

B.2.1 The Electric and Magnetic Currents Occupy Different Volumes

If the external field $B$ were due to moving electric charges (in an electrically neutral “wire”), rather than to static magnetic charges, then there would be some “hidden” mechanical momentum associated with electric currents. Supposing these currents to form a magnetic dipole $m_e$, and that the electric field $E$ due to the magnetic currents of $p_m$ is sufficiently uniform over the magnetic dipole $m_e$, this additional “hidden” mechanical momentum would be given by eq. (9). Then, the total “hidden” mechanical momentum would be, combining eqs. (33) and (9),

$$P_{\text{hidden}} = - \frac{p_m \times B_{at}}{c} + \frac{m_e \times E_{at}}{c}$$

$$= - \frac{p_m}{c} \times \frac{3(m_e \cdot \hat{r})\hat{r} - m_e}{r^3} + \frac{m_e}{c} \times \frac{3(p_m \cdot \hat{r})\hat{r} - p_m}{r^3}$$

$$= \frac{2p_m \times m_e}{cr^3} + \frac{3\hat{r} \times [\hat{r} \times (p_m \times m_e)]}{cr^3} = \frac{3[(p_m \times m_e) \cdot \hat{r}]\hat{r} - (p_m \times m_e)}{cr^3},$$  

(34)

where $r$ is the distance between $p_m$ and $m_e$. This result (with functional form like that of the electromagnetic fields of the dipoles $p_m$ and $m_e$) is listed in the table appended to the link to [11], which follows from eq. (66) of that paper,

$$P_{\text{hidden}} = - P_{\text{EM}} = \frac{p_m \times \nabla V_m}{c} - \nabla \left( \frac{p_m \cdot A_e}{c} \right),$$  

(35)

where $V_m$ is the scalar potential associated with magnetic charges and $A_e$ is the vector potential associated with electric currents.\textsuperscript{12,13}

In the present example, we suppose that the magnetic currents of $p_m$ are magnetically neutral, such that $V_m = 0$. Then, eq. (35) implies,

$$P_{\text{hidden}} = - \nabla \left( \frac{p_m \cdot A_e}{c} \right) = - \frac{p_m \times B_e}{c} - \left( \frac{p_m \cdot \nabla}{c} \right) \frac{A_e}{c}. \quad (36)$$

That is, $-(p_m \cdot \nabla)A_e/c$ is the “hidden” mechanical momentum associated with the electric currents.

For example, suppose the external magnetic field is due to a loop of electric current with magnetic moment $m_e$, for which its magnetic field has vector potential $A_e = m_e \times r/r^3$, where $r$ is the vector from the magnetic dipole to the observer. Then,

$$- \left( p_m \cdot \nabla \right) \frac{A_e}{c} = - \left( p_m \cdot \nabla \right) \left( m_e \times \frac{r}{cr^3} \right) = m_e \times - \left( p_m \cdot \nabla \right) \frac{r}{cr^3} = m_e \times - \nabla \left( \frac{p_m \cdot r}{cr^3} \right)$$

$$= \frac{m_e \times E_m}{c}. \quad (37)$$

$$P_{\text{hidden}} = - \frac{p_m \times B_e}{c} + \frac{m_e \times E_m}{c}. \quad (38)$$

as expected, which confirms the consistency of the analysis of [11] for electric and magnetic dipoles that occupy different volumes.

\textsuperscript{12}B_e = \nabla \times A_e, and for steady magnetic currents as considered in [11], $B_m = - \nabla V_m$.

\textsuperscript{13}If $A_e = 0$ but $V_m \neq 0$, we recover eq. (33) from eq. (35).
B.2.2 The Electric and Magnetic Currents Occupy the Same Volume

A special case is that the two dipoles \( p_m \) and \( m_e \) occupy the same volume, instead of being external to one another as assumed above. However, we might still expect that

\[
\mathbf{P}_{\text{hidden}} = -\frac{\mathbf{p}_m \times \mathbf{B}_{\text{at } p}}{c} + \frac{\mathbf{m}_e \times \mathbf{E}_{\text{at } m}}{c}.
\]  

(39)

We consider the case of dipoles due to uniform densities \( \mathbf{P}_m \) and \( \mathbf{M}_e \) of electric polarization due to magnetic currents and magnetic polarization due to electric currents, all inside a sphere of radius \( a \).

The internal fields of these current-based dipoles are

\[
\mathbf{E}_m(r < a) = \frac{2\mathbf{p}_m}{a^3}, \quad \mathbf{B}_e(r < a) = \frac{2\mathbf{m}_e}{a^3}.
\]  

(40)

Using these in eq. (39), we find that

\[
\mathbf{P}_{\text{hidden}} = -\frac{4\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}.
\]  

(41)

As a check, we now compute the “hidden” mechanical momentum according to the relation (36).

The vector potential of the uniform magnetic field inside the sphere has only an azimuthal component, \( \mathbf{A}_{e,\phi}(r < a) = \frac{r \mathbf{B}_e}{2} \), with respect to the direction of \( \mathbf{m}_e \) and \( \mathbf{B}_e \). A delicacy in the argument is to suppose for the time being that \( \mathbf{p}_m \) is just an element of the polarization density \( \mathbf{P}_m \), a distance \( r \) from the center of the sphere. Then, at the location of the element \( \mathbf{p}_m \), we have

\[
-\nabla (\mathbf{p}_m \cdot \mathbf{A}_e) = \mathbf{p}_m \times (\nabla \times \mathbf{A}_e) - (\mathbf{p}_m \cdot \nabla)\mathbf{A}_e = -\mathbf{p}_m \times \mathbf{B}_e - (\mathbf{p}_m \cdot \hat{r}) \frac{\mathbf{B}_e}{2} \hat{\phi},
\]  

(42)

since \( \mathbf{A}_e \) has only a \( \phi \) component, which depends only on \( r \). This result is independent of \( r \), so we can sum up the contributions from all elements \( \mathbf{p}_m \), and again consider the symbol \( \mathbf{p}_m \) to represent the total electric dipole moment. Now,

\[
(\mathbf{p}_m \cdot \hat{r}) \frac{\mathbf{B}_e}{2} \hat{\phi} = \frac{\mathbf{B}_e \times \mathbf{p}_m}{2} = -\frac{\mathbf{p}_m \times \mathbf{B}_e}{2},
\]  

so altogether,

\[
\mathbf{P}_{\text{hidden}} = -\nabla \left( \frac{\mathbf{p}_m \cdot \mathbf{A}_e}{c} \right) = -\frac{\mathbf{p}_m \times \mathbf{B}_e}{c} + \frac{\mathbf{p}_m \times \mathbf{B}_e}{2c} = -\frac{\mathbf{p}_m \times \mathbf{B}_e}{2c} = -\frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c},
\]  

(44)

which differs from eq. (41) by a factor of 4.

To clarify the situation, we recall that in static examples like the present, the “hidden” mechanical momentum should be equal and opposite to the electromagnetic field momentum (so that the total momentum of the system is zero). A computation of the field momentum for the present example is given in Appendix A.1.3 of [25], with the result,

\[
\mathbf{P}_{EM} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \, d\text{Vol} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}.
\]  

(45)

This suggests that the computation of “hidden” mechanical momentum based on eq. (35) is valid, while (at least for the case of two dipole in the same volume) that based on eq. (39) is not. It remains surprising to this author that eq. (39) fails in the present case.
References


http://arxiv.org/abs/1303.7456


*Zum ‘Kleinschen Paradoxon’*, Z. Phys. 73, 547 (1931),
