Currents in a Conducting Sheet with a Hole

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1 Problem

Current enters an infinite plane conducting sheet at some point p and leaves at infinity. A circular hole, exclusive of p, is cut anywhere in the sheet. Show that the potential difference between any two points on the edge of the hole is twice that between the same two points before the hole was cut [1].

2 Solution

We take the hole to have radius a, which is less than the distance b from the center of the hole to the point p where the current enters the sheet.

We use a polar coordinate system with its origin at the center of the hole, and point p at $(r, \theta) = (b, 0)$.

The electric potential ϕ is taken to be $\phi = \phi_0 + \phi_1$, where ϕ_0 is the potential in the absence of the hole, and ϕ_1 is a correction that vanishes at large radius r (and is finite at $r = 0$).

In the absence of the hole, the current density $\mathbf{J}_0$ flows radially outward from point p. Let $\rho = \sqrt{r^2 + b^2 - 2rb\cos\theta}$ be the distance from point p to the point (r, θ), and $\hat{\boldsymbol{\rho}}$ be a unit vector pointing away from p. Then the current density $\mathbf{J}_0$ is given by

$$\mathbf{J}_0 = \frac{I}{2\pi\rho}\hat{\boldsymbol{\rho}} = \sigma\mathbf{E}_0 = -\sigma\boldsymbol{\nabla}\phi_0, \tag{1}$$

where σ is the electrical conductivity of the sheet. Solving for the potential ϕ_0, we find

$$\phi_0 = -\frac{I}{2\pi\sigma}\ln\rho + K. \tag{2}$$

Note that the potential ϕ_0 is the same as that due to a line charge $q = I/4\pi\sigma$ (in Gaussian units) perpendicular to the (r, θ) plane at point p. Indeed, following sec. 4-10 of [1] or sec. 4.02 of [2], we can rewrite ϕ_0 (ignoring the constant K) as

$$\phi_0(r,\theta) = \frac{I}{2\pi\sigma}\begin{cases} \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n\cos n\theta - \ln b, & r < b, \\ \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{b}{r}\right)^n\cos n\theta - \ln r, & r > b. \end{cases} \tag{3}$$

The hole perturbs the potential ϕ_0 by the addition of potential ϕ_1. This correction to the potential vanishes for large r, is finite at $r = 0$, and is symmetric in angle θ. Hence, we can write (ignoring the constant term in the potential)

$$\phi_1(r,\theta) = \begin{cases} \sum_{n=1}^{\infty}A_n\left(\frac{r}{a}\right)^n\cos n\theta, & r < a, \\ \sum_{n=1}^{\infty}A_n\left(\frac{a}{r}\right)^n\cos n\theta, & r > a. \end{cases} \tag{4}$$

The existence of the nonconducting hole implies that the radial component of the current density, and hence also the radial component of the electric field, vanishes at $r = a^+$. Using the form of eq. (3) for $r < b$ (since $a < b$) and that of eq. (4) for $r > a$ we obtain the constraint,

$$0 = -E_r(r = a^+) = \frac{\partial\phi(r = a^+)}{\partial r} = \frac{I}{2\pi\sigma}\sum_{n=1}^{\infty}\frac{1}{a}\left(\frac{a}{b}\right)^n \cos n\theta - \sum_{n=1}^{\infty}\frac{n}{a}A_n \cos n\theta. \qquad (5)$$

Hence, the Fourier coefficients A_n are given by

$$A_n = \frac{I}{2n\pi\sigma}\left(\frac{a}{b}\right)^n. \qquad (6)$$

The potential ϕ_1 is therefore given by

$$\phi_1(r,\theta) = \frac{I}{2\pi\sigma}\begin{cases} \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n \cos n\theta, & r < a, \\ \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a^2/b}{r}\right)^n \cos n\theta, & r > a. \end{cases} \qquad (7)$$

In particular, on the rim of the hole (as well as inside the hole) the potential ϕ_1 is given by

$$\phi_1(r \le a, \theta) = \phi_0(r \le a, \theta) + \frac{I}{2\pi\sigma}\ln b, \qquad (8)$$

and so,

$$\phi(r \le a, \theta) = 2\phi_0(r \le a, \theta) + \frac{I}{2\pi\sigma}\ln b, \qquad (9)$$

Hence, the potential difference between any two points on the rim of the hole (or within the hole) is twice that in the absence of the hole.

Inside the hole, the functional form of the potential is that same as that in its absence, so the equipotentials are circles centered on point p, and the electric field lines radiate from point p. Since the electric field within the sheet is tangential to the rim, there must be a distribution of fixed charges along the rim to support the field lines that cross the hole.

The potential and fields within the sheet at $r > a$ can be understood via an image model with a source and a sink with $r < a$. From eq. (3) we can write

$$\phi_1(r > a, \theta) = \frac{I}{2\pi\sigma}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a^2/b}{r}\right)^n \cos n\theta - \ln r\right] + \frac{I}{2\pi\sigma}\ln r. \qquad (10)$$

Comparing with eq. (3) for $r > b$, we see that the potential $\phi_1(r > a)$ can be thought of a due to a current source located at point $p' = (a^2/b, 0)$ in a conducting sheet with no hole, plus a current sink at the origin. The image source and sink have magnitudes identical to that of the current source at point p. The electric field at an arbitrary point in the sheet is therefore given by

$$\mathbf{E} = \frac{I}{2\pi\sigma}\left(\frac{\hat{\boldsymbol{\rho}}}{\rho} + \frac{\hat{\boldsymbol{\rho}}'}{\rho'} - \frac{\hat{\mathbf{r}}}{r}\right), \qquad (11)$$

where $\hat{\boldsymbol{\rho}}'$ is the unit vector along the line from point p' to the arbitrary point, and ρ' is the distance between these two points. With a small effort one can verify that the electric field (11) within the sheet and near the rim of the hole is purely azimuthal, and twice that of the azimuthal component of the field due to the original source at point p,

$$\mathbf{E}(r = a^+, \theta) = \frac{I}{2\pi\sigma} \frac{2b\,|\sin\theta|}{\rho^2}\,\hat{\boldsymbol{\theta}} = 2\frac{I}{2\pi\sigma}\frac{\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\theta}}}{\rho}\,\hat{\boldsymbol{\theta}}. \tag{12}$$

3 Appendix: Solution Using Eq. (2) Rather Than (3)

If we did not recall the expansion (3) of the logarithmic potential (2), we could evaluate the constraint (5) at the rim of the hole as

$$0 = E_r(r = a^+) = -\frac{\partial\phi(r = a^+)}{\partial r} = \frac{I}{2\pi\sigma}\frac{a - b\cos\theta}{a^2 + b^2 - 2ab\cos\theta} + \sum_{n=1}^{\infty}\frac{n}{a}A_n\cos n\theta. \tag{13}$$

The Fourier coefficients A_n are then given by

$$\begin{aligned} A_n &= -\frac{aI}{2n\pi^2\sigma}\int_0^{2\pi}\frac{a - b\cos\theta}{a^2 + b^2 - 2ab\cos\theta}\cos n\theta\; d\theta \\ &= -\frac{I}{n\pi^2\sigma}\frac{a^2}{a^2 + b^2}\int_0^{\pi}\frac{\cos n\theta - \frac{b}{2a}\cos(n+1)\theta - \frac{b}{2a}\cos(n-1)\theta}{1 - \frac{2ab}{a^2+b^2}\cos\theta}\, d\theta \\ &= \frac{I}{2n\pi\sigma}\left(\frac{a}{b}\right)^n, \end{aligned} \tag{14}$$

using Dwight 858.536 [3], which tells us that for $b > a$,

$$\int_0^{\pi}\frac{\cos n\theta}{1 - \frac{2ab}{a^2+b^2}\cos\theta}\, d\theta = \frac{\pi}{\sqrt{1 - \left(\frac{2ab}{a^2+b^2}\right)^2}}\left(\frac{1 - \sqrt{1 - \left(\frac{2ab}{a^2+b^2}\right)^2}}{\frac{2ab}{a^2+b^2}}\right)^n = \pi\frac{b^2 + a^2}{b^2 - a^2}\left(\frac{a}{b}\right)^n. \tag{15}$$

We now wish to show that the azimuthal electric field at $r = a^+$ due to the potential ϕ_1 is the same as that due to potential ϕ_0, *i.e.*,

$$-\frac{\partial\phi_0(a,\theta)}{\partial\theta} = -\frac{\partial\phi_1(a,\theta)}{\partial\theta}. \tag{16}$$

From eq. (2) we have

$$-\frac{\partial\phi_0(a,\theta)}{\partial\theta} = \frac{I}{2\pi\sigma}\frac{b\sin\theta}{a^2 + b^2 - 2ab\cos\theta}, \tag{17}$$

and from eq. (4) we have

$$-\frac{\partial\phi_1(a,\theta)}{\partial\theta} = \sum_{n=0}^{\infty}\frac{n}{a}A_n\sin n\theta. \tag{18}$$

If eq. (16) is to be valid, eqs. (17) and (18) imply that the Fourier coefficients A_n can also be calculated according to

$$\begin{aligned} A_n &= \frac{aI}{2n\pi^2\sigma} \int_0^{2\pi} \frac{b \sin\theta}{a^2 + b^2 - 2ab\cos\theta} \sin n\theta \; d\theta \\ &= \frac{I}{2n\pi^2\sigma} \frac{ab}{a^2+b^2} \int_0^{\pi} \frac{\cos(n-1)\theta - \cos(n+1)\theta}{1 - \frac{2ab}{a^2+b^2}\cos\theta} \, d\theta \\ &= \frac{I}{2n\pi\sigma} \left(\frac{a}{b}\right)^n . \end{aligned} \tag{19}$$

Since calculations (14) and (19) give the same results for A_n we again conclude that the potential between any two points on the rim of the hole is twice that in its absence.

4 References

[1] Problem 7-3 of W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, Reading, MA, 1962).

[2] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, New York, 1968).

[3] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, New York, 1961).