An Off-Axis Neutrino Beam

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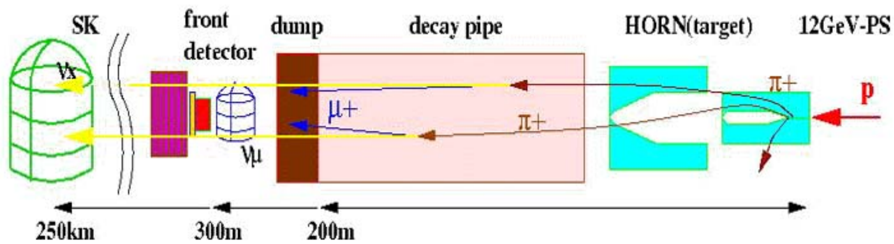
1 Problem

A typical high-energy neutrino beam is made from the decay of π mesons that have been produced in proton interactions on a target, as sketched in the figure below.



Suppose that only positively charged particles are collected by the "horn". The main source of neutrinos is then the decay $\pi^+ \to \mu^+\nu_\mu$.

1. Give a simple estimate of the relative number of other types of neutrinos than ν_μ in the beam.

2. If the decay pions have energy $E_\pi \gg m_\pi$, what is the characteristic angle θ_C of the decay neutrinos with respect to the direction of the π^+?

3. If a neutrino is produced with energy $E_\nu \gg m_\pi$, what is the maximum angle $\theta_{\rm max}(E_\nu)$ between it and the direction of its parent pion (which can have any energy)? What is the maximum energy E_ν at which a neutrino can be produced in the decay of a pion if it appears at a given angle θ with respect to the pion's direction?

   Parts 4 and 6 explore consequences of the existence of these maxima.

4. Deduce an analytic expression for the energy-angle spectrum $d^2N/dE_\nu d\Omega$ for neutrinos produced at angle $\theta \le \theta_C$ to the proton beam. You may suppose that $E_\nu \gg m_\pi$, that the pions are produced with an energy spectrum $dN/dE_\pi \propto (E_p - E_\pi)^5$, where E_p is the energy of the proton beam, and that the "horn" makes all pion momenta parallel to that of the proton beam.

5. At what energy $E_{\nu,\rm peak}$ does the neutrino spectrum peak for $\theta = 0$?

6. Compare the characteristics of a neutrino beam at $\theta = 0$ with an off-axis beam at angle θ such that $E_{\nu,\rm max}(\theta)$ is less than $E_{\nu,\rm peak}(\theta = 0)$.

Facts: $m_\pi = 139.6$ MeV/c^2, $\tau_\pi = 26$ ns, $m_\mu = 105.7$ MeV/c^2, $\tau_\mu = 2.2$ μs. In this problem, neutrinos can be taken as massless.

2 Solution

In this solution we use units where $c = 1$.

1. Besides the ν_μ from the decay $\pi^+ \to \mu^+\nu_\mu$, the beam will also contain $\bar{\nu}_\nu$ and ν_e from the subsequent decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$. Both of these decays occur (primarily) in the "decay pipe" shown in the figure. As both the pions and muons of relevance are relativistic in this problem, they both have about the same amount of time to decay before they are absorbed in the "dump". Hence, the ratio of number of muon decays to pion decays is roughly the inverse of the ratio of their lifetimes, *i.e.*, about 0.01. Our simple estimate is therefore,

$$\frac{N_{\nu_e}}{N_{\nu_\mu}} = \frac{N_{\bar{\nu}_\mu}}{N_{\nu_\mu}} \approx 0.01. \tag{1}$$

   Experts may note that an additional source of ν_e is the decay $\pi^+ \to e^+\nu_e$ at the level of 10^{-4}. Also, K^+ mesons will be produced by the primary proton interaction at a rate about 10% that of π^+. About 65% of K^+ decays are to $\mu^+\nu_\mu$, which add to the main ν_μ beam, but about 5% of the decays are to $\pi^+\pi^0\nu_e$, which increases the ν_e component of the neutrino beam by about $0.1 \times 0.05 = 0.005$.

2. Parts 2-6 of this problem are based on the kinematics of charged pion decay, which are closely related to kinematic features of neutral pion decay, $\pi^0 \to \gamma\gamma$ [1].

   Experts may guess that the characteristic angle of the decay neutrinos with respect to the parent pion is $\theta_C = 1/\gamma_\pi = m_\pi/E_\pi$. The details of the derivation are needed in part 3.

   We consider the decay $\pi \to \mu\nu$ in the rest frame of the pion (in which quantities will be labeled with the superscript $\star$) and transform the results to the lab frame.

   Energy-momentum conservation can be written as the 4-vector relation,

$$\pi = \mu + \nu, \tag{2}$$

   where the squares of the 4-vectors are the particle masses, $\pi^2 = m_\pi^2$, $\mu^2 = m_\mu^2$ and $\nu^2 = 0$. As we are not concerned with details of the muon, it is convenient to rewrite eq. (2) as

$$\mu = \pi - \nu, \tag{3}$$

   and square this to find

$$m_\mu^2 = m_\pi^2 - 2(\pi \cdot \nu). \tag{4}$$

   In the rest frame of the pion, its 4-vector can be written

$$\pi = (m_\pi, 0, 0, 0). \tag{5}$$

   Taking the z axis to be the direction of the pion in the lab frame, the 4-vector of the (massless) neutrino in the pion rest frame can be written as

$$\nu = (E_\nu^\star, E_\nu^\star \sin\theta^\star, 0, E_\nu^\star \cos\theta^\star), \tag{6}$$

since the energy and momentum of a massless particle are equal. The 4-vector product $(\pi \cdot \nu) = \pi_0 \nu_0 - \pi_i \nu_i$ is therefore

$$(\pi \cdot \nu) = m_\pi E_\nu^\star. \tag{7}$$

Hence, from eq. (4) the energy of the neutrino in the pion rest frame is

$$E_\nu^\star = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.8 \text{ MeV}, \tag{8}$$

using the stated facts.

We can now transform the neutrino 4-vector (6) to the lab frame, using the Lorentz boost $\gamma_\pi = E_\pi / m_\pi$,

$$\begin{aligned} \nu &= (E_\nu, E_\nu \sin\theta, 0, E_\nu \cos\theta) \\ &= (\gamma_\pi E_\nu^\star (1 + \beta_\pi \cos\theta^\star), E_\nu^\star \sin\theta^\star, 0, \gamma_\pi E_\nu^\star (\beta_\pi + \cos\theta^\star)). \end{aligned} \tag{9}$$

The pion has spin zero, so the decay is isotropic in the pion rest frame. A relation for the angle θ between the neutrino and its parent pion can be obtained from the 1 and 3 components of eq. (9),

$$\tan\theta = \frac{E_\nu^\star \sin\theta^\star}{\gamma_\pi E_\nu^\star (\beta_\pi + \cos\theta^\star)}. \tag{10}$$

The characteristic angle of the decay in the lab frame is usefully associated with decays at $\theta^\star = 90°$ in the pion rest frame. Thus,

$$\tan\theta_C = \frac{1}{\gamma_\pi \beta_\pi}. \tag{11}$$

When $E_\pi \gg m_\pi$ then $\gamma_\pi \gg 1$, $\beta_\pi \approx 1$, and

$$\theta_C \approx \frac{1}{\gamma_\pi} = \frac{m_\pi}{E_\pi} \ll 1. \tag{12}$$

3. We now consider the lab angle (10) between the neutrino and its parent pion with emphasis on the neutrino energy rather than the pion energy. If $E_\nu \gg m_\pi$, then $E_\pi \gg m_\pi$ also, so $\gamma_\pi \gg 1$ and $\beta_\pi \approx 1$. Then we can write

$$\tan\theta \approx \frac{E_\nu^\star \sin\theta^\star}{\gamma_\pi E_\nu^\star (1 + \cos\theta^\star)} \approx \frac{E_\nu^\star \sin\theta^\star}{E_\nu}, \tag{13}$$

using the time component of eq. (9). Since $\sin\theta^\star$ cannot exceed unity, we see that there is a maximum lab angle θ relative to the direction of the pion at which a neutrino of energy E_ν can appear, namely

$$\theta_{\max} \approx \frac{E_\nu^\star}{E_\nu} \approx \frac{30 \text{ MeV}}{E_\nu}, \tag{14}$$

which is small for $m_\pi \ll E_\nu$.

If instead, the angle θ is given, eq. (13) also tells us that

$$E_\nu \approx \frac{E_\nu^\star \sin\theta^\star}{\tan\theta} \leq \frac{E_\nu^\star}{\tan\theta}. \tag{15}$$

4. We desire the neutrino spectrum in terms of the laboratory quantities E_ν, θ and ϕ. We expect that the spectrum is uniform in the azimuthal angle ϕ. We are given the energy spectrum $dN/dE_\pi \propto (E_p - E_\pi)^5$ of the parent pions, and we have deduced that the spectrum is isotropic in the pion rest frame, *i.e.*, flat in $\cos\theta^\star$. Hence, we seek the transformation

$$\frac{d^2N}{dE_\nu d\Omega} \propto \frac{d^2N}{dE_\nu d\cos\theta} = \frac{d^2N}{dE_\pi d\cos\theta^\star} J(E_\pi, \cos\theta^\star; E_\nu, \cos\theta) \propto (E_p - E_\pi)^5 J, \qquad (16)$$

where the Jacobian is given by

$$J(E_\pi, \cos\theta^\star; E_\nu, \cos\theta) = \begin{vmatrix} \frac{\partial E_\pi}{\partial E_\nu} & \frac{\partial \cos\theta^\star}{\partial E_\nu} \\ \frac{\partial E_\pi}{\partial \cos\theta} & \frac{\partial \cos\theta^\star}{\partial \cos\theta} \end{vmatrix}. \qquad (17)$$

The "exact" form of the Jacobian is somewhat lengthy, so we will simplify to the extent we can by noting that when $E_\nu \gg m_\pi$, the parent pion has $E_\pi \gg m_\pi$ also, and so $\beta_\pi \approx 1$. Also, part 3 tells us that θ is very small for any value of $\theta^\star$.

We already have relation (13) between E_ν, $\tan\theta$ and $\sin\theta^\star$, so we can write

$$\cos\theta^\star = \sqrt{1 - \sin^2\theta^\star} \approx \sqrt{1 - \frac{E_\nu^2}{E_\nu^{\star 2}} \tan^2\theta} = \sqrt{1 - \frac{E_\nu^2}{E_\nu^{\star 2}} \left(\frac{1}{\cos^2\theta} - 1\right)}. \qquad (18)$$

Thus,

$$\frac{\partial \cos\theta^\star}{\partial E_\nu} \approx -\frac{\frac{E_\nu}{E_\nu^{\star 2}} \tan^2\theta}{\sqrt{1 - \frac{E_\nu^2}{E_\nu^{\star 2}} \tan^2\theta}} \approx -\frac{E_\nu \theta^2}{E_\nu^{\star 2} \cos\theta^\star}, \qquad (19)$$

for small θ, and

$$\frac{\partial \cos\theta^\star}{\partial \cos\theta} \approx \frac{\frac{E_\nu^2}{E_\nu^{\star 2} \cos^3\theta}}{\sqrt{1 - \frac{E_\nu^2}{E_\nu^{\star 2}} \tan^2\theta}} \approx \frac{E_\nu^2}{E_\nu^{\star 2} \cos\theta^\star}. \qquad (20)$$

We can also use time components of eq. (9) to write

$$\gamma_\pi = \frac{E_\pi}{m_\pi} = \frac{E_\nu}{E_\nu^\star (1 + \beta_\pi \cos\theta^\star)} \approx \frac{E_\nu}{E_\nu^\star (1 + \cos\theta^\star)} \qquad (21)$$

Hence,

$$\frac{\partial E_\pi}{\partial E_\nu} \approx \frac{m_\pi}{E_\nu^\star (1 + \cos\theta^\star)} - \frac{m_\pi E_\nu}{E_\nu^\star (1 + \cos\theta^\star)^2} \frac{\partial \cos\theta^\star}{\partial E_\nu} \approx \frac{E_\pi}{E_\nu} + \frac{E_\pi^2 \theta^2}{m_\pi E_\nu^\star \cos\theta^\star}, \qquad (22)$$

and

$$\frac{\partial E_\pi}{\partial \cos\theta} \approx -\frac{m_\pi E_\nu}{E_\nu^\star (1 + \cos\theta^\star)^2} \frac{\partial \cos\theta^\star}{\partial \cos\theta} \approx -\frac{E_\pi^2 E_\nu}{m_\pi E_\nu^\star \cos\theta^\star}. \qquad (23)$$

The Jacobian (17) is therefore

$$J \approx \begin{vmatrix} \frac{E_\pi}{E_\nu} + \frac{E_\pi^2\theta^2}{m_\pi E_\nu^\star \cos\theta^\star} & -\frac{E_\nu\theta^2}{E_\nu^{\star 2}\cos\theta^\star} \\ -\frac{E_\pi^2 E_\nu}{m_\pi E_\nu^\star \cos\theta^\star} & \frac{E_\nu^2}{E_\nu^{\star 2}\cos\theta^\star} \end{vmatrix} = \frac{E_\pi E_\nu}{E_\nu^{\star 2}\cos\theta^\star}, \tag{24}$$

and hence the neutrino spectrum can be written from eq. (16) as

$$\frac{d^2N}{dE_\nu d\cos\theta} \propto (E_p - E_\pi)^5 \frac{E_\pi E_\nu}{\cos\theta^\star}. \tag{25}$$

Because the factor $\cos\theta^\star$ in the denominator of the Jacobian can go to zero, it is possible that the neutrino flux is higher for nonzero values of the lab angle θ.

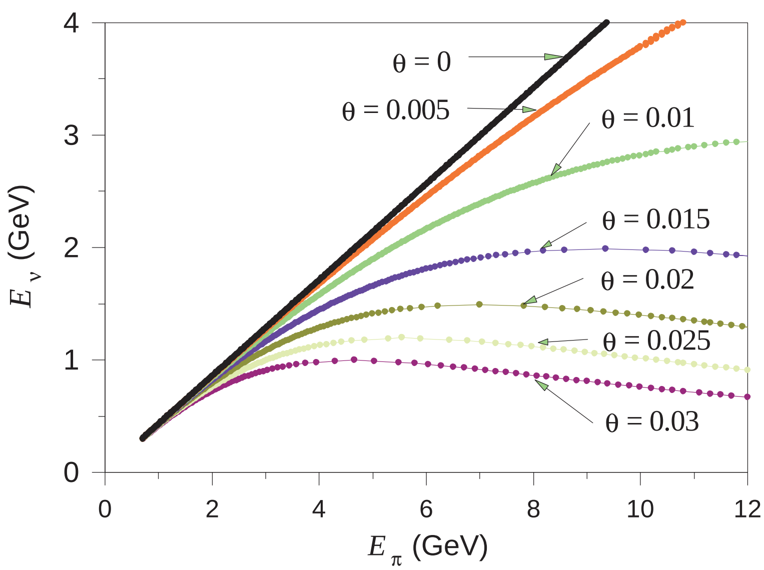
5. On the axis, $\theta = 0$, $\theta^\star = 0$, and $E_\pi = m_\pi E_\nu/2E_\nu^\star \approx 2E_\nu$ according to eq. (21). In this case, the neutrino spectrum (25) is

$$\frac{d^2N(\theta = 0)}{dE_\nu d\cos\theta} \propto \left(E_p - \frac{m_\pi E_\nu}{2E_\nu^\star}\right)^5 E_\nu^2. \tag{26}$$

The peak of the spectrum occurs at

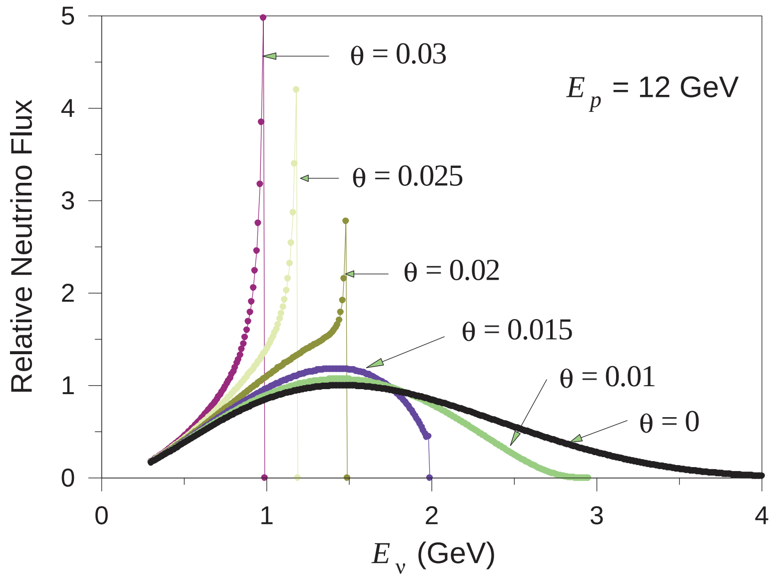
$$E_{\nu,\text{peak}} = \frac{4E_\nu^\star}{7m_\pi} E_p \approx \frac{E_p}{8}. \tag{27}$$

6. For an off-axis neutrino beam (at a nonzero value of angle θ) we must evaluate the spectrum (25) using relations (18) and (21). This is readily done numerically. For example, a plot of the pion energy E_π needed to produce a neutrino of energy E_ν at various angles θ is shown below.



As expected from part 3, we see that for a given angle θ, there is a maximum possible neutrino energy, and as the neutrino energy approaches this value, a large range of pion energies contributes to a small range of neutrino energies. This will result in an enhancement of the neutrino spectrum. If we desire the enhancement at a particular neutrino energy, we should look for the neutrinos close to the angle $\theta_{\max}$ given in eq. (14), which is independent of the proton/pion energy.

A numerical evaluation of the neutrino spectrum (25) for several values of angle θ with respect to the proton/pion beam is shown below.



We see that the spectrum of neutrinos at a nonzero angle is peaked at a lower energy, and is narrower, than that at zero degrees, due to the existence of a maximum possible neutrino energy (15) in decays at a given angle to the direction of the parent pion. This effect is especially prominent when $E_{\nu,\max}(\theta) \approx (30\text{ MeV})/\theta$ is less than $E_{\nu,\text{peak}}(\theta = 0)$, as then there is a substantial rate of higher energy pions all of which decay into a narrow band of neutrino energies at this angle.

The spectral narrowing of an off-axis neutrino beam remains in more complete calculation [2, 3] that include the nonzero transverse momenta of the pions before and after passing through the "horn", although the spectrum will not have such hard edges, and the favored angle-energy combination is $\theta \approx (50\text{ MeV})/E_\nu$.

In sum, the existence of a maximum energy for neutrinos that decay at a given angle to their parent pions implies that many different pion energies contribute to the this neutrino energy, which enhances the neutrino spectrum at this angle-energy combination, $\theta \approx (30\text{-}50\text{ MeV})/E_\nu$.

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References

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