Maxwell and Special Relativity
Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(May 26, 2014; updated December 22, 2017)

It is now commonly considered that Maxwell’s equations [1] in vacuum implicitly contain the special theory of relativity.\(^1\)

For example, these equations imply that the speed \(c\) of light in vacuum is related by
\[
\frac{1}{\sqrt{\epsilon_0\mu_0}},
\]
(1)

where the constants \(\epsilon_0\) and \(\mu_0\) can be determined in any (inertial) frame via electrostatic and magnetostatic experiments (nominally in vacuum).\(^2\) Even in æther theories, the velocity of the laboratory with respect to the hypothetical æther should not affect the results of these static experiments,\(^3\) so the speed of light should be the same in any (inertial) frame. Then, the theory of special relativity, as developed in [2], follows from this remarkable fact.

Maxwell does not appear to have crisply drawn the above conclusion, that the speed of light is independent of the velocity of the observer, but he did make arguments in Arts. 599 and 770 of [3] that correspond to the low-velocity approximation to special relativity, as pointed out in sec. 5 of [6]. These arguments contrast with claims [7, 8] that Maxwell tacitly assumed that Galilean relativity [9] applied to his electrodynamics.

The notion of Galilean electrodynamics seems to have been developed only in 1973 [9]. In this view there are no electromagnetic waves, and only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics. In fact, there are two variants of Galilean electrodynamics, so-called electric Galilean relativity in which the transformations between two inertial frames with relative velocity \(v\) are (sec. 2.2 of [9], but given here in Gaussian units)
\[
\rho'_e = \rho_e, \quad J'_e = J_e - \rho_e v \quad (c |\rho_e| \gg |J_e|),
\]
\[
E'_e = E_e, \quad B'_e = B_e - \frac{v}{c} \times E_e \quad f_e = \rho_e E_e \quad \text{(electric)},
\]
(2)

and so-called magnetic Galilean relativity (sec. 2.3 of [9]) with transformations
\[
\rho'_m = \rho_m - \frac{v}{c^2} \cdot J_m \quad J'_m = J_m \quad (c |\rho_e| \ll |J_e|),
\]
\[
E'_m = E_m + \frac{v}{c} \times B_m, \quad B'_m = B_m \quad f_m = \rho_m \left(E_m + \frac{v}{c} \times B_m\right) \quad \text{(magnetic)}. \quad \text{(3)}
\]

---

\(^1\)Maxwell’s electrodynamics was the acknowledged inspiration to Einstein in his 1905 paper [2].
\(^2\)Equation (1) is a transcription into SI units of the discussion in sec. 80 of and sec. 758 of [3].
\(^3\)As discussed in [4], examples of a “static” current-carrying wire involve effects of order \(v^2/c^2\) where \(v\) is the speed of the moving charges of the current. A consistent view of this in the rest frame of the moving charges requires special relativity. These arguments could have been made as early as 1820, but it took 85 years for them to be fully developed.

It is now sometimes said that electricity plus special relativity implies magnetism, but a more historical view is that (static) electricity plus magnetism implies special relativity. This theme is emphasized in [5].

\(^4\)This ansatz is a weak form of Einstein’s Principle of Relativity.

\(^5\)In Galilean electrodynamics the symbol \(c\) does not represent the speed of light (as light does exist in this theory), but only the function \(1/\sqrt{\epsilon_0\mu_0}\) of the (static) permittivity and permeability of the vacuum.
For comparison, the low-velocity limit of special relativity has the transformations,
\[ \rho'_s \approx \rho_s - \frac{v}{c^2} \cdot J_s, \quad J'_s \approx J_s - \rho_s v, \]
\[ E'_s \approx E_s + \frac{v}{c} \times B_s, \quad B'_s \approx B_s - \frac{v}{c} \times E_s \]
(special relativity, \( v \ll c \)). \tag{4}

\section{1 Articles 598-599 of Maxwell’s \textit{Treatise}}

In Arts. 598-599 of his \textit{Treatise} [3], Maxwell argues that an electric charge \( q \) which moves with velocity \( v \) in electric and magnetic fields \( E \) and \( B = \mu H \) experiences an “electromotive intensity,” \textit{i.e.}, an electromagnetic force,\(^7\)
\[ F = q \left( E + \frac{v}{c} \times B \right). \tag{5} \]
This is now known as the Lorentz force,\(^8\) and it seems seldom noted that Maxwell gave this form, likely because he disguised it by writing (in electromagnetic units), eq. (10) of Art. 599,\(^9\)
\[ \mathcal{E} = \mathcal{H} \times \mathcal{B} - \dot{\mathcal{A}} - \nabla \Psi, \tag{6} \]
where \( \mathcal{E} \) is the “electromotive intensity,” \textit{i.e.}, \( F/q \), \( \mathcal{B} \) is the velocity \( v \), \( \mathcal{B} \) is the magnetic field \( B \), \( \mathcal{A} \) is the vector potential and \( \Psi \) is the scalar potential. Then, eq. (5) follows noting that the electric field is given (in emu) by \( -\partial A/\partial t - \nabla \Psi \),\(^{10}\) and that \( v \times B \) in emu becomes \( v/c \times B \) in Gaussian units.

We can also interpret Maxwell’s \( \mathcal{E} \) as the electric field \( E' \) in the frame of the moving circuit, such that Maxwell’s transformation of the electric field is,\(^{11}\)
\[ E' = E + \frac{v}{c} \times B. \tag{7} \]

The transformation (7) is compatible with both magnetic Galilean relativity, eq. (3), and the low-velocity limit of special relativity, eq. (4). These two version of relativity differ as to the transformation of the magnetic field. In particular, if \( B = 0 \) while \( E \) were due to a single

\(^6\)Electromagnetic waves can exist in low-velocity approximation to special relativity, and, of course, propagate in vacuum with speed \( c \).

\(^7\)Maxwell did not hold a view of an electric charge as a “particle,” but rather as a state of “displaced” æther.

\(^8\)Lorentz actually advocated the form \( F = q(D + v \times H) \) in eq. (V), p. 21, of [10], although he seems mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [11]. That is, Lorentz considered \( D \) and \( H \), rather than \( E \) and \( B \), to be the microscopic electromagnetic fields.

\(^9\)This result also appeared in sec. 65 of [1] (1864), where \( v/c \times \mu H \) was called the “electromotive force.”

\(^{10}\)This assumes that Maxwell’s \( \dot{\mathcal{A}} \) corresponds to \( \partial A/\partial t \), and not to the convective derivative \( DA/Dt = \partial A/\partial t + (v \cdot \nabla) A \).

\(^{11}\)A more direct use of Faraday’s law, without invoking potentials, to deduce the electric field in the frame of a moving circuit was made in sec. 9-3, p. 160, of [12], which argument appeared earlier in sec. 86, p. 398, of [13]. An extension of this argument to deduce the full Lorentz transformation of the electromagnetic fields \( E \) and \( B \) is given in Appendix B below.
electric charge at rest (in the unprimed frame), then magnetic Galilean relativity predicts that the moving charge/observer would consider the magnetic field $\mathbf{B}'$ to be zero, whereas it is nonzero according to special relativity.

Maxwell does not comment in Art. 599 on the magnetic field as observed by the moving charge. He does make some remarks on the kinematic relations between two frames of reference with relative velocity $\mathbf{v}$ in Arts. 600-601, assuming Galilean relativity for these, with the main (and well-known) conclusion being that force is the same in both frames, $\mathbf{F}' = \mathbf{F}$ (as is also true in the low-velocity approximation to special relativity).

2 Articles 769-770 of Maxwell’s Treatise

Faraday considered that a moving electric charge generates a magnetic field [14] (as quoted in [15]). Maxwell also argued for this in Arts. 769-770 of [3], where his verbal argument can be transcribed as

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (8)$$

for the magnetic field experienced by a fixed observer due to a moving charge. Maxwell noted that this is a very small effect, and claimed that it had never been observed. However, the magnetic field of a moving charge had been detected in 1876 by Rowland [15, 16] (while working in Helmholtz’ lab in Berlin). The form (8) was verified (in theory) more explicitly by J.J. Thomson in 1881 [17] for uniform speed $v \ll c$, and for any $v < c$ by Heaviside [18] and by Thomson [19] in 1889 (which latter two works gave the full special-relativistic form for $\mathbf{E}$ as well).

If $\mathbf{v}$ represents the velocity of a moving observer relative to a fixed electric charge, then eq. (8) implies that the magnetic field experienced by the moving observed would be

$$\mathbf{B}' = -\frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (9)$$

This corresponds to the low-velocity limit (4) of special relativity, but not to the prediction (3) of magnetic Galilean relativity (or to the prediction (2) of electric Galilean relativity). It is therefore wrong to argue [7, 8] that Maxwell and his followers (such as Thomson, Heaviside and FitzGerald [20, 21]) tacitly used Galilean relativity for transformation of the field, although they did use Galilean transformations of spacetime coordinates and mechanical quantities.\(^\text{13}\)

This note was stimulated by e-discussions with Dragan Redžić.

---

\(^\text{12}\)It was noted by Larmor in 1884 on p. 12 of [37] that Maxwell’s analysis also implies effects of order $v^2/c^2$, although Larmor ignored these.

\(^\text{13}\)See Appendix C below for discussion of a “pre-relativistic” argument by J.J. Thomson.
A Maxwell’s Derivation of the Lorentz Force Law

A.1 In On Faraday’s Lines of Force

On p. 47 of [25], Maxwell wrote a version of Ohm’s Law as \( F = IK \) for an electrical circuit of resistance \( K \) that carries current \( I \). He calls \( F \) the electro-motive force, which is consistent with a more contemporary notation,

\[
\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = IR \quad (F = IK),
\]

where \( \mathcal{E} = F \) is the electromotive “force” (with dimensions of electric potential (voltage) rather than of force), \( \mathbf{E} \) is the electrical field and \( d\mathbf{l} \) is line element inside the wire of the circuit.

On p. 53, Maxwell introduced the (free/conduction) electric current-density vector \( \mathbf{J} = (a_2, b_2, c_2) \), the electric scalar potential \( \Psi = p_2 \), and the electric field \( \mathbf{E} = (\alpha_2, \beta_2, \gamma_2) \), writing in his eq. (A),

\[
\mathbf{E} = \mathbf{E}_{\text{other}} - \nabla \Psi \quad \left( \alpha_2 = X_2 - \frac{dp_2}{dx}, \text{ etc.} \right),
\]

with \( \mathbf{E}_{\text{other}} = (X_2, Y_2, Z_2) \) being a possible contribution to the electric field not associated with a scalar potential. To possible confusion, Maxwell called the vector \( \mathbf{E} \) an electro-motive force, which term he also used for the scalar \( \mathcal{E} \).

Also on p. 53, Maxwell introduced the electrical resistivity \( \rho = k_2 \) (reciprocal of the electrical conductivity \( \sigma = 1/\rho \), so the Ohm’s Law can be written as Maxwell’s eq. (B),

\[
\mathbf{E} = \rho \mathbf{J} = \frac{\mathbf{J}}{\sigma} \quad (\alpha_2 = k_2 a_2, \text{ etc.}),
\]

On p. 54, Maxwell noted that for any closed curve eq. (11) implies

\[
\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{E}_{\text{other}},
\]

He also introduces the concept of the flux (conduction) \( \mathbf{E} \cdot d\mathbf{S} \) of a field \( \mathbf{E} \) across a surface element \( d\mathbf{S} \), and noted (Gauss’ Law) that for a closed surface,

\[
\int \mathbf{E} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{E} d\text{Vol},
\]

He indicated in his eq. (C) that he will often write the divergence of a vector field as \( 4\pi \rho \).

On p. 54 Maxwell also introduced magnetic phenomena, and emphasized a formal parallel with electric phenomena. He labeled the magnetic field \( \mathbf{H} \) as \( (\alpha_1, \beta_1, \gamma_1) \) and the magnetic (induction) field \( \mathbf{B} \) as \( (a_1, b_1, c_1) \), called the (relative) magnetic permeability \( \mu \) the reciprocal of the resistance to magnetic induction, \( k_1 \), and noted (in words) that the parallel to our eq. (12), his eq. (B), is

\[
\mathbf{H} = \frac{\mathbf{B}}{\mu} \quad (\alpha_1 = k_1 a_1, \text{ etc.}),
\]
and that in the relation $\nabla \cdot \mathbf{B} = 4\pi \rho_m$, $\rho_m$ is the density of real magnetic matter.

On p. 56, Maxwell stated Ampère’s Law in the form,\(^{14}\)

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{free}} \quad \left( a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}, \text{ etc.} \right),$$

(16)

and on p. 57 he noted that the divergence of eq. (16) is zero, so that his discussion is limited to closed currents that obey $\nabla \cdot \mathbf{J} = 0$ (i.e., to magnetostatics). Indeed, he added: in fact we know little of the magnetic effects of any current that is not closed.\(^{15}\)

There followed an interlude on various theorems, some due to Green [33], and also Helmholtz’ theorem [34] that “any” vector field $\mathbf{E}$ can be related to a scalar potential $\Psi$ and a vector potential $\mathbf{A}$ as $\mathbf{E} = \nabla \Psi + \nabla \times \mathbf{A}$, where $\Psi = 0$ if $\nabla \cdot \mathbf{E} = 0$ and $\mathbf{A} = 0$ if $\nabla \times \mathbf{E} = 0$.

After this, Maxwell considered the energy stored in the magnetic field. On p. 63, he first argued that if the magnetic field were due to a density $\rho_m$ of magnetic charges, the field could be deduced from a magnetic scalar potential $\Psi_m$ and the energy stored in the field during the assembly of this configuration could be written\(^{16}\)

$$U_m = \frac{1}{2} \int \rho_m \Psi_m d\text{Vol}. \quad (17)$$

He then noted that this form can be transformed to

$$U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{8\pi}, d\text{Vol}. \quad (18)$$

He next argued that since this form does not include any trace of the origin of the magnetic field, it should also hold if the field is due to electrical currents, it can be transformed to

$$U_m = \int \frac{\mathbf{J} \cdot \mathbf{A}}{2c}, d\text{Vol}, \quad (19)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$.\(^{17}\) The rate of change of energy is the magnetic field is $-\mathbf{J} \cdot \mathbf{E}$,\(^{18}\) so, by taking the time derivative of eq. (19), and noting that $\mathbf{A} = (\alpha_0, \beta_0, \gamma_0)$ scales linearly with $\mathbf{J}$, he infers that the electro-motive force due to the action of magnets and currents is

$$\mathbf{E}_{\text{induced}} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad \left( a_2 = -\frac{1}{4\pi} \frac{d\alpha_0}{dt}, \text{ etc.} \right)$$

(20)

\(^{14}\)This is probably the first statement of Ampère’s Law as a differential equation.

\(^{15}\)This statement can be regarded as a precursor to Maxwell’s later vision (first enunciated in eq. (112), p. 19, of [29]) that all currents are closed if one considers the “displacement-current” (density) $1/4\pi) d\mathbf{D}/dt$ in addition to the conduction-current density $\mathbf{J}_{\text{free}}$. But it also indicates that Maxwell chose not to consider the notion of moving charged particle as elements of an electrical current, as advocated by Weber (1843) [31] (see p. 88 of the English translation) as a way of understanding Ampère’s expression for the force between two current loops.

For discussion of the “displacement current” of a uniformly moving charge, see [32].

\(^{16}\)It seems to this author that Maxwell omitted a factor of 1/2 in throughout his discussion on pp. 63-64.

\(^{17}\)Maxwell did not seem to suppose in [25] that there is no real magnetic matter, so his $\mathbf{B}$ was also related to a scalar potential, and his version of eq. (19) has and additional term related to possible magnetic charges and the scalar potential.

\(^{18}\)Most contemporary discussions of magnetic field energy start from the relation and work towards eq. (19) and then (18).
On p. 66, he stated this equation in words as: Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction. Here, Maxwell describes the vector potential $A$ as the sl electro-tonic intensity, following Faraday.

At the bottom of p. 64, Maxwell made a statement that proved to be problematic for his later efforts towards the “Lorentz” force law: If $\alpha_0$ be expressed as a function of $x$, $y$, $z$, and $t$, and if $x$, $y$, $z$ be the co-ordinates of a moving particle, then the electro-motive force measured in the direction of $x$ is

$$\alpha_2 = -\frac{1}{4\pi} \left( \frac{d\alpha_0}{dx} + \frac{d\alpha_0}{dy} \frac{dy}{dt} + \frac{d\alpha_0}{dz} \frac{dz}{dt} + \frac{d\alpha_0}{dt} \right), \quad \mathbf{E}_{\text{induced}} = -\frac{1}{c} \left( \frac{\partial A}{\partial t} + (\mathbf{v} \cdot \nabla) A \right), \quad (21)$$

Here, Maxwell claimed that the electric field experienced by a moving particle should be computed using the convective derivative of the vector potential, and not just the partial time derivative.

Faraday’s Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [25]: the electro-motive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit, as a summary of Faraday’s comments in [26].

A.2 In On Physical Lines of Force

Maxwell deduced (via an energy argument!) the differential form of Faraday’s law, $\nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B} / \partial t$, in eq. (54) of [28] (although he used $\mu \mathbf{H}$ rather than $\mathbf{B}$ for the magnetic field as the latter was only invented by W. Thomson in 1871, eq. (r), p. 401 of [35]). Surprisingly, Maxwell did not give this differential form either in [1] or in his Treatise [3].

A.3 In A Dynamical Theory of the Electromagnetic Field

In sec. 24 of [1] Maxwell considered an electrical circuit $A$ that carries current $I_A$, and another circuit $B$ that carries current $I_B$, and noted that the magnetic flux, $\Phi_m$, through circuit $A$ is given by the first equation on p. 468,

$$\Phi_m = \int_A \mathbf{B} \cdot d\mathbf{S} = LI_A + MI_B,$$  \quad (22)

where $\mathbf{B}$ is the magnetic field, $d\mathbf{S}$ is an element of the area of a surface bounded by circuit $A$, $L$ is the self inductance of circuit $A$, and $M$ is the mutual inductance between circuits $A$ and $B$. Maxwell called this flux the “momentum,” or the “reduced momentum” of the circuit.

In sec. 50, Maxwell gave a verbal statement of Faraday’s Law:

1st, If any closed curve be drawn in the field, the value of $M$ for that curve will be expressed by the number of lines of force which pass through that closed curve.

2ndly. If this curve be a conducting circuit and be moved through the field, an electromotive force will act in it, represented by the rate of decrease of the number of lines passing through the curve.

We transcribe this into symbols (in Gaussian units) as
\[ E = -\frac{1}{c} \frac{d\Phi_m}{dt}, \]  

(23)

where \( E \) is the electromotive force, which is a scalar.

However, in sec. 56, Maxwell used the term “electromotive force” in a different way, to describe a vector, \( \mathbf{E} = (P, Q, R) \): \( P \) represents the difference of potential per unit of length in a conductor placed in the direction of \( x \) at the given point. This appears to mean that

\[ \mathbf{E} = (P, Q, R) = \nabla \Psi, \]  

(24)

where \( \Psi \) is Maxwell’s symbol for the electric scalar potential. If so, this is the first mention of an aspect of the electric field \( \mathbf{E} \) in [1], although he had introduced the electric “displacement” \( \mathbf{D} = (f, g, h) \) in sec. 55, along the with “displacement-current” (density), \( (1/4\pi)\frac{d\mathbf{D}}{dt} \) in his eq. (A),

\[ \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \frac{1}{4\pi} \frac{d\mathbf{D}}{dt} \left( (p', q', r) = (p, q, r) + \frac{d(f, g, h)}{dt} \right). \]  

(25)

Maxwell did not use separate symbols for partial and total derivatives, so that there can be some ambiguity as to his meaning when his equation describe moving systems.

In sec. 57, Maxwell introduced the vector potential \( \mathbf{A} = (F, G, H) \), but called it the “electromagnetic momentum.” In his eq. (29), Maxwell identified \(-\frac{d\mathbf{A}}{dt}\) with the part of the electromotive force which depends on the motion of magnets or currents. Thus, we might now presume that Maxwell’s \( \mathbf{E} = (P, Q, R) \) is the electric field,

\[ \mathbf{E} = -\nabla \Psi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \]  

(26)

but this conclusion may be premature.

In eq. (29) of sec. 58, Maxwell gave the relation

\[ \Phi_m = \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{S}, \]  

(27)

and called this the “total momentum” (which we must remember to distinguish from the “electromagnetic momentum” \( \mathbf{A} \)).

He also noted in sec. 58 that

\[ \oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S}. \]  

(28)

In sec. 59, Maxwell introduced the magnetic field \( \mathbf{H} = (\alpha, \beta, \gamma) \).

In sec. 60, Maxwell introduced the (relative) permeability \( \mu \), calling it the “coefficient of magnetic induction.”

In eq. (B) of sec. 61, Maxwell gave the “Equations for Magnetic Force,”

\[ \mu \mathbf{H} = \nabla \times \mathbf{A}. \]  

(29)

We use the symbol \( \mathbf{B} \) for Maxwell’s \( \mu \mathbf{H} \).\(^{19}\)

\(^{19}\)The symbol \( \mathbf{B} \) for the quantity \( \mathbf{H} + 4\pi \mathbf{M} \), where \( \mathbf{M} \) is the magnetization density, was introduced by W. Thomson in 1871, eq. (r), p. 401 of [35], and appears in sec. 399 of Maxwell’s Treatise [3].
In eq. (C) of sec. 62, Maxwell gives a version of Ampère's Law,

\[ \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}, \]

but does not clarify here whether the current density \( \mathbf{J} \) is only that of conduction currents, or refers to the total current that he had introduced in his eq. (A) of sec. 56, our eq. (25).

In eq. (32), sec. 63 of [1], Maxwell stated the “electromotive force” in an electrical circuit is related by\(^{20}\)

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l}. \]

Supposing this circuit, A, carries current \( I_A \), and another circuit, B, carries current \( I_B \), Maxwell, in his eq. (33), reminded us the magnetic flux through circuit A is given by our eq. (22)

\[ \Phi_m = \oint_A \mathbf{A} \cdot d\mathbf{l} = LI_A + MI_B, \]

as he had previously discussed in sec. 24. Then, his eq. (34) states that

\[ \mathcal{E} = -\frac{1}{c} \frac{d}{dt} (LI_A + MI_B) \left( = -\frac{1}{c} \int \frac{d\mathbf{A}}{dt} \cdot d\mathbf{l} \right), \]

so comparison with our eq. (31) leads to the inference, Maxwell’s eq. (35), that if there is no motion of the circuit A,

\[ \mathcal{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi, \]

where \( \Psi \) could be any scalar function. But, the discussion in his sec. 56 led Maxwell to identify \( \Psi \) of eq. (34) as the electrical scalar potential.

In eq. (34), we have written \( \partial \mathbf{A}/\partial t \), while Maxwell wrote \( d\mathbf{A}/dt \), in that for an observer (at rest) of a circuit at rest, use of a convective derivative is not appropriate.

In sec. 64, Maxwell deduced the force on a bar the slides on a U-shaped rail, while carrying a current, with the entire system in an external magnetic field. He gave no figure in [1], but the figure below is associated with his discussion of this example in Arts. 594-597 of [3]. \( C \) represents a battery that drives the current in the circuit.

\(^{20}\)This meaning of the term “electromotive force” is still in use today. However, Maxwell also used the term “electromotive force” in sec. 65 of [1] to describe the force \( \mathbf{v}/c \times \mu \mathbf{H} \) on a moving, unit charge in a magnetic field, referring to his eq. (D).
Maxwell desired a very general discussion in sec. 64, so he considered a circuit whose plane was not perpendicular to any of the \( x, y \) or \( z \) axes, which makes his description rather intricate. Here, we suppose the circuit lies in the \( x-z \) plane, with the sliding piece, \( AB \), of length \( a \) parallel to the \( x \)-axis, and the long arms of the U-shaped rail parallel to the \( z \)-axis, at, say \( x = 0 \) and \( a \). The velocity \( v_z = dz/dt \) of the sliding bar is in the \( z \)-direction, and the uniform, external magnetic field is in the \( y \)-direction.

As in sec. 63, Maxwell considered changes in the magnetic flux through the circuit, \( \oint \mathbf{A} \cdot d\mathbf{l} \), our eq. (27), to infer the strength of his vector \( \mathbf{E} \). The part of the line integral over the sliding bar changes at rate,

\[ a \frac{dA_x}{dz} \frac{dz}{dt}, \]

as indicated in the first equation on p. 485. Because the length of the circuit in \( z \) is increasing, the line integral also changes at rate

\[ \frac{dz}{dt} [A_z(x = 0) - A_z(x = a)] = -a \frac{dA_z}{dx} \frac{dz}{dt}, \]

as given in the second equation on p. 485. Hence, the total rate of change of magnetic flux, given in the third and fourth equations on p. 485, is

\[ \frac{d\Phi_m}{dt} = av_z \left( \frac{dA_x}{dz} - \frac{dA_z}{dx} \right) = av_z B_y = -c \mathbf{E} = -c \oint \mathbf{E} \cdot d\mathbf{l}. \]  

Maxwell considered that eq. (37) describes a contribution to the electromotive force \( \mathbf{E} \) beyond that in eq. (34), which additional contribution would be localized to the component \( \mathbf{E}_x \) along the sliding bar (of length \( a \)), \( i.e., \oint \mathbf{E} \cdot d\mathbf{l} = a \mathbf{E}_x \). Hence, he concluded in his eq. (36) that

\[ \mathbf{E}_x = -\frac{v_z B_y}{c} \left( = -\mu \beta \frac{dz}{dt} \right), \quad i.e., \quad \mathbf{E} = \frac{v}{c} \times \mathbf{B}, \]

is the part of \( \mathbf{E} \) due to the motion of the sliding bar.

Finally, in sec. 65, Maxwell stated that the total electromotive force on a moving conductor is his eq. (D),

\[ \mathbf{E} = \frac{v}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi, \]

recalling his eq. (35), our eq. (34). Again, we have written \( \partial \mathbf{A}/\partial t \) where Maxwell wrote \( d\mathbf{A}/dt \).

Note that Maxwell’s argument in sec. 65 does not address the mechanical force on the sliding bar, \( I \mathbf{a} \times \mathbf{B}/c \), which is the subject of most present discussions of this example.

### A.4 In Maxwell’s Treatise

#### B Appendix: From Faraday’s Law to the Lorentz Transformation of the Electromagnetic Fields

This section is based on [23] (1979).

---

21In eqs.(35)-(38), the quantities \( A_x, A_z \) and \( B_y \) are evaluated at the location of the sliding bar.

22On p. 485, the equations of Magnetic Force (8) should read: the equations of Magnetic Force (B).

23Note that convective....
B.1 Force on a Moving Circuit

For a circuit at rest, the integral form of Faraday’s law can be written as

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{1}{c} \frac{d\Phi_B}{dt} = -\frac{1}{c} \frac{d}{dt} \oint \mathbf{A} \cdot d\mathbf{l} \],

(40)

where \( \mathcal{E} \) is the electromotive force in the circuit, \( \mathbf{E} \) is the electric field, \( d\mathbf{l} \) is an element of length along the circuit, \( d\mathbf{S} \) is an element of area of a surface bounded by the current loop that generates magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \), \( \mathbf{A} \) is the vector potential (Faraday’s electrotonic state\(^{24} \)), and \( \Phi_B = \int \mathbf{B} \cdot d\mathbf{S} \) is the magnetic flux through the circuit.\(^{25,26} \)

We now consider a circuit that moves with velocity \( \mathbf{v} \) in the lab. An inference from Galilean relativity (which implies Newton’s First Law\(^{27} \)) is that an observer moving with

\(^{24}\)Faraday introduced his electrotonic state in Art. 60 of [24].

\(^{25}\)Equation (40) also implies the relation

\[ \mathbf{E}_{\text{induced}} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \],

(41)

for the electric field induced by a changing magnetic field (due to a changing current in the circuit).

\(^{26}\)Faraday’s Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [25]: the electro-motive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit, as a summary of Faraday’s comments in [20].

Maxwell did not give the mathematical form (40) in [25]), but he did give (the equivalent of) eq. (41) on p. 64 (\( \alpha_2 = -\frac{1}{4\pi} \frac{d\Phi}{dt} \), etc.). Then, on p. 66 he stated this equation in words as: Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction.

Maxwell deduced (via an energy argument!) the differential form of Faraday’s law, \( \nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B}/\partial t \), in eq. (54) of [28] (although he used \( \mu \mathbf{H} \) rather than \( \mathbf{B} \) for the magnetic field as the latter was only invented by W. Thomson in 1871, eq. (r), p. 401 of [35]). Surprisingly, Maxwell did not give this differential form either in [1] or in his Treatise [3].

Digression: On p. 64 of [25], Maxwell deduced that the electric field induced by changing currents at a point at rest in the lab is that given in eq. (41) above. Then, he stated that for a moving (charged) “particle” with velocity \( \mathbf{v} \), the field it experiences should be computed using the convective derivative, \(-c \mathbf{E}_{\text{on moving charge}} = D\mathbf{A}/Dt = \partial \mathbf{A}/\partial t + (\mathbf{v} \times \nabla)\mathbf{A} \).

While we now consider electric charge to be a phenomenon separate from the electromagnetic field, Maxwell considered charge (density) to be an aspect of “displacement” in the æther, \( \rho = \nabla \cdot \mathbf{D} \). In this view, it seems natural to suppose that a moving charge samples an electromagnetic field in a manner analogous to a particle moving through a fluid, where the convective derivative describes the time dependence of, for example, pressure and density its experiences.

Later, in eq. (D), sec. 65, p. 485, of [1] and eq. (B), Art. 598, p. 239 (and also eq. (10), Art. 599, p. 241) of [3], Maxwell realized that the force on a moving charge should include the term \( \mathbf{v} \cdot \mathbf{c} \times \mathbf{B} \) (and not \( \mathbf{H} \)), and managed to arrive at the correct “Lorentz” force law despite his use of the convective derivative (in eq. (2) of Art. 598 of [3]). However, Helmholtz, eq. (5‘), p. 309 of [36], argued that the term \( \mathbf{v} \cdot \mathbf{c} \times \mathbf{B} \) should be accompanied by the additional term \(-\nabla(\mathbf{A} \cdot \mathbf{v}/c) \), which claim was seconded on p. 12 of [37] (1884) and on p. 273 of [38] (1888). These claims may have had the effect that Maxwell’s derivation of the force on a moving “particle,” when corrected/clarified, was not considered to yield the Lorentz force law. For example, when J.J. Thomson edited the 3rd edition of Maxwell’s Treatise he added a comment, p. 260 of [3], casting doubt Maxwell’s analysis of the force on a moving charge. This is unfortunate in that Lorentz, eq. (V), sec. 12 of [10] and eq. (23) of [11], wrote the force law as \( \mathbf{F} = q(\mathbf{D} + \mathbf{v} \times \mathbf{H}) \), which is not correct, although this was clarified only in 1944 by experiments [39] on the motion of high-energy particles penetrating magnetized steel.

\(^{27}\)Law I. Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is
velocity \( v \) measures fields \( E' \) and \( B' \), and the integral form of Faraday’s law for this observer would be

\[
E' = \oint E' \cdot dl' = \oint E' \cdot dl = -\frac{1}{c'} \frac{d}{dt'} \int B' \cdot dS' = -\frac{1}{c} \frac{d}{dt} \int B' \cdot dS,
\]

(42)

where the integrals are performed over the circuit in the moving frame, where it is at rest. That is, according to Galilean relativity, the moving observer measures length and time to be the same as for an observer at rest, and the constant \( c \), which is determined by experiments performed at rest in any frame, has the same value in any (inertial) frame.

We next suppose that \( B' = B \), i.e., observers in the lab frame and in the moving frame assign the same value to the magnetic field. We now consider how an observer at rest in the lab frame might describe the moving observer’s calculation, of \( (d/dt) \int B' \cdot dS \) for a circuit at rest in his frame, as

\[
\int \frac{DB}{Dt} \cdot dS,
\]

(43)

over the circuit at time \( t \) in the lab frame. Of course, \( DB/ Dt \) does not equal \( \partial B/ \partial t \) as it must incorporate effects of the motion of the circuit in the lab frame.\(^{28}\)

Referring to the figure below, which is in the lab frame, and where the direction of a surface element \( dS \) is related to the line element \( dl \) by the right-hand rule,

![Diagram](image)

we can express the time derivative of the magnetic flux through the moving circuit in lab-frame quantities as,

\[
\frac{d}{dt} \int_{\text{moving circuit}} B \cdot dS \approx \frac{1}{dt} \left[ \int_{t+dt} B_{t+dt} \cdot dS_{t+dt} - \int_t B_t \cdot dS_t \right]. \tag{44}
\]

We have, approximately, that

\[
B_{t+dt} = B_t + \frac{\partial B_t}{\partial t} dt,
\]

(45)

so

\[
\frac{d}{dt} \int_{\text{moving circuit}} B \cdot dS \approx \int_{t+dt} \frac{\partial B_t}{\partial t} \cdot dS_{t+dt} + \frac{1}{dt} \left[ \int_{t+dt} B_t \cdot dS_{t+dt} - \int_t B_t \cdot dS_t \right]. \tag{46}
\]

\(^{28}\)Such issues occur frequently in fluid dynamics.

\(\text{compelled to change that state by forces impressed thereon; p. 83 of [40].}\)
We now play a famous trick, and consider the integral over the entire surface of the volume swept out by the circuit during time interval \(dt\), taking \(dS\) to be directed out of this volume,

\[
\int_{\text{entire surface}} B_t \cdot dS = \int_{t+dt} B_t \cdot dS_{t+dt} - \int_t B_t \cdot dS_t + \int_{\text{side}} B_t \cdot dS_{\text{side}}. \tag{47}
\]

By Gauss' theorem,

\[
\int B_t \cdot dS = \int \nabla \cdot B_t \, d\text{Vol} = \int (\nabla \cdot B_t) \, v \, dt \cdot dS_t, \tag{48}
\]

and an area element on the “sides” of the surface is related by \(dS_{\text{side}} = d\mathbf{l} \times v \, dt\), such that

\[
\int_{\text{side}} B_t \cdot dS_{\text{side}} = \oint B_t \cdot d\mathbf{l} \times v \, dt = -dt \oint B_t \times d\mathbf{l} = dt \int \nabla \times (B_t \times v) \cdot dS_t. \tag{49}
\]

We can now rewrite eq. (47) as

\[
\int_{t+dt} B_t \cdot dS_{t+dt} - \int_t B_t \cdot dS_t = dt \int (\nabla \cdot B_t) \, v \cdot dS_t + dt \int \nabla \times (B_t \times v) \cdot dS_t. \tag{50}
\]

Then, recalling eq. (46), and taking the limit as \(dt \to 0\), we have that

\[
\frac{d}{dt} \int \, B \cdot dS = \int \frac{\partial B}{\partial t} \cdot dS + \int (\nabla \cdot B) \, v \cdot dS + \int \nabla \times (B \times v) \cdot dS \equiv \int \frac{DB}{Dt} \cdot dS, \tag{51}
\]

where

\[
\frac{DB}{Dt} = \frac{\partial B}{\partial t} + (\nabla \cdot B) \, v + \nabla \times (B \times v) = \frac{\partial B}{\partial t} + (v \cdot \nabla)B \tag{52}
\]

is the convective derivative.

Faraday’s Law for the moving circuit, eq. (42), can now be written as

\[
\mathcal{E}' = \oint_{\text{moving circuit}} \mathbf{E}' \cdot d\mathbf{l} = \int_{\text{moving circuit}} \nabla \times \mathbf{E}' \cdot dS = -\frac{1}{c} \int_{\text{fixed circuit}} \frac{DB}{Dt} \cdot dS
\]

\[
= -\frac{1}{c} \int_{\text{fixed circuit}} \left[ \frac{\partial B}{\partial t} + \nabla \times (B \times v) \right] \cdot dS, \tag{53}
\]

noting that \(\nabla \cdot B = 0\). This holds at any fixed time \(t\), so we infer that

\[
\nabla \times \left( \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B} \right) = -\frac{1}{c} \frac{\partial B}{\partial t} = \nabla \times \mathbf{E}, \tag{54}
\]

and hence,

\[
\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}. \tag{55}
\]

That is, the force on the moving circuit is given by the Lorentz force law.

This argument, which did not use potentials, but did involve a convective derivative, is perhaps what Maxwell’s Art. 598 of [3] could/should have been. Instead, this argument may have been first given in sec. 86, p. 398 of [13] (1904). See also sec. 9-3, p 160 of [12].
B.2 Magnetic Field According to a Moving Observer

The preceding section was based on the assumption that the magnetic field is the same for an observer on the moving circuit as for one at rest in the lab. However, we could consider a moving, mathematical loop (not associated with a physical electric current) in a region of vacuum, away from the sources of laboratory fields $E$ and $B$. Instead of emphasizing Faraday’s Law, and supposing that $B' = B$ according to a moving observer, we could emphasize Maxwell’s extension of Ampère’s Law, which in vacuum and in the lab frame reads,\(^{29}\)

\[ \nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t}, \quad \int_{\text{fixed loop}} \nabla \times B \cdot dS = \oint_{\text{fixed loop}} B \cdot dl = \frac{1}{c} \frac{d}{dt} \int_{\text{fixed loop}} E \cdot dS, \tag{57} \]

and suppose that for a moving loop, $E' = E$. Then, an argument parallel to that of the preceding section would imply that the magnetic field according to the moving observer is

\[ B' = B - \frac{v}{c} \times E, \tag{58} \]

where the change of sign compared to eq. (55) is due to the difference in signs between the Faraday’s Law and Maxwell version of Ampère’s law.

B.3 Both E and B are Different for Observers in the Lab and in a Moving Frame

The previous two sections have assumed that either $E$ or $B$ is the same for observers in the lab and in a moving frame. But, it is much more plausible that both $E$ and $B$ have different values in different frames (for the same physical configuration).

If eqs. (55) and (58) were the correct general relations for the fields $E'$ and $B'$ according to an observer with velocity $v$ in the lab, we would expect that the transformation from fields in the moving frame to the lab frame, which latter has velocity $v' = -v$ with respect to the former, would be obtained by exchanging primed and unprimed quantities,

\[ E = E' - \frac{v}{c} \times B', \tag{59} \]
\[ B = B' + \frac{v}{c} \times E'. \tag{60} \]

We could then check for consistency of these transformations by, form example, starting from eq. (60), slightly rearranged, and then using eq. (59), also slightly rearranged,

\[ B' = B - \frac{v}{c} \times E' = B - \frac{v}{c} \times \left( E + \frac{v}{c} \times B' \right) \]
\[ \left( 1 - \frac{v^2}{c^2} \right) B' = B - \frac{v}{c} \times E - \left( \frac{v}{c} \cdot B' \right) \frac{v}{c}. \tag{61} \]

\(^{29}\)Ampère’s law in the form

\[ \nabla \times H = \frac{4\pi}{c} J_{\text{free}}, \tag{56} \]

is not due to Ampère himself, but was first given by Maxwell in eq. (9), p. 171, of [27] (1861).
Similarly, we would find,
\[
\left( 1 - \frac{v^2}{c^2} \right) E' = E + \frac{v}{c} \times B - \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}.
\] (62)

Thus our inferences in secs. B.1-2 for the transformations of the fields to a moving frame lead to inconsistencies at order \( v^2/c^2 \) (although they are valid at order \( v/c \)).

A partial remedy would be to “split” the factor \( 1 - v^2/c^2 \) between the transformations and their inverses,
\[
E' = \frac{1}{\sqrt{1 - v^2/c^2}} \left( E + \frac{v}{c} \times B \right) + ?, \quad E = \frac{1}{\sqrt{1 - v^2/c^2}} \left( E' - \frac{v}{c} \times B' \right) + ?, \quad (63)
\]
\[
B' = \frac{1}{\sqrt{1 - v^2/c^2}} \left( B - \frac{v}{c} \times E \right) + ?, \quad B = \frac{1}{\sqrt{1 - v^2/c^2}} \left( B' + \frac{v}{c} \times E' \right) + ?, \quad (64)
\]
but it is less obvious how to deal with the “extra” terms \((v/c \cdot E)v/c\) and \((v/c \cdot B)v/c\) in eqs. (61)-(62). An inspired “guess” would be to include these forms in the transformations, but with an as-yet-undetermined coefficient \( \alpha \). Introducing the notation
\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}},
\] (65)
we consider the transformations,
\[
E' = \gamma \left( E + \frac{v}{c} \times B \right) - \alpha \left( \frac{v}{c} \cdot E \right) \frac{v}{c}, \quad (66)
\]
\[
B' = \gamma \left( B - \frac{v}{c} \times E \right) - \alpha \left( \frac{v}{c} \cdot B \right) \frac{v}{c}. \quad (67)
\]
The inverse transformations are again obtained by swapping primed and unprimed quantities, and changing \( v \) to \( -v \),
\[
E = \gamma \left( E' - \frac{v}{c} \times B' \right) - \alpha \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}, \quad (68)
\]
\[
B = \gamma \left( B' + \frac{v}{c} \times E' \right) - \alpha \left( \frac{v}{c} \cdot B' \right) \frac{v}{c}. \quad (69)
\]
To determine \( \alpha \), we rearrange eq. (68)-(69),
\[
E' = \frac{E}{\gamma} + \frac{v}{c} \times B' + \alpha \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}, \quad (70)
\]
\[
B' = \frac{B}{\gamma} - \frac{v}{c} \times E' + \alpha \left( \frac{v}{c} \cdot B' \right) \frac{v}{c}, \quad (71)
\]
and then use eq. (71) in (70) to find,
\[
\frac{E'}{\gamma^2} = \frac{1}{\gamma} \left( E + \frac{v}{c} \times B \right) + \left( \frac{\alpha}{\gamma} - 1 \right) \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}. \quad (72)
\]

14
We also have from eq. (66) that
\[ \mathbf{v} / c \cdot \mathbf{E}' = \left( \gamma - \frac{v^2}{c^2} \right) \frac{\mathbf{v}}{c} \cdot \mathbf{E}, \]  
so that eq. (72) can be rewritten as
\[ \mathbf{E}' = \gamma \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) + \gamma^2 \left( \frac{\alpha}{\gamma} - 1 \right) \left( \gamma - \frac{v^2}{c^2} \right) \left( \frac{\mathbf{v}}{c} \cdot \mathbf{E}' \right) \frac{\mathbf{v}}{c}. \]  
This should be the same as eq. (68), which gives a quadratic equation for \( \alpha \),
\[ \alpha = \gamma^2 \left( \frac{\alpha}{\gamma} - 1 \right) \left( \gamma - \frac{v^2}{c^2} \right), \quad \alpha^2 - 2\gamma^2 \frac{v^2}{c^2} \alpha + \gamma^2 \frac{v^2}{c^2} = 0, \quad \alpha = \frac{v^2}{c^2} (\gamma \pm 1). \]  
For the transformations to have the trivial form when \( \mathbf{v} = 0 \), we take the negative root, \( \alpha = (c^2/v^2)(\gamma - 1) \).

The self-consistent transformations of the electromagnetic fields from the lab frame to a moving frame, deduced from use of Faraday’s and Ampère’s Laws (as formulated by Maxwell), are
\[ \mathbf{E}' = \gamma \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - (\gamma - 1)(\mathbf{v} \cdot \mathbf{E}) \mathbf{\hat{v}}, \quad \mathbf{B}' = \gamma \left( \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \right) - (\gamma - 1)(\mathbf{v} \cdot \mathbf{B}) \mathbf{\hat{v}}, \] 
\[ \mathbf{E} = \gamma \left( \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}' \right) - (\gamma - 1)(\mathbf{v} \cdot \mathbf{E}') \mathbf{\hat{v}}, \quad \mathbf{B} = \gamma \left( \mathbf{B}' + \frac{\mathbf{v}}{c} \times \mathbf{E}' \right) - (\gamma - 1)(\mathbf{v} \cdot \mathbf{B}') \mathbf{\hat{v}}, \]  
as first deduced by Einstein, sec. 6 of [2].\(^{30}\) Of course, the present analysis does not yield the insight that there also is a transformation of spacetime coordinates between the two frames.

If we decompose the field vectors into components parallel and perpendicular to velocity \( \mathbf{v} \), \( \mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp \) where \( \mathbf{E}_\parallel = (\mathbf{v} \cdot \mathbf{E}) \mathbf{\hat{v}} \), then
\[ \mathbf{E}'_\parallel = \mathbf{E}_\parallel, \quad \mathbf{E}'_\perp = \gamma \left( \mathbf{E}_\perp + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad \mathbf{B}'_\parallel = \mathbf{B}_\parallel, \quad \mathbf{B}'_\perp = \gamma \left( \mathbf{B}_\perp - \frac{\mathbf{v}}{c} \times \mathbf{E} \right), \] 
\[ \mathbf{E}_\parallel = \mathbf{E}'_\parallel, \quad \mathbf{E}_\perp = \gamma \left( \mathbf{E}'_\perp - \frac{\mathbf{v}}{c} \times \mathbf{B}' \right), \quad \mathbf{B}_\parallel = \mathbf{B}'_\parallel, \quad \mathbf{B}_\perp = \gamma \left( \mathbf{B}'_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}' \right). \]  

The low-velocity approximations to these transformations are
\[ \mathbf{E}' \approx \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{B}' \approx \mathbf{B}_\perp - \frac{\mathbf{v}}{c} \times \mathbf{E}, \] 
\[ \mathbf{E} \approx \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}', \quad \mathbf{B} \approx \mathbf{B}_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}'. \]  
as previously found in secs. B.1-2.

\(^{30}\)Einstein’s derivation was based on the assumptions that Maxwell’s equations have the same form in both the lab frame and a moving (inertial) frame. In particular, he used Faraday’s and Ampère’s Laws in empty space,
\[ \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial c t, \quad \nabla \times \mathbf{B} = \partial \mathbf{E} / \partial c t, \quad \text{and} \quad \nabla' \times \mathbf{E}' = -\partial \mathbf{B}' / \partial c t', \quad \nabla' \times \mathbf{B}' = \partial \mathbf{E}' / \partial c t', \]  
together with the transformation of coordinates \((c t, \mathbf{x})\) to \((c t', \mathbf{x}')\) to deduce eqs. (79)-(80).
B.4 Comments

This Appendix shows that arguments using convective (time) derivatives can, with considerable effort, lead to the full, relativistic transformations of fields from the lab frame to a moving frame, although most straightforward use of the convective derivative only yields the low-velocity transformation.

B.4.1 Use of Potentials to Compute the Fields

As also remarked in footnote 16 above, Maxwell made an argument based on potentials, rather than fields, in Art. 598 of [3], and while he arrived at the correct low-velocity approximation via discussion of the convective derivative of the vector potential, he seems to have “fudged” some intermediate steps if we are to regard his symbol Ψ as the electrical scalar potential, as he claims it to be. It is felicitous that no “fudging” is needed if one considers the fields rather than the potentials (sec. 5.1 above), as perhaps first done by Abraham, sec. 86, p. 398 of [13] (1904).

We elaborate on this topic by deducing how the transform of potentials for low-velocity, and the resulting transform of the fields if they are then deduced from the transformed potentials.

In any (inertial) frame the fields \( E \) and \( B \) are related the scalar potential \( V \) and the vector potential \( A \) (in some gauge) by

\[
E = -\nabla V - \frac{\partial A}{\partial ct}, \quad \text{and} \quad B = \nabla \times A,
\]  

(83)

provided the derivatives are taken with respect to the coordinates of that frame.

The electromagnetic potentials comprise a 4-vector \((V, A)\), so the Lorentz transformations of the potentials from the lab frame to the primed frame that has velocity \( \mathbf{v} \) with respect to the lab frame are

\[
V' = \gamma \left( V - \frac{\mathbf{v}}{c} \cdot A \right) \approx V - \frac{\mathbf{v}}{c} \cdot A,
\]  

(84)

\[
A' = A + (\gamma - 1)(A \cdot \hat{v}) \hat{v} - \gamma \frac{\mathbf{v}}{c} V \approx A - \frac{\mathbf{v}}{c} V,
\]  

(85)

where the approximations hold for low velocity.

For the derivatives, we note that \((\partial/\partial ct, -\nabla)\) is a 4-vector,\(^{31}\) so its transform is

\[
\frac{\partial}{\partial ct'} = \gamma \left( \frac{\partial}{\partial ct} - \frac{\mathbf{v}}{c} \cdot (-\nabla) \right) \approx \frac{\partial}{\partial ct} + \frac{\mathbf{v}}{c} \cdot \nabla, \quad \frac{\partial}{\partial t'} \approx \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,
\]  

(86)

\[
-\nabla' = -\nabla + (\gamma - 1)(-\nabla \cdot \hat{v}) \hat{v} - \gamma \frac{\mathbf{v}}{c} \frac{\partial}{\partial ct} \approx -\nabla,
\]  

(87)

where we neglect terms of order \( 1/c^2 \). Note that the low-velocity approximation to the time derivative in the moving frame is the convective derivative in terms of lab-frame quantities,

\(^{31}\)Strictly, \((V, A)\) is a covariant 4-vector and \((\partial/\partial ct, -\nabla)\) is a contravariant 4-vector. These distinctions are unimportant is special relativity for inertial frames, but are significant when considering accelerated frames. See, for example, [41].
The fields in the moving frame can now be computed as

\[ E' = -\nabla'V' - \frac{\partial A'}{\partial ct'} \approx -\nabla(V - \frac{v}{c} \cdot A) - \left( \frac{\partial}{\partial ct} + \frac{v}{c} \cdot \nabla \right) \left( A - \frac{v}{c} V \right) \]

\[ \approx -V - \frac{\partial A}{\partial ct} + \nabla \left( \frac{v}{c} \cdot A \right) - \left( \frac{v}{c} \cdot \nabla \right) A \]

\[ = E + \left( \frac{v}{c} \cdot \nabla \right) A + \frac{v}{c} \times (\nabla \times A) - \left( \frac{v}{c} \cdot \nabla \right) A = E + \frac{v}{c} \times B, \tag{88} \]

\[ B' = \nabla' \times A' \approx \nabla \times \left( A - \frac{v}{c} V \right) = B + \nabla \times \left( \frac{v}{c} (-V) \right) = B - \frac{v}{c} \times (-\nabla V) \]

\[ = B - \frac{v}{c} \times E - \frac{v}{c} \times \frac{\partial A}{\partial ct} \approx B - \frac{v}{c} \times E. \tag{89} \]

Thus, using the potentials and the various low-velocity Lorentz transformations, we recover the low-velocity forms (81)-(82) for the electromagnetic fields.\textsuperscript{32}\n
B.4.2 Lorentz Force

Another felicitous result is that the low-velocity transform of the electric field from lab frame to a moving frame has the same form as the lab-frame Lorentz force (per unit charge), \( \mathbf{F}/q = E + v/c \times B \), for arbitrary velocity. Of course, neither Maxwell (1873) nor Lorentz (1892) were aware of this happy result of the Lorentz transformation of 4-force (Minkowski force).

C Appendix: J.J. Thomson (1880)

While Arts. 599 and 769-770 of Maxwell’s Treatise are consistent with the low-velocity limit of special relativity, this was not evident at the time, when electromagnetism was generally interpreted in an æther theory. For example, in his first research paper, J.J. Thomson \textsuperscript{22} used Arts. 598-599 of Maxwell’s Treatise to reach a peculiar conclusion as to the speed of light in a dielectric medium that has velocity \( v \) with respect to the frame of the æther.

In the present section, quantities in the ether frame will be unprimed, while a quantity in the frame of the moving dielectric will be denoted with a ‘.\textsuperscript{33}

\textsuperscript{32}Hence, it seems to this author that Maxwell's correct expression in sec. (65) of \cite{1} and Art. 598 of \cite{3}, for the low-velocity field experienced by a moving circuit could well have been deduced by a valid argument, despite the doubts cast on this by Helmholtz, and Thomson.

Thomson felt that his objection was validated by the example of a rotating, conducting sphere is in uniform external magnetic field, in his note on p. 260 of \cite{3}. However, this example involves and accelerated frame, for which the magnetic field is considered to be the same by a rotating observer and one at rest (see, for example, \cite{41}).

If we had supposed that the magnetic field, and the vector potential were that same in the lab frame and in the moving frame, then eq. (88) above would read \( E' \approx E + v/c \times B + (v \cdot \nabla)A/c \). So, while the assumption that \( B' = B \) permits one to deduce the correct, low-velocity electric-field transformation via consideration of Faraday’s Law for moving circuit (as in sec. B.1 above), use of this assumption in an argument based on potentials does lead to an erroneous result.

\textsuperscript{33}The dielectric medium could be vacuum.
Thomson begins with Maxwell’s eq. (7) and notes that \( \mathbf{B} = \nabla \times \mathbf{A} \) where the vector potential obeys \( \nabla \cdot \mathbf{A} = 0 \).

Thomson’s eqs. (1)-(3) correspond to relating the electric displacement field in the moving frame by

\[
\mathbf{D}' = \varepsilon_0 (\mathbf{E}' + \mathbf{P}'),
\]

where \( \mathbf{P}' \) is the electric polarization field in the moving frame. His eq. (4) is more properly then

\[
\nabla' \cdot \mathbf{D}' = 0.
\]

Thomson’s goal is to deduce a wave equation for \( \mathbf{D}' \), and to infer from this the speed of light in the moving frame. To this end, he takes the curl of eq. (90), assuming that \( \nabla' \times \mathbf{P}' = 0 \) and that the dielectric medium is in uniform motion such that derivatives of the velocity \( \mathbf{v} \) are zero,

\[
\nabla' \times \mathbf{D}' = \varepsilon_0 \nabla' \times \mathbf{E}' = \nabla' \times \mathbf{E} + \mathbf{v} \left( \frac{\mathbf{V}}{c} \cdot \nabla' \right) \mathbf{B} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} - \left( \frac{\mathbf{V}}{c} \cdot \nabla \right) \mathbf{B}.
\]

Thomson then supposes that the effect of taking derivatives with respect to spacetime coordinates \( x, y, z \) and \( t \) is the same in the ether frame and in the moving frame. That is, he assumes that Galilean relativity relates these coordinates in the two frames. Note that this assumption was not needed in the interpretation of Maxwell’s Arts. 599 and 769-770 (although Maxwell did use this assumption in his Arts. 600-601).

With this tacit assumption of Galilean relativity for \( (x, y, z, t) \), eq. (92) can be written

\[
\nabla \times \mathbf{D}' = \nabla \times \mathbf{E} + \mathbf{v} \left( \frac{\mathbf{V}}{c} \cdot \nabla \right) \mathbf{B} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} - \left( \frac{\mathbf{V}}{c} \cdot \nabla \right) \mathbf{B},
\]

in that \( \nabla \cdot \mathbf{B} = 0 \) (while in special relativity, \( \nabla' \cdot \mathbf{B} \neq 0 \)).

It was not appreciated in 1880, and perhaps not until the work of LeBellac and Levy-Leblond in 1973 [9], that Maxwell’s equations are not consistent with Galilean relativity. When Maxwell’s eq. (7) applies, the appropriate modifications to Maxwell’s equations to be compatible with Galilean relativity are those of so-called magnetic Galilean relativity (sec. 2.3 of [9]),

\[
\nabla \cdot \mathbf{D}_m = 4\pi \rho_m, \quad \nabla \cdot \mathbf{B}_m = 0, \quad \nabla \times \mathbf{E}_m = -\frac{1}{c} \frac{\partial \mathbf{B}_m}{\partial t}, \quad \nabla \times \mathbf{H}_m = \frac{4\pi}{c} \mathbf{J}_m
\]

where \( \rho \) and \( \mathbf{J} \) and the volume densities of “free” charge and currents, and there is no “displacement current” and no electromagnetic waves. While the velocity \( c \) has a value equal to the speed of light in vacuum it is to be deduced from static experiments and is not related to (nonexistent) wave propagation in Galilean relativity.

Thomson considered a nonmagnetic dielectric medium in which \( \mathbf{B} = \mathbf{H} \), with no free charge or current densities, and supposed that in this case

\[
\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t},
\]

18
whereas in a consistent Galilean view of this case, $\nabla \times \mathbf{B} = 0$. He then took the curl of eq. (93), writing (his eq. (10))

$$\nabla^2 \mathbf{D}' = \frac{1}{c^2} \frac{\partial \mathbf{D}}{\partial t^2} - \left( \frac{\mathbf{v}}{c^2} \cdot \nabla \right) \frac{\partial \mathbf{D}}{\partial t}, \quad (96)$$

In the magnetic Galilean relativity of [9] this would be just $\nabla^2 \mathbf{D}' = 0$, corresponding to instantaneous propagation of electromagnetic effects.

At this point Thomson seems to have assumed that $\mathbf{D}' = \mathbf{D}$ even though his argument began with Maxwell’s relation (7) in which $E' \neq E$. Assuming a wavefunction $\mathbf{D}' = \mathbf{D} = D_0 e^{i(kx-\omega t)}$ and $\mathbf{v} = v \hat{x}$, the wave equation (96) leads to the dispersion relation

$$\omega^2 - vk\omega - k^2 c^2 = 0, \quad v' = \frac{\omega}{k} = \frac{1}{2} \left( v \pm \sqrt{1 + \frac{v^2}{8c^4}} \right). \quad (97)$$

Only the positive root could make sense, leading to\(^{34}\)

$$v' \approx c + \frac{v}{2} \quad (v \ll c), \quad (98)$$

which was interpreted as the speed of the waves in the moving dielectric (even if that dielectric were vacuum).

While this result makes no sense from a “modern” perspective, it illustrates how naïve assumptions about relativity and Maxwell’s equations can lead to peculiar conclusions.

References


\(^{34}\)For sound waves we expect $v' = c-v$. That the sign in eq. (98) is “wrong” seems to have gone unnoticed.


http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_pm_27_324_89.pdf


[33] G. Green, *Mathematical Papers* (Macmillan, 1871),

[34] H. Helmholtz, "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbel-
bewegungen entsprechen," Crelles J. 55, 25 (1858),


Elektrodynamische Kräfte in bewegten Leitern," J. Reine Angew. Math. 78, 273 (1874),
p. 309,

Mag. 17, 1 (1884),
http://physics.princeton.edu/~mcdonald/examples/EM/larmor_pm_17_1_84.pdf

[38] H.W. Watson, "Note on the Electromotive Force in Moving Conductors," Phil. Mag. 25,
271 (1888),

http://physics.princeton.edu/~mcdonald/examples/EM/rasetti_pr_66_1_44.pdf
This paper mentions the earlier history of erratic results on this topic.

[40] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686),

http://physics.princeton.edu/~mcdonald/examples/rotating_EM.pdf

22