Maxwell and Special Relativity

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(May 26, 2014; updated January 1, 2018)

It is now commonly considered that Maxwell’s equations [1] in vacuum implicitly contain the special theory of relativity.\(^1\)

For example, these equations imply that the speed \(c\) of light in vacuum is related by\(^2\)

\[
c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}, \tag{1}
\]

where the constants \(\epsilon_0\) and \(\mu_0\) can be determined in any (inertial) frame via electrostatic and magnetostatic experiments (nominally in vacuum).\(^3\) Even in æther theories, the velocity of the laboratory with respect to the hypothetical æther should not affect the results of these static experiments,\(^4\) so the speed of light should be the same in any (inertial) frame. Then, the theory of special relativity, as developed in [2], follows from this remarkable fact.

Maxwell does not appear to have crisply drawn the above conclusion, that the speed of light is independent of the velocity of the observer, but he did make arguments in Arts. 599-600 and 770 of [6] that correspond to the low-velocity approximation to special relativity, as pointed out in sec. 5 of [9]. These two arguments also correspond to use of the two types of Galilean electrodynamics [10], as noted in [11, 12].

The notion of Galilean electrodynamics seems to have been developed only in 1973 [10]. In this concept there are no electromagnetic waves, but only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics as a whole. In fact, there are two variants of Galilean electrodynamics, so-called electric Galilean relativity (for weak magnetic fields) in which the transformations between two inertial frames with relative velocity \(v\) are (sec. 2.2 of [10], given here in Gaussian units, as will be used in the rest of this note)\(^5\)

\[
\begin{align*}
\rho'_e &= \rho_e, & J'_e &= J_e - \rho_e v, & (c|\rho_e| \gg |J_e|), & \Psi'_e = \Psi_e, & A'_{ae} &= \frac{v}{c} \Psi_e, \\
E'_e &= E_e, & B'_e &= B_e - \frac{v}{c} \times E_e & f_e &= \rho_e E_e \quad \text{(electric)},
\end{align*}
\]

where \(\rho\) and \(J\) are the electric charge and current densities, \(\Psi\) and \(A\) are the electromagnetic scalar and vector potentials, \(\mathbf{E} = -\nabla \Psi - \partial \mathbf{A} / \partial t\) is the electric field, \(\mathbf{B} = \nabla \times \mathbf{A}\) is the magnetic field, \(c|\rho_e|\) and \(c|J_e|\) are the characteristic electric and magnetic fields, \(v\) is the relative velocity of the two frames, \(c\) is the speed of light, and \(\mathbf{f}\) is the force density. This expression for \(\mathbf{f}\) is the Galilean law of motion for electric charges.

\(^1\)Maxwell’s electrodynamics was the acknowledged inspiration to Einstein in his 1905 paper [2].

\(^2\)Equation (1) is a transcription into SI units of the discussion in sec. 80 of and sec. 758 of [6].

\(^3\)As reviewed in [7], examples of a “static” current-carrying wire involve effects of order \(v^2/c^2\) where \(v\) is the speed of the moving charges of the current. A consistent view of this in the rest frame of the moving charges requires special relativity. These arguments could have been made as early as 1820, but it took 85 years for them to be fully developed.

\(^4\)It is now sometimes said that electricity plus special relativity implies magnetism, but a more historical view is that (static) electricity plus magnetism implies special relativity. This theme is emphasized in, for example, [8].

\(^5\)This ansatz is a weak form of Einstein’s Principle of Relativity.

\(^6\)In Galilean electrodynamics the symbol \(c\) does not represent the speed of light (as light does exist in this theory), but only the function \(1/\sqrt{\epsilon_0 \mu_0}\) of the (static) permittivity and permeability of the vacuum.
magnetic (induction) field, and so-called magnetic Galilean relativity (for weak electric fields, sec. 2.3 of [10]) with transformations

$$\rho'_m = \rho_m - \frac{v}{c^2} \cdot J_m, \quad J'_m = J_m, \quad (c |\rho_e| \ll |J_e|), \quad \Psi'_m = \Psi_m - \frac{v}{c} \cdot A, \quad A'_m = A_m,$$

$$E'_m = E_m + \frac{v}{c} \times B_m, \quad B'_m = B_m \quad f_m = \rho_m \left(E_m + \frac{v}{c} \times B_m\right) \quad \text{(magnetic).}$$

For comparison, the low-velocity limit of special relativity has the transformations,$^6$

$$\rho'_s \approx \rho_s - \frac{v}{c^2} \cdot J_s, \quad J'_s \approx J_s - \rho_s v, \quad \Psi'_s \approx \Psi_s - \frac{v}{c} \cdot A_s, \quad A'_s \approx A_s - \frac{v}{c} \Psi_s,$$

$$E'_s \approx E_s + \frac{v}{c} \times B_s, \quad B'_s \approx B_s - \frac{v}{c} \times E_s \quad \text{(special relativity, } v \ll c).$$

### 1 Articles 598-599 of Maxwell’s Treatise

In Arts. 598-599 of his Treatise [6], Maxwell argued that an electric charge $q$ which moves with velocity $v$ in electric and magnetic fields $E$ and $B = \mu H$ experiences an electromotive intensity, i.e., a vector electromagnetic force,$^7$

$$F = q \left(E + \frac{v}{c} \times B\right).$$

This is now known as the Lorentz force,$^8^9$ and it seems seldom noted that Maxwell gave this form, likely because he disguised it by writing (in electromagnetic units), eq. (10) of Art. 599,$^10$

$$\mathcal{E} = \mathcal{B} \times \mathcal{B} - \dot{\mathcal{A}} - \nabla \Psi,$$

where $\mathcal{E}$ is the electromotive intensity, i.e., $F/q$, $\mathcal{B}$ is the velocity $v$, $\mathcal{B}$ is the magnetic field $B$, $\mathcal{A}$ is the vector potential and $\Psi$ is the scalar potential. Then, eq. (5) follows noting that the electric field is given (in emu) by $-\partial \mathcal{A}/\partial t - \nabla \Psi$,$^{11}$ and that $v \times B$ in emu becomes $v/c \times B$ in Gaussian units.

In his Arts. 598-599, Maxwell considered a lab-frame view of a moving circuit. However, we can also interpret Maxwell’s $\mathcal{E}$ as the electric field $E'$ in the frame of the moving circuit,

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$^6$Electromagnetic waves can exist in low-velocity approximation to special relativity, and, of course, propagate in vacuum with speed $c$.

$^7$Maxwell did not hold a view of an electric charge as a “particle,” but rather as a state of “displaced” ether.

$^8$Lorentz actually advocated the form $F = q (D + v \times H)$ in eq. (V), p. 21, of [13], although he seems mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [14]. That is, Lorentz considered $D$ and $H$, rather than $E$ and $B$, to be the microscopic electromagnetic fields.

$^9$It is generally considered that Heaviside first gave the Lorentz force law (5) for electric charges in [15], but the key insight is already visible for the electric case in [16] and for the magnetic case in [17].

$^{10}$This result also appeared in eq. (77), p. 343, of [19] (1861), and in eq. (D), sec. 65, p. 485, of [1] (1864), where $\mathcal{E}$ was called the electromotive force. The evolution of Maxwell’s thoughts on the “Lorentz” force are traced in Appendix A below. See also [22, 23, 24].

$^{11}$This assumes that Maxwell’s $\mathcal{A}$ corresponds to $\partial \mathcal{A}/\partial t$, and not to the convective derivative $D \mathcal{A}/Dt = \partial \mathcal{A}/\partial t + (v \cdot \nabla) \mathcal{A}$. 
such that Maxwell’s transformation of the electric field is,

\[ E' = E + \frac{v}{c} \times B. \quad (7) \]

The transformation (7) is compatible with both magnetic Galilean relativity, eq. (3), and the low-velocity limit of special relativity, eq. (4). These two versions of relativity differ as to the transformation of the magnetic field. In particular, if \( B = 0 \) while \( E \) were due to a single electric charge at rest (in the unprimed frame), then magnetic Galilean relativity predicts that the moving charge/observer would consider the magnetic field \( B' \) to be zero, whereas it is nonzero according to special relativity.

These themes were considered by Maxwell in Arts. 600-601, under the heading: *On the Modification of the Equations of Electromotive Intensity when the Axes to which they are referred are moving in Space*, which we review in sec. 2 below.

### 1.1 Details

In Art. 598, Maxwell started from the integral form of Faraday’s law, that the (scalar) *electromotive force* \( \mathcal{E} \) in a circuit is related to the rate of change of the magnetic flux through it by his eqs. (1)-(2),

\[ \mathcal{E} = -\frac{1}{c} \frac{d\Phi_m}{dt} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{1}{c} \frac{d}{dt} \oint \mathbf{A} \cdot d\mathbf{l} = -\frac{1}{c} \oint \left( \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} \right) \cdot d\mathbf{l}, \quad (8) \]

where the last form, involving the convective derivative, holds for a circuit that moves with velocity \( \mathbf{v} \) with respect to the lab frame. In his discussion leading to eq. (3) of Art. 598, Maxwell argued for the equivalent of use of the vector-calculus identity

\[ \nabla (\mathbf{v} \cdot \mathbf{A}) = (\mathbf{v} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{v}), \quad (9) \]

which implies for the present case,

\[ (\mathbf{v} \cdot \nabla) \mathbf{A} = -\mathbf{v} \times (\nabla \times \mathbf{A}) + \nabla (\mathbf{v} \cdot \mathbf{A}) = -\mathbf{v} \times \mathbf{B} + \nabla (\mathbf{v} \cdot \mathbf{A}), \quad (10) \]

\[ \mathcal{E} = \oint \left( \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \cdot d\mathbf{l} = \oint \mathbf{\mathcal{E}} \cdot d\mathbf{l}, \quad (11) \]

since \( \oint \nabla (\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{l} = 0 \). Our eq. (11) corresponds to Maxwell’s eqs. (4)-(5), from which we infer that the vector *electromotive intensity* \( \mathcal{E} \) has the form

\[ \mathcal{E} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi, \quad (12) \]

for some scalar field \( \Psi \), that Maxwell identified with the electric scalar potential.

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12A more direct use of Faraday’s law, without invoking potentials, to deduce the electric field in the frame of a moving circuit was made in sec. 9-3, p. 160, of [25], which argument appeared earlier in sec. 86, p. 398, of [26]. An extension of this argument to deduce the full Lorentz transformation of the electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) is given in Appendix B below.
If it were clear that $\Psi$ is indeed the electric scalar potential, then Maxwell should be credited with having “discovered” the “Lorentz” force law. However, Helmholtz [eq. (5$^d$), p. 309 of [27] (1874)], Larmor [p. 12 of [28] (1884)], Watson [p. 273 of [29] (1888)], and J.J. Thomson [in his editorial note on p. 260 of [6] (1892)] argued that our eq. (10) leads to

$$E = \oint \left[ \frac{v}{c} \times B - \frac{1}{c} \frac{\partial A}{\partial t} - \nabla \left( \frac{v}{c} \cdot A \right) \right] \cdot dl,$$

(13)

so Maxwell’s eq. (D) of Art. 598 and eq. (10) of Art. 599 should really be written as

$$\mathcal{E} = \frac{v}{c} \times B - \frac{1}{c} \frac{\partial A}{\partial t} - \nabla \left( \frac{v}{c} \cdot A + \Psi \right),$$

(14)

where $\Psi$ is the electric scalar potential. It went unnoticed by these authors that use of eq. (14) rather than (12) would destroy the elegance of Maxwell’s argument in Arts. 600-601 (discussed in sec. 2 below), as well as that Maxwell’s earlier derivations of our eq. (12), on pp. 340-342 of [19] and in secs. 63-65 of [1], used different methods which did not suggest the possible presence of a term $-\nabla (v \cdot A/c)$ in our eq. (12). However, the practical effect of these doubts by illustrious physicists was that Maxwell has not been credited for having deduced the “Lorentz” force law, which became generally accepted only in the 1890’s.

The view of this author is that Maxwell did deduce the “Lorentz” force law, although in a manner that was “not beyond a reasonable doubt.”

2 Articles 600-601 of Maxwell’s Treatise

In Art. 600, Maxwell considered a moving point with respect to two coordinate systems, the lab frame where $x = (x, y, z)$, and a frame moving with uniform velocity $v$ respect to the lab in which the coordinates of the point are $x' = (x', y', z')$, with quantities in the two frames related by Galilean transformations. Noting that a force has the same value in both frames, Maxwell deduced that the “Lorentz” force law has the same form in both frames, provided the electric scalar potential $\Psi'$ in the moving frame is related to lab-frame quantities by

$$\Psi' = \Psi - \frac{v}{c} \cdot A.$$

(15)

This is the form according to the low-velocity Lorentz transformation (4), and also to the transformations of magnetic Galilean electrodynamics (3), which latter is closer in spirit to Maxwell’s arguments in Arts. 600-601.

2.1 Details

In Art. 600, Maxwell considered both translations and rotations of the moving frame, but we restrict our discussion here to the case of translation only, with velocity $v = (u, v, w) = (\delta x/dt, \delta y/dt, \delta z/dt)$ with respect to the lab.14 Maxwell labeled the velocity of the moving

13 These derivations of Maxwell are reviewed in Appendix A below.
14 For discussion of electrodynamics in a rotating frame (in which one must consider “fictitious” charges and currents, see, for example, [31].

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point with respect to the moving frame by \( u' = dx'/dt' = dx/dt \), while he called labeled its velocity with respect to the lab frame by \( u = dx/dt \). Then, Maxwell stated the velocity transformation to be, eq. (1) of Art. 600,

\[
\mathbf{u}' = \mathbf{u} - \mathbf{v}, \quad \text{i.e.,} \quad \mathbf{u} = \mathbf{v} + \mathbf{u}' \quad \left( \frac{dx}{dt} = \frac{\delta x}{\delta t} + \frac{dx'}{dt'} \right),
\]

which corresponds to the Galilean coordinate transformation,

\[
x' = x - vt. \quad t' = t, \quad \nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.
\]

Maxwell next considered the transformation of the time derivative of the vector potential \( \mathbf{A} = (F, G, H) \) in his eq. (3), Art. 600,

\[
\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} \quad \left( \frac{dF'}{dt} = \frac{dF}{dx} \frac{\delta x}{\delta t} + \frac{dF}{dy} \frac{\delta y}{\delta t} + \frac{dF}{dz} \frac{\delta z}{\delta t} + \frac{dF}{dt} \right),
\]

which tacitly assumed that \( \mathbf{A}' = \mathbf{A} \), and hence that \( \mathbf{B}' = \mathbf{B} \). In eqs. (4)-(7) of Art. 600, Maxwell argued for the equivalent of use of the vector-calculus identity

\[
\nabla (\mathbf{v} \cdot \mathbf{A}) = (\mathbf{v} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{v}),
\]

which implies for the present case,

\[
(\mathbf{v} \cdot \nabla) \mathbf{A} = -\mathbf{v} \times (\nabla \times \mathbf{A}) + \nabla (\mathbf{v} \cdot \mathbf{A}) = -\mathbf{v} \times \mathbf{B} + \nabla (\mathbf{v} \cdot \mathbf{A}),
\]

\[
\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \times \mathbf{B} + \nabla (\mathbf{v} \cdot \mathbf{A}).
\]

Then, in eqs. (8)-(9) of Art. 600, Maxwell combined his eq. (B) of Art. 598 with our eqs. (16) and (21) to write the \textit{electromotive force} \( \mathbf{E} \) as, in the notation of the present section,

\[
\mathbf{E} = \frac{\mathbf{u}}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi = \frac{\mathbf{u'}}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t'} - \nabla \left( \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right).
\]

Finally, since a force has the same value in two frames related by a Galilean transformation, Maxwell inferred that the \textit{electromotive force} \( \mathbf{E}' \) in the moving frame can be written as

\[
\mathbf{E}' = \frac{\mathbf{u'}}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t'} - \nabla \left( \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) = \frac{\mathbf{u'}}{c} \times \mathbf{B}' - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t'} - \nabla' \left( \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) = \frac{\mathbf{u'}}{c} \times \mathbf{B}' - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t'} - \nabla' \Psi = \frac{\mathbf{u'}}{c} \times \mathbf{B}' + \mathbf{E'},
\]

\[15\text{Equation (2) of Art. 600 refers to rotations of a rigid body about the origin of the moving frame.}

\[16\text{While this assumption does not correspond to the low-velocity Lorentz transformation of the field between inertial frames, it does hold for the transformation from an inertial frame to a rotating frame. Faraday considered rotating magnets in [32], and in sec. 3090, p. 31, concluded that No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it. That is, } \mathbf{B}' = \mathbf{B} \text{ relates the magnetic field in an inertial and a rotating frame. Possibly, this might have led Maxwell to infer a similar result for a moving inertial frame as well.}
where the electric scalar potential $\Psi'$ in the moving frame is related to lab-frame quantities by

$$\Psi' = \Psi - \frac{v}{c} \cdot A.$$  \hspace{1cm} (24)

This is the form according to the low-velocity Lorentz transformation (4), and also to the transformations of magnetic Galilean electrodynamics (3), which latter is closer in spirit to Maxwell’s arguments in Arts. 600-601.

Further, the force $\mathbf{F}'$ on a moving electric charge $q$ in the moving frame is given by the “Lorentz” form,

$$\mathbf{F}' = \left( \mathbf{E}' + \frac{\mathbf{u}' \times \mathbf{B}'}{c} \right),$$ \hspace{1cm} (25)

which has the same form eq. (5) in the lab frame. As Maxwell stated at the beginning of Art. 601: It appears from this that the electromotive intensity is expressed by a formula of the same type, whether the motions of the conductors be referred to fixed axes or to axes moving in space.

2.2 Articles 602-603. The “Biot-Savart” Force Law

In articles 602-603, Maxwell considered the force on a current element $Idl$ in a circuit at rest in a magnetic field $\mathbf{B}$, and deduced the “Biot-Savart” form,$^{17}$

$$d\mathbf{F} = \frac{Id}{c} \times \mathbf{B}, \quad \mathbf{F} = \oint \frac{Id}{c} \times \mathbf{B}.$$  \hspace{1cm} (26)

3 Articles 769-770 of Maxwell’s Treatise

Faraday considered that a moving electric charge generates a magnetic field [33] (as quoted in [34]). Maxwell also argued for this in Arts. 769-770 of [6], where his verbal argument can be transcribed as

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E},$$  \hspace{1cm} (27)

for the magnetic field experienced by a fixed observer due to a moving charge. Maxwell noted that this is a very small effect, and claimed (1873) that it had never been observed.$^{18}$

If $\mathbf{v}$ represents the velocity of a moving observer relative to a fixed electric charge, then eq. (27) implies that the magnetic field experienced by the moving observer would be

$$\mathbf{B}' = -\frac{\mathbf{v}}{c} \times \mathbf{E},$$ \hspace{1cm} (28)

$^{17}$Maxwell had argued for this in his eqs. (12)-14), p. 172, of [18] (1861), which is the first statement of the “Biot-Savart” force law in terms of a magnetic field. Biot and Savart [30] discussed the force on a magnetic “pole” due to an electric circuit, and had no concept of the magnetic field.

$^{18}$The magnetic field of a moving charge was detected in 1876 by Rowland [34, 35] (while working in Helmholtz’ lab in Berlin). The form (27) was verified (in theory) more explicitly by J.J. Thomson in 1881 [36] for uniform speed $v \ll c$, and for any $v < c$ by Heaviside [15] and by Thomson [37] in 1889 (which latter two works gave the full special-relativistic form for $\mathbf{E}$ as well).
This corresponds to the low-velocity limit (4) of special relativity, and to form (2) of electric Galilean relativity.

It remains that while Maxwell used Galilean transformations as the basis for his considerations of fields and potentials in moving frames, he was rather deft in avoiding the contradictions between “Galilean electrodynamics” and his own vision. For a contrast, in which use of Galilean transformations for electrodynamics by J.J. Thomson [38] (1880) led to a result in disagreement with Nature (unrecognized at the time), see Appendix C below.19

Maxwell did not note the incompatibility of his use of Galilean transformations in his Arts. 601-602 and 769-770 with his system of equations for the electromagnetic fields, but if he had, he might have mitigated this issue by deduction of self-consistent transformations for both \( E \) and \( B \) between the lab frame and a uniformly moving frame, as in Appendix B below.

This note was stimulated by e-discussions with Dragan Redžić.

A Maxwell’s Derivations of the Lorentz Force Law

Maxwell published his developments of the theory of electrodynamics in four steps, On Faraday’s Lines of Force [41] (1856), On Physical Lines of Force [18, 19, 20, 21] (1861-61), A Dynamical Theory of the Electromagnetic Field [1] (1864), and in his Treatise on Electricity and Magnetism [3, 4, 5, 6] (1873). In this Appendix we review his arguments related, in a broad sense, to the issue of the “Lorentz” force law.

A.1 In On Faraday’s Lines of Force [41]

For a discussion of Maxwell’s thoughts in 1855, which culminated in the publication [41], see [42].

A.1.1 Theory of the Conduction of Current Electricity and On Electro-motive Forces

On p. 46 of [41], Maxwell stated: According to the received opinions we have here a current of fluid moving uniformly in conducting circuits, which oppose a resistance to the current which has to be overcome by the application of an electro-motive force at some part of the circuit.

He continued on pp. 46-47: When a uniform current exists in a closed circuit it is evident that some other forces must act on the fluid besides the pressures. For if the current were due to difference of pressures, then it would flow from the point of greatest pressure in both directions to the point of least pressure, whereas in reality it circulates in one direction constantly. We must therefore admit the existence of certain forces capable of keeping up a constant current in a closed circuit. Of these the most remarkable is that which is produced by chemical action. A cell of a voltaic battery, or rather the surface of separation of the fluid of the cell and the zinc, is the seat of an electro-motive force which can maintain a current in opposition to the resistance of the circuit. If we adopt the usual convention in speaking of

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19 For a deft use by FitzGerald (1882) of Maxwell’s “Galilean” arguments to deduce a result in agreement with special relativity as \( v \to c \), see [39, 40].
electric currents, the positive current is from the fluid through the platinum, the conducting circuit, and the zinc, back to the fluid again.

Here, Maxwell seemed to accept the received opinions\textsuperscript{20} that electric current is a fluid; and actually two counterpropagating fluids.

A.1.2 Ohm’s Law, Electromotive Force and the Electric Field

On p. 47 of [41], Maxwell wrote a version of Ohm’s Law as \( F = IK \) for an electrical circuit of resistance \( K \) that carries current \( I \). He calls \( F \) the electro-motive force, which is consistent with a more contemporary notation,

\[
\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = IR \quad (F = IK),
\]

where \( \mathcal{E} = F \) is the electromotive “force” (with dimensions of electric potential (voltage) rather than of force), \( \mathbf{E} \) is the electrical field and \( d\mathbf{l} \) is line element inside the wire of the circuit.

On p. 53, Maxwell introduced the (free/conduction) electric current-density vector \( \mathbf{J} = (a_2, b_2, c_2) \), the electric scalar potential \( \Psi = p_2 \), and the electric field \( \mathbf{E} = (\alpha_2, \beta_2, \gamma_2) \), writing in his eq. (A),

\[
\mathbf{E} = \mathbf{E}_{\text{other}} - \nabla \Psi \quad \left( \alpha_2 = X_2 - \frac{dp_2}{dx}, \text{etc.} \right),
\]

with \( \mathbf{E}_{\text{other}} = (X_2, Y_2, Z_2) \) being a possible contribution to the electric field not associated with a scalar potential. To possible confusion, Maxwell called the vector \( \mathbf{E} \) an electro-motive force, which term he also used for the scalar \( \mathcal{E} \).

Also on p. 53, Maxwell introduced the electrical resistivity \( \varrho = k_2 \) (reciprocal of the electrical conductivity \( \sigma = 1/\varrho \), so the Ohm’s Law can be written as Maxwell’s eq. (B),

\[
\mathbf{E} = \varrho \mathbf{J}_{\text{free}} = \frac{\mathbf{J}_{\text{free}}}{\sigma} \quad (\alpha_2 = k_2 a_2, \text{etc.}),
\]

On p. 54, Maxwell noted that for any closed curve eq. (30) implies

\[
\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{E}_{\text{other}},
\]

He also introduces the concept of the flux (conduction) \( \mathbf{E} \cdot d\mathbf{S} \) of a field \( \mathbf{E} \) across a surface element \( d\mathbf{S} \), and noted (Gauss’ Law) that for a closed surface,

\[
\int \mathbf{E} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{E} d\text{Vol},
\]

He indicated in his eq. (C) that he will often write the divergence of a vector field as \( 4\pi \rho \).

\textsuperscript{20}Maxwell cited French translations of papers by Kirchoff [43] and Quincke [44]. He seemed unaware that Kirchhoff had also published an English version of his paper [43].
A.1.3 The Magnetic Fields H and $\mu H = B$

On p. 54 Maxwell also introduced magnetic phenomena, and emphasized a formal parallel with electric phenomena. He labeled the magnetic field $H$ as $(\alpha_1, \beta_1, \gamma_1)$ and the magnetic (induction) field $B$ as $(a_1, b_1, c_1)$.\(^{21}\) called the (relative) magnetic permeability $\mu$ the reciprocal of the resistance to magnetic induction, $k_1$, and noted (in words) that the parallel to our eq. (31), his eq. (B), is

$$H = \frac{B}{\mu} \quad (\alpha_1 = k_1 a_1, \ etc.), \quad (34)$$

and that in the relation $\nabla \cdot B = 4\pi \rho_m$, $\rho_m$ is the density of real magnetic matter.

A.1.4 Ampère’s Law

On p. 56, Maxwell stated Ampère’s Law in the form,\(^{22}\)

$$\nabla \times H = \frac{4\pi}{c} J_{\text{free}} \left( a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}, \ etc. \right), \quad (39)$$

and on p. 57 he noted that the divergence of eq. (39) is zero, so that his discussion is limited to closed currents that obey $\nabla \cdot J = 0$ (i.e., to magnetostatics). Indeed, he added: in fact

\(^{21}\)Maxwell did not use the symbol $B$ for the magnetic (induction) field until 1873, in his Treatise [6], when he followed W. Thomson (1871), eq. (r), p. 401 of [45], who first defined $B = H + 4\pi M$, where $M$ is the density of magnetization.

\(^{22}\)This is probably the first statement of Ampère’s Law as a differential equation.

Ampère did not have a concept of fields, but discussed the force between two circuits, carrying currents $I_1$ and $I_2$, on pp. 21-24 of [46] (1822) and inferred that this could be written (here in vector notation) as

$$F_{\text{on } 1} = \oint_{I_1} \oint_{I_2} d^2 F_{\text{on } 1} = I_1 I_2 [3(\hat{r} \cdot \hat{r})(\hat{r} \cdot d\hat{r}) - 2 (\hat{r} \cdot d\hat{r})] \frac{\hat{r}}{c^2 r^2} = -d^2 F_{\text{on } 2}, \quad (35)$$

where $\hat{r} = \hat{r}_1 - \hat{r}_2$ is the distance from a current element $I_2 d\hat{r}_2$ at $\hat{r}_2$ to element $I_1 d\hat{r}_1$ at $\hat{r}_1$.

Ampère also noted that

$$dl_1 = \frac{\partial \hat{r}}{\partial l_1} dl_1, \quad \hat{r} \cdot dl_1 = \hat{r} \cdot \frac{\partial \hat{r}}{\partial l_1} dl_1 = \frac{1}{2} \frac{\partial^2 r}{\partial l_1^2} dl_1 = r \frac{\partial r}{\partial l_1} dl_1, \quad dl_2 = -\frac{\partial \hat{r}}{\partial l_2} dl_2, \quad \hat{r} \cdot dl_2 = -\frac{\partial r}{\partial l_2} dl_2, \quad (36)$$

where $l_1$ and $l_2$ measure distance along the corresponding circuits in the directions of their currents. Then,

$$dl_1 \cdot dl_2 = -dl_1 \cdot \frac{\partial l_2}{\partial l_2} dl_2 = -\frac{\partial}{\partial l_2} (r \cdot dl_1) dl_1 dl_2 = -\frac{\partial}{\partial l_2} \left( r \frac{\partial r}{\partial l_1} \right) dl_1 dl_2 = - \left( \frac{\partial r}{\partial l_1} + r \frac{\partial^2 r}{\partial l_1 \partial l_2} \right) dl_1 dl_2, \quad (37)$$

and eq. (35) can also be written in forms closer to that used by Ampère,

$$d^2 F_{\text{on } 1} = I_1 I_2 dl_1 dl_2 \left[ 2 r \frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \right] \frac{\hat{r}}{c^2 r^2} = 2 I_1 I_2 dl_1 dl_2 \frac{\partial^2 r}{\partial l_1 \partial l_2} \frac{\sqrt{r}}{c^2 \sqrt{r}} = -d^2 F_{\text{on } 2}. \quad (38)$$

The integrand $d^2 F_{\text{on } 1}$ of eq. (35) has the appeal that it changes sign if elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton’s third law. However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton’s gravitational force, and Coulomb’s law for the static force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, i.e., a pair of moving electric charges.
we know little of the magnetic effects of any current that is not closed.\footnote{This statement can be regarded as a precursor to Maxwell’s later vision (first enunciated in eq. (112), p. 19, of \cite{20}) that all currents are closed if one considers the “displacement-current” (density) $1/4\pi\frac{d\mathbf{D}}{dt}$ in addition to the conduction-current density $\mathbf{J}_{\text{free}}$. But it also indicates that Maxwell chose not to consider the notion of moving charged particle as elements of an electrical current, as advocated by Weber (1843) \cite{47} (see p. 88 of the English translation) as a way of understanding Ampère’s expression for the force between two current loops.}

### A.1.5 Helmholtz’ Theorem

There followed an interlude on various theorems, some due to Green \cite{49}, and also Helmholtz’ theorem \cite{50} that “any” vector field $\mathbf{E}$ can be related to a scalar potential $\Psi$ and a vector potential $\mathbf{A}$ as $\mathbf{E} = \nabla \Psi + \nabla \times \mathbf{A}$, where $\Psi = 0$ if $\nabla \cdot \mathbf{E} = 0$ and $\mathbf{A} = 0$ if $\nabla \times \mathbf{E} = 0$.

### A.1.6 Magnetic Field Energy and the Electric Field Induced by a Changing Current

After this, Maxwell considered the energy stored in the magnetic field. On p. 63, he first argued that if the magnetic field were due to a density $\rho_m\rho_1$ of magnetic charges, the field could be deduced from a magnetic scalar potential $\Psi_m = p_1$ and the energy stored in the field during the assembly of this configuration could be written

\[ U_m = \frac{1}{2} \int \rho_m \Psi_m \, d\text{Vol} \quad \left( Q = \int \int \int \rho_1 p_1 \, dx dy dz \right). \tag{40} \]

He then noted that this form can be transformed to

\[ U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{8\pi} \, d\text{Vol} \quad \left( Q = \frac{1}{4\pi} \int \int \int (a_1 \alpha_1 + b_1 \beta_1 + c_1 \gamma_1) \, dx dy dz \right). \tag{41} \]

He next argued that since this form does not include any trace of the origin of the magnetic field, it should also hold if the field is due to electrical currents, it can be transformed to

\[ U_m = \int \frac{\mathbf{J} \cdot \mathbf{A}}{2c} \, d\text{Vol} \quad \left( Q = \frac{1}{4\pi} \int \int \int \left\{ p_1 p_1 - \frac{1}{4\pi} (a_0 \alpha_2 + \beta_0 \beta_2 + \gamma_0 \gamma_2) \right\} \, dx dy dz \right). \tag{42} \]

where $\mathbf{B} = \nabla \times \mathbf{A}$,\footnote{It seems to this author that Maxwell omitted a factor of $1/2$ in throughout his discussion on pp. 63-64.}\footnote{Maxwell seems to have made a sign error in his integration by parts of the integrand $\mathbf{B} \cdot \nabla$, so his $\mathbf{B}$ was also related to a scalar potential, and his version of eq. (42) has an additional term related to possible magnetic charges and the scalar potential.}\footnote{W. Thomson, p. 63 of \cite{51} (1846), described the electrical force due to a unit charge at the origin exerted at the point $(x, y, z)$ as $r/r^3$, without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term “electric field” were used by Thomson.}
The time rate of change of energy in the magnetic field is $-\mathbf{J} \cdot \mathbf{E}$, so by taking the time derivative of eq. (42), and noting that $\mathbf{A} = (\alpha_0, \beta_0, \gamma_0)$ scales linearly with $\mathbf{J}$, he infers (on p. 54) that the electro-motive force due to the action of magnets and currents is

$$E_{\text{induced}} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \left( \alpha_2 = -\frac{1}{4\pi} \frac{d\alpha_0}{dt}, \text{ etc.} \right).$$  \hspace{1cm} (43)$$

On p. 66, he stated this equation in words as: Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction. Here, Maxwell describes the vector potential $\mathbf{A}$ as the electro-tonic intensity, following Faraday [53].

A.1.7 The Electro-Motive Force on a Moving Particle

At the bottom of p. 64, Maxwell made a statement that anticipated his later efforts towards the “Lorentz” force law: If $\alpha_0$ be expressed as a function of $x$, $y$, $x$, and $t$, and if $x$, $y$, $z$ be the co-ordinates of a moving particle, then the electro-motive force measured in the direction of $x$ is

$$\alpha_2 = -\frac{1}{4\pi} \left( \frac{d\alpha_0}{dx} \frac{dx}{dt} + \frac{d\alpha_0}{dy} \frac{dy}{dt} + \frac{d\alpha_0}{dz} \frac{dz}{dt} \right), \quad \mathbf{e} = -\frac{1}{c} \left( \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} \right),$$

where we use the symbol $\mathbf{e} = (\alpha_2, \beta_2, \gamma_2)$ for Maxwell's vector electromotive force, which is not necessarily the same as the lab-frame electric field $\mathbf{E}$. Here, Maxwell claimed that the electric field experienced by a moving particle should be computed using the convective derivative of the vector potential, and not just the partial time derivative.

If Maxwell had persisted in following the consequences of this claim, he could have deduced, via a vector-calculus identity, that the electro-motive force experienced by a moving particle is

$$\mathbf{e} = -\frac{1}{c} \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) + \mathbf{v} \times (\nabla \times \mathbf{A}) \right) = \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{1}{c} \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) \right).$$  \hspace{1cm} (45)

If Maxwell had further considered that the electric field can have a term deducible from a scalar potential $\Psi$, then he might have claimed that the total electro-motive force on a moving particle is

$$\mathbf{e} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \nabla(\Psi + \mathbf{v} \cdot \mathbf{A}) - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \hspace{1cm} (46)$$

---

Thomson immediately continued with the example of a small magnet, i.e., a “point” magnetic dipole $\mathbf{m}$, whose scalar potential is $\Phi = \mathbf{m} \cdot \mathbf{r}/r^3$, noting that the magnetic force $(X, Y, Z) = -\nabla \Phi = \mathbf{B}$ on a unit magnetic pole $p$ can also be written as $\nabla \times \mathbf{A}$ where $\mathbf{A} = (\alpha, \beta, \gamma) = \mathbf{m} \times \mathbf{r}/r^3$ (although Thomson did not assign a symbol to the vector $\mathbf{A}$). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [52] implied this result, and is generally credited with the invention of the vector potential although the relation $\mathbf{B} = \nabla \times \mathbf{A}$ is not evident in this paper).

28 Most contemporary discussions of magnetic field energy start from the relation and work towards eq. (42) and then (41).

29 Maxwell seems to have made another sign error, in his discussion of the time rate of change of the field energy, such that his version of our eq. (43) had the correct sign.
We will see below that Maxwell did not follow the path sketched above, but made important variants thereto, while others in the late 1800’s (Helmholtz [27], Larmor [28], Watson [29], J.J. Thomson in his Appendix to Chap. IX of [6], p. 260) argued that he should have proceeded as above.

A.1.8 Faraday’s Law

Faraday’s Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [41]: the electro-motive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit, as a summary of Faraday’s comments in [32]. We express this as the equation (for a circuit at rest in the lab where $\mathbf{E} = \mathbf{E}$),

$$
\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d\Phi_m}{dt} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}.
$$

A.2 In On Physical Lines of Force [18, 19, 20, 21]

A.2.1 The Magnetic Stress Tensor, Ampère’s Law and the Biot-Savart Force Law

In [18] (1861), Maxwell considered a (linear) magnetic medium with uniform (relative) permeability $\mu$ to be analogous to a fluid filled with vortices, which led him, on p. 168, to deduce/invent the stress tensor $T_{ij} = p_{ij}$ of the magnetic field,

$$
T_{ij} = \frac{\mu H_i H_j}{4\pi} - \delta_{ij} p_m,
$$

where $\mathbf{H} = (\alpha, \beta, \gamma)$ is the magnetic field, and $p_m = p_1$ is a magnetic pressure. The volume force density $\mathbf{f}$ in the medium is then given by Maxwell’s eq (3),

$$
\mathbf{f} = \nabla \cdot \mathbf{T},
$$

and the $x$-component of this force, $f_x = X$, is given by Maxwell’s eqs. (4)-(5),

$$
\begin{align*}
 f_x &= \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \\
 &= \frac{1}{4\pi} \left( 2\mu H_x \frac{\partial H_x}{\partial x} - 4\pi \frac{\partial p_m}{\partial x} + \mu H_x \frac{\partial H_y}{\partial y} + H_y \frac{\partial H_z}{\partial y} + \mu H_x \frac{\partial H_z}{\partial z} + H_z \frac{\partial H_x}{\partial y} \right) \\
 &= \frac{H_x}{4\pi} \nabla \cdot \mathbf{B} + \frac{\mu}{8\pi} \frac{\partial H^2}{\partial x} - \frac{B_y}{4\pi} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) - \frac{B_z}{4\pi} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial y} \right) - \frac{\partial p_m}{\partial x} \\
 &= \frac{H_x}{4\pi} \nabla \cdot \mathbf{B} + \frac{\mu}{8\pi} \frac{\partial H^2}{\partial x} - \frac{B_y}{4\pi} \left( \nabla \times \mathbf{H} \right)_z + \frac{B_z}{4\pi} \left( \nabla \times \mathbf{H} \right)_y - \frac{\partial p_m}{\partial x},
\end{align*}
$$

where Maxwell introduced $\mathbf{B} = \mu \mathbf{H}$ as the magnetic induction field. In his eq. (6), p. 168, Maxwell stated that
$$\nabla \cdot \mathbf{B} = 4\pi \rho_m \left( \frac{d}{dx} \mu_\alpha + \frac{d}{dy} \mu_\beta + \frac{d}{dz} \mu_\gamma = 4\pi m \right), \quad (51)$$

where $\rho_m = m$ is the density of “imaginary magnetic matter.” In his eq. (9), p. 171, Maxwell stated that

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{free}} \left[ \frac{1}{4\pi} \left( \frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) = p, \text{ etc.} \right], \quad (52)$$

where $\mathbf{J}_{\text{free}} = (p, q, r)$ is the density of (free) electric current. He did not reference his discussion of our eq. (39) in his paper [41]), but arrived at eq. (52) via an argument that involved both magnetic poles and electric currents. Thus, the force density (50) can be rewritten as, Maxwell’s eqs. (12)-(14), p. 172,

$$\mathbf{f} = \rho_m \mathbf{B} + \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{B} + \nabla \frac{\mu H^2}{8\pi} - \nabla p_m. \quad (53)$$

The second term of eq. (53) is (this author believes) the first statement of what is now commonly called the Biot-Savart force law for a free electric-current density,

$$\mathbf{F} = \int \frac{\mathbf{J}_{\text{free}} \times \mathbf{B}}{c} \, d\text{Vol}, \quad (54)$$

in terms of a magnetic field (of which Biot and Savart [30] had no conception).\footnote{In 1845, Grassmann [54] argued that although Ampère claimed [46] that his force law was uniquement déduite de l’expérience, it included the assumption that it obeyed Newton’s third law. He noted that Ampère’s law (35) implies that the force is zero for parallel current elements whose lie of centers makes angle $\cos^{-1} \sqrt{2}/3$ to the direction of the currents, which seemed implausible to him. Grassmann claimed that, unlike Ampère, he would make no “arbitrary” assumptions, but in effect he assumed that there is no magnetic force between collinear current elements, which leads to a force law

$$\mathbf{F}_{\text{on 1}} = \int_{\mathbb{1}} \int_{\mathbb{2}} d^2 \mathbf{F}_{\text{on 1}}, \quad d^2 \mathbf{F}_{\text{on 1}} = I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{c^2 r^2}, \quad (55)$$

in vector notation (which, of course, Grassmann did not use). While $d^2 \mathbf{F}_{\text{on 1}}$ is not equal and opposite to $d^2 \mathbf{F}_{\text{on 2}}$, Grassmann showed that the total force on circuit 1 is equal and opposite to that on circuit 2, $\mathbf{F}_{\text{on 1}} = -\mathbf{F}_{\text{on 2}}$.

Grassmann’s result is now called the Biot-Savart force law,

$$\mathbf{F}_{\text{on 1}} = \int_{\mathbb{1}} I_1 d\mathbf{l}_1 \times \frac{\mathbf{B}_2}{c}, \quad \mathbf{B}_2 = \int_{\mathbb{2}} \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{c r^2}, \quad (56)$$

although Grassmann did not identify the quantity $\mathbf{B}_2$ as the magnetic field.

\footnote{If the permeability $\mu$ is nonuniform, the third and fourth terms combine to yield the term $(H^2/8\pi) \nabla \mu$, as noted by Helmholtz (1881) [55].}}
A.2.2 Magnetic Field Energy

On p. 63 of [41], Maxwell had deduced that the energy stored in the magnetic field can be computed according to our eq. (41) via an argument that supposed the existence of magnetic charges (monopoles) and a corresponding magnetic scalar potential. In Prop VI, pp. 286-288 of [19], Maxwell used his model of molecular vortices to deduce the same result (given in his eqs. (45)-(46), p. 288, and again in his eq. (51), p. 289),

$$U_m = \int \frac{B \cdot H}{8\pi} \, d\text{Vol} \quad \left( E = \frac{1}{8\pi} \mu (\alpha^2 + \beta^2 + \gamma^2) V \right). \quad (57)$$

A.2.3 Faraday’s Law

In Prop. VII, pp. 288-291 of [19], Maxwell considered the time derivative of the magnetic field energy, written in his eq. (52), p. 289, as

$$\frac{dU_m}{dt} = \int \frac{H}{4\pi} \cdot \frac{\partial B}{\partial t} \, d\text{Vol} \quad \left[ \frac{dE}{dt} = \frac{1}{4\pi} \mu V \left( \alpha \frac{d\alpha}{dt} + \beta \frac{d\beta}{dt} + \gamma \frac{d\gamma}{dt} \right) \right], \quad (58)$$

He also introduced the electromotive force \( E = (P, Q, R) \) on a unit electric charge, and argued, using his model of molecular vortices, that the (electro)magnetic field does work on an electric current density \( J_{\text{free}} \) at rate,

$$\frac{dU_m}{dt} = - \int J_{\text{free}} \cdot E \, d\text{Vol}. \quad (59)$$

In cases like the present where the electromotive force \( E \) is that on a hypothetical unit charge at rest in the lab, it is the same as the electric field \( E \), whose symbol will be used in this section.\(^{33}\) Maxwell next gave an argument in his eqs. (48-50) equivalent to using our eq. (52) to find

$$\frac{dU_m}{dt} = - \frac{c}{4\pi} \int E \cdot (\nabla \times H) \, d\text{Vol} = - \frac{c}{4\pi} \int H \cdot (\nabla \times E) \, d\text{Vol}. \quad (60)$$

Comparing our eqs. (58) and (60), we infer,\(^{34}\) as in Maxwell’s eq. (54), p. 290, that

$$\nabla \times E = - \frac{1}{c} \frac{\partial B}{\partial t} \left( \frac{dQ}{dz} - \frac{dR}{dy} = \mu \frac{d\alpha}{dt}, \text{etc.} \right). \quad (61)$$

This is the first statement of Faraday’s law as a (vector) differential equation. Surprisingly, Maxwell did not give this differential form either in [1] or in his Treatise [6].

\(^{32}\)Strictly, Maxwell wrote in his eq. (47), p. 289, that \( dE/dt \) is a surface integral rather than a volume integral.

\(^{33}\)In sec. A.2.5, which concerns moving charges, we will use the symbol \( E \) for Maxwell’s electromotive force vector.

\(^{34}\)This argument ignores a possible contribution to the field energy from the electric field.
A.2.4 Electric Field outside a Toroidal Coil with a Time-Varying Current

We digress slightly to note that on pp. 338-339 of [19], Maxwell considered a toroidal coil in his Fig. 3. If this coil carries an electric current, there is no exterior magnetic field even in the case of a time-dependent current (if one neglects electromagnetic radiation, whose existence Maxwell reported only in the next paper, [20], in his series *On Physical Lines of Force*). However, a changing current induces an external electric field, which seems like action at a distance. Maxwell noted that while the external magnetic field is zero, the external vector potential $\mathbf{A}$ (*electrotonic state*) is not, and the external electric field is related to the time derivative of $\mathbf{A}$. It seems that the vector potential $\mathbf{A}$ had “physical reality” for Maxwell, which view was later extended to a quantum context by Aharonov and Bohm [56, 57].

![Fig. 3.](image)

A.2.5 Electromotive Force in a Moving Body

In Prop. XI, pp. 340-341 of [19], Maxwell considered a body that might be deforming, translating, and/or rotating, and discussed the resulting changes in the magnetic field $\mathbf{H}$. On one hand, he stated in his eq. (70), p. 341, that

$$
\delta \mathbf{H} = (\delta \mathbf{x} \cdot \nabla) \mathbf{H} + \delta t \frac{\partial \mathbf{H}}{\partial t} \left( \frac{\delta \alpha}{dx} + \frac{\delta \alpha}{dy} \delta y + \frac{\delta \alpha}{dz} \delta z + \frac{\delta \alpha}{dt} \delta t \right),
$$

which uses the convective derivative. On the other hand, he stated before his eq. (69): The variation of the velocity of the vortices in a moving element is due to two causes—the action of the electromotive forces, and the change of form and position of the element. The whole variation of $\alpha$ is therefore

$$
\delta \alpha = \frac{1}{\mu} \left( \frac{dQ}{dz} - \frac{dR}{dy} \right) \delta t + \alpha \frac{d}{dx} \delta x + \beta \frac{d}{dy} \delta x + \gamma \frac{d}{dz} \delta x \left( \delta \mathbf{H} = -\frac{c}{\mu} \nabla \times \mathbf{E} \delta t + (\mathbf{H} \cdot \nabla) \delta \mathbf{x} \right). (63)
$$

If we accept this relation, then we can follow Maxwell that for an incompressible medium, whose velocity field obeys $\nabla \cdot \mathbf{v} = 0$, and if $\nabla \cdot \mathbf{H} = 0$ (which Maxwell stated to hold in the absence of free magnetism), then his eqs. (69)-(70) do lead to his eq. (76), p. 342,

$$
\frac{d}{dz} \left( Q + \mu \gamma \frac{dx}{dt} - \mu \alpha \frac{dz}{dt} - \frac{dG}{dt} \right) - \frac{d}{dy} \left( R + \mu \alpha \frac{dy}{dt} - \mu \beta \frac{dx}{dt} - \frac{dH}{dt} \right) = 0
$$

---

351 believe that our eq. (62) holds for a body that has translated by $\delta \mathbf{x}$, without rotation or deformation.

36In this section, which considers the electromotive force on a moving, unit charge, we use the symbol $\mathbf{E}$ for this, rather than the symbol $\mathbf{E}$.

37The term $(\mathbf{H} \cdot \nabla) \delta \mathbf{x}$ was motivated by Maxwell’s Props. IX and X, pp. 340-341 of [19], but is not evident to this author.
\[
\n\nabla \times \left( \mathbf{e} - \frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0.
\]

(64)

where here Maxwell wrote the vector potential (electrotonic components) as \( \mathbf{A} = -(F, G, H) \). This leads to Maxwell’s eq. (77),

\[
\mathbf{e} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \nabla \Psi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad \left( P = \mu \gamma \frac{dy}{dt} - \mu \beta \frac{dz}{dt} + \frac{dF}{dt} - \frac{d\Psi}{dx} \text{, etc.} \right)
\]

(65)

where the “function of integration” \( \Psi \) was interpreted by Maxwell as the electric scalar potential (electric tension).

The sense of Maxwell’s derivation is that \( q \mathbf{e} \) would be the force experienced (in the lab frame) by an electric charge in the moving body, i.e.,

\[
\mathbf{F} = q \mathbf{e} = q \left( \frac{\mathbf{v}}{c} \times \mathbf{B} - \nabla \Psi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \quad \text{("Lorentz")},
\]

(66)

using the relation \( \mathbf{E} = -\nabla \Psi - \partial \mathbf{A}/\partial t \) for the electric field in the lab frame. Then, eq. (66) is the first statement of the “Lorentz” force law. However, Maxwell’s argument seemed to have had little impact, perhaps due to the doubtful character of his argument leading to our eq. (63).

If the body were in uniform motion with velocity \( \mathbf{v} \), \( \mathbf{e} \) could be interpreted as the electric field \( \mathbf{E}' \) experienced by an observer moving with the body. Then,

\[
\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B},
\]

(67)

which is the low-velocity Lorentz transformation of the electric field \( \mathbf{E} \).

### A.2.6 Faraday’s Law, Revisited

On p. 343 of [19], Maxwell considered a moving conductor, and moving circuit, in a magnetic field, with no electric field in the lab frame. In his eqs. (78)-(79) he applied his eq. (77) to a segment of a moving conductor, finding

\[
\mathbf{e}' \cdot d\mathbf{l} = \frac{\mathbf{v}}{c} \times \mathbf{B} \cdot d\mathbf{l} = -\mathbf{B} \cdot \frac{\mathbf{v}}{c} \times d\mathbf{l} = -\frac{1}{c} \mathbf{B} \cdot \frac{d\mathbf{S}}{dt} \quad \left[ e = S(Pl + Qm + Rn) = S \mu \alpha \left( m \frac{dz}{dt} - n \frac{dy}{dt} \right) \right],
\]

(68)

where \( \mathbf{e} \) is the electromotive force vector with respect to the moving conductor, \( d\mathbf{S}/dt = dx/dt \times d\mathbf{l} \) is the area swept out by the moving line element, \( d\mathbf{l} \) of the conductor in unit time.

In the case of a moving, closed circuit, the total (scalar) electromotive force is then,

\[
\mathcal{E} = \oint \mathbf{e} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{1}{c} \frac{d\Phi_m}{dt},
\]

(69)

i.e., the total electromotive force in a closed conductor is measured by the change of the number of lines of force which pass through it; and this is true whether the change be produced by the motion of the conductor or by any external cause.
Since the above argument applies only to the case of a moving circuit, it does not demonstrate Maxwell's claim that our eq. (69) also holds for a circuit at rest with the number of lines of force which pass through it changing due to an external cause, i.e., a time variation of the magnetic field $B$. Presumably, Maxwell meant for the reader to recall his discussion on pp. 338-339 of [19] (sec. A.2.4 above): This experiment shows that, in order to produce the electromotive force, it is not necessary that the conducting wire should be placed in a field of magnetic force, or that lines of magnetic force should pass through the substance of the wire or near it. All that is required is that lines of force should pass through the circuit of the conductor, and that these lines of force should vary in quantity during the experiment.

A.2.7 Electric Currents in the Model of Molecular Vortices

On p. 13 of [20], Maxwell stated: According to our theory, the particles which form the partitions between the cells constitute the matter of electricity. The motion of these particles constitutes an electric current; the tangential force with which the particles are pressed by the matter of the cells is electromotive force, and the pressure of the particles on each other corresponds to the tension or potential of the electricity. Similarly, on p. 86 of [21], Maxwell stated: in this paper I have regarded magnetism as a phenomenon of rotation, and electric currents as consisting of the actual translation of particles.

Maxwell illustrated this vision in his Fig. 2, along with the description: Let $A B$, $P_1$. V. fig. 2, represent a current of electricity in the direction from $A$ to $B$.

![Fig. 2](image)

A contemporary version of this view of electric currents in magnetic matter is that there exist a “bound” current density therein, related by

$$J_{\text{bound}} = c \nabla \times M,$$

where $M$ is the density of magnetization (i.e., of Ampèrian magnetic dipoles, which are “molecular” current loops). However, this relation does not appear in On Physical Lines of Force, where Maxwell seemed to have supposed that his vision applied to all media, including “vacuum,” and not just to magnetic matter.
A.2.8 Displacement Current and Electromagnetic Waves

The most novel aspect of Maxwell’s paper *On Physical Lines of Force* was his introduction of the “displacement current,” and his deduction that the equations of electromagnetism then imply the existence of electromagnetic waves that propagate with the speed of light.

On p. 14 of [20], his discussion reads:

Electromotive force acting on a dielectric produces a state of polarization of its parts similar in distribution to the polarity of the particles of iron under the influence of a magnet, and, like the magnetic polarization, capable of being described as a state in which every particle has its poles in opposite conditions. In a dielectric under induction, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively, and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another.

The effect of this action on the whole dielectric mass is to produce a general displacement of the electricity in a certain direction. This displacement does not amount to a current, because when it has attained a certain value it remains constant, but it is the commencement of a current, and its variations constitute currents in the positive or negative direction, according as the displacement is increasing or diminishing. The amount of the displacement depends on the nature of the body, and on the electromotive force; so that if \( h \) is the displacement (in the \( z \)-direction), \( R \) the electromotive force, and \( E \) a coefficient depending on the nature of the dielectric, \( R = -4\pi E^2 h \); and if \( r \) is the value of the electric current (in the \( z \)-direction) due to displacement,

\[
 r_{\text{displacement}} = \left( -\frac{1}{4\pi} \right) \frac{dR}{dt} = \frac{1}{4\pi E^2} \frac{dR}{dt}, \tag{71}
\]

where it seems to this author that a minus sign should be inserted in Maxwell’s original version of our eq. (71).

In the above, the *electromotive force* vector \( (P, Q, R) = E \) is the electric field, \( E^2 = 1/\epsilon \) where \( \epsilon \) is the relative permittivity (dielectric constant), the *displacement* vector \( (f, g, h) = -D/4\pi \) is proportional to our present electric field vector \( D \), and \( (p, q, r) = J_{\text{free}} \) is the free current density. Maxwell’s relation \( R = -4\pi E^2 h \) (repeated in his eq. (105), p. 18, of [18]) is equivalent to

\[
 E = \frac{D}{\epsilon}. \tag{72}
\]

Then, Maxwell’s expression for the *electric current due to displacement* is equivalent to

\[
 J_{\text{displacement}} = \frac{1}{4\pi} \frac{dD}{dt}. \tag{73}
\]

On p. 19 of [20], Maxwell argued that a variation of displacement is equivalent to a current, and this current \( r_{\text{displacement}} \) of our eq. (71) must be taken into account in equations (9) [our eq. (52)] and added to \( r \). The three equations then become

\[
 p = \frac{1}{4\pi} \left( \frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) - \frac{1}{E^2} \frac{dP}{dt}, \quad \nabla \times \mathbf{H} = \frac{4\pi}{c} J_{\text{free}} + \frac{1}{c} \frac{dD}{dt}. \tag{74}
\]
This is the first statement of Maxwell’s “fourth” equation as we know it today.

Maxwell next noted that the equation of continuity for free charge and current densities is, his eq. (113), p. 19 of \[20\],

$$\nabla \cdot \mathbf{J}_{\text{free}} + \frac{\partial \rho_{\text{free}}}{\partial t} = 0 \quad \left( \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} + \frac{de}{dt} = 0 \right), \quad (75)$$

where $e = \rho_{\text{free}}$ is the free charge density. On taking the divergence of eq. (74) we arrive at Maxwell’s eqs. (114)-(115),

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = 4\pi \frac{\partial \rho_{\text{free}}}{\partial t}, \quad \nabla \cdot \mathbf{D} = 4\pi \rho_{\text{free}} \left[ e = \frac{1}{4\pi E^2} \left( \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \right) \right], \quad (76)$$

which is the first statement of Maxwell’s “first” equation.\(^{38}\)

In Prop. XVI, p. 22 of \[20\], Maxwell considered the rate of propagation of transverse vibrations through the elastic medium of which the cells are composed, and found the constant $c$ in our equation (74) to have a value remarkably close to the speed of light in vacuum, and concluded that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.

### A.3 In A Dynamical Theory of the Electromagnetic Field [1]

#### A.3.1 Scalar Electromotive Force and the Integral Form of Faraday’s Law

In sec. 24 of [1] Maxwell considered an electrical circuit A that carries current $I_A = u$, and another circuit B that carries current $I_B = v$, and noted that the magnetic flux, $\Phi_m$, through circuit A is given by the first equation on p. 468,

$$\Phi_m = \int_A \mathbf{B} \cdot d\mathbf{S} = LI_A + MI_B \quad (Lu + Mv), \quad (77)$$

where $\mathbf{B}$ is the magnetic field, $d\mathbf{S}$ is an element of the area of a surface bounded by circuit A, $L$ is the self inductance of circuit A, and $M$ is the mutual inductance between circuits A and B. Maxwell called this flux the momentum, or the reduced momentum of the circuit.

In sec. 50, Maxwell gave a verbal statement of Faraday’s Law:

1st. If any closed curve be drawn in the field, the value of $M$ for that curve will be expressed by the number of lines of force which pass through that closed curve.

2ndly. If this curve be a conducting circuit and be moved through the field, an electromotive force will act in it, represented by the rate of decrease of the number of lines passing through the curve.

We transcribe this into symbols (in Gaussian units) as

$$\mathcal{E} = -\frac{1}{c} \frac{d\Phi_m}{dt}, \quad (78)$$

where $\mathcal{E}$ is the scalar electromotive force.

\(^{38}\)Nowadays it is more common to argue that Maxwell’s “first” and “fourth” equations together imply the continuity equation (75).
A.3.2 Vector Electromagnetic Force and Displacement Current

However, in sec. 56, Maxwell used the term electromotive force in a different way, to describe a vector, \( \mathbf{E} = (P, Q, R) \): \( P \) represents the difference of potential per unit of length in a conductor placed in the direction of \( x \) at the given point. This appears to mean that
\[
\mathbf{E} = (P, Q, R) = -\nabla \Psi,
\]
where \( \Psi \) is Maxwell’s symbol for the electric scalar potential. If so, this is the first mention of an aspect of the electric field \( \mathbf{E} \) in [1], although he had introduced the electrical displacement \( \mathbf{D} = (f, g, h) \) in sec. 55, along with “displacement-current” (density), \( (1/4\pi) d\mathbf{D}/dt \) in his eq. (A),
\[
\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \frac{1}{4\pi} \frac{d\mathbf{D}}{dt} \left( (p', q', r) = (p, q, r) + \frac{d(f, g, h)}{dt} \right),
\]
where the free current density \( \mathbf{J}_{\text{free}} = (p, q, r) \) was introduced in sec. 54, and the total motion of electricity is \( \mathbf{J}_{\text{total}} = (p', q', r') \). Maxwell did not use separate symbols for partial and total derivatives, so that there can be some ambiguity as to his meaning when his equations describe moving systems.

A.3.3 Vector Potential aka Electromagnetic Momentum

In sec. 57, Maxwell introduced the vector potential \( \mathbf{A} = (F, G, H) \), but called it the electromagnetic momentum. In his eq. (29), Maxwell identified \(-d\mathbf{A}/dt\) with the part of the electromotive force which depends on the motion of magnets or currents. Thus, we might now presume that Maxwell’s \( \mathbf{E} = (P, Q, R) \) of his sec. 56 is the electric field,
\[
\mathbf{E} = -\nabla \Psi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},
\]
but this conclusion may be premature.

A.3.4 Magnetic Flux aka Electromagnetic Momentum of a Circuit

In eq. (29) of sec. 58, Maxwell gave the relation for the magnetic flux \( \Phi_m \) through a circuit, the number of lines of magnetic force which pass through it,
\[
(\Phi_m = \oint \mathbf{B} \cdot d\mathbf{S} =) \oint \mathbf{A} \cdot d\mathbf{l} \left[ \int \left( \frac{dF}{ds} + \frac{dG}{ds} + \frac{dH}{ds} \right) ds \right],
\]
and called this the total electromagnetic momentum (which we must remember to distinguish from the electromagnetic momentum \( \mathbf{A} \)).

He also noted in sec. 58 that
\[
(\Phi_m = \oint \mathbf{A} \cdot d\mathbf{l} = \int) \nabla \times \mathbf{A} \cdot d\mathbf{S} \left[ \left( \frac{dH}{dy} - \frac{dG}{dz} \right) dy dz \right],
\]
is the number of lines of magnetic force which pass through the area \( dy \, dz \).
A.3.5 The Magnetic Fields H and B and Maxwell’s Fourth Equation

In sec. 59, Maxwell introduced the magnetic field \( \mathbf{H} = (\alpha, \beta, \gamma) \).

In sec. 60, Maxwell introduced the (relative) permeability \( \mu \), calling it the coefficient of magnetic induction.

In eq. (B) of sec. 61, Maxwell gave the Equations for Magnetic Force,

\[ \mu \mathbf{H} = \nabla \times \mathbf{A} \quad \left[ \mu \alpha = \left( \frac{dH}{dy} - \frac{dG}{dz} \right), \text{ etc.} \right] . \quad (84) \]

We use the symbol \( \mathbf{B} \) for Maxwell’s \( \mu \mathbf{H} \).

In eq. (C) of sec. 62, Maxwell gives a version of Ampère’s Law,

\[ \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{total}} \quad \left( \frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p', \text{ etc.} \right) , \quad (85) \]

recalling from our eq. (80) that Maxwell’s vector \( (p', q', r') \) is the total current density \( \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{displacement}} \).

A.3.6 Electromotive Force in a Circuit at Rest

In eq. (32), sec. 63 of [1], Maxwell stated that the electromotive force acting round an electrical circuit is related by

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} \quad \left[ \xi = \int \left( P \frac{dx}{ds} + Q \frac{dy}{ds} + R \frac{dz}{ds} \right) ds \right] . \quad (86) \]

Supposing this circuit, \( \mathcal{A} \), carries current \( I_A = u \), and another circuit, \( \mathcal{B} \), carries current \( I_B = v \), Maxwell, in his eq. (33), reminded us the magnetic flux through circuit \( \mathcal{A} \) is given by our eq. (77)

\[ \Phi_m = \oint_{A} \mathbf{A} \cdot d\mathbf{l} = LI_A + MI_B \quad \left[ \int \left( F \frac{dx}{ds} + G \frac{dy}{ds} + H \frac{dz}{ds} \right) ds = Lu + Mv \right] , \quad (87) \]

as he had previously discussed in sec. 24. Then, his eq. (34) states that

\[ \mathcal{E} = -\frac{1}{c} \frac{d}{dt} (LI_A + MI_B) \quad \left( = \frac{1}{c} \int \frac{d\mathbf{A}}{dt} \cdot d\mathbf{l} \right) , \quad (88) \]

so comparison with our eq. (86) leads to the inference, Maxwell’s eq. (35), that if there is no motion of the circuit \( \mathcal{A} \),

\[ \mathcal{E} = -\frac{1}{c} \frac{\partial}{\partial t} (LI_A + MI_B) \quad \left( = -\frac{1}{c} \int \frac{d\mathbf{A}}{dt} \cdot d\mathbf{l} \right) , \quad (89) \]

where \( \Psi \) could be any scalar function. But, the discussion in his sec. 56 led Maxwell to identify \( \Psi \) of eq. (89) as the electrical scalar potential.

In eq. (89), we have written \( \partial \mathbf{A} / \partial t \), while Maxwell wrote \( d\mathbf{A} / dt \), in that for an observer (at rest) of a circuit at rest, use of a convective derivative is not appropriate.

39 The symbol \( \mathbf{B} \) for the quantity \( \mathbf{H} + 4\pi \mathbf{M} \), where \( \mathbf{M} \) is the magnetization density, was introduced by W. Thomson in 1871, eq. (r), p. 401 of [45], and appears in sec. 399 of Maxwell’s Treatise [6].

40 This meaning of the term electromotive force is still in use today. However, Maxwell also used the term electromotive force in sec. 65 of [1] to describe the force \( \mathbf{v}/c \times \mu \mathbf{H} \) on a moving, unit charge in a magnetic field, referring to his eq. (D).
A.3.7 The Vector Electromotive Force on a Moving Conductor

In sec. 64, Maxwell deduced the force on a bar the slides on a U-shaped rail, while carrying a current, with the entire system in an external magnetic field. He gave no figure in [1], but the figure below is associated with his discussion of this example in Arts. 594-597 of [6]. C represents a battery that drives the current in the circuit.

Maxwell desired a very general discussion in sec. 64, so he considered a circuit whose plane was not perpendicular to any of the \(x\), \(y\) or \(z\) axes, which makes his description rather intricate. Here, we suppose the circuit lies in the \(x\)-\(z\) plane, with the sliding piece, \(AB\), of length \(a\) parallel to the \(x\)-axis, and the long arms of the U-shaped rail parallel to the \(z\)-axis, at, say \(x = 0\) and \(a\). The velocity \(v_z = dz/dt\) of the sliding bar is in the \(z\)-direction, and the uniform, external magnetic field is in the \(y\)-direction.

As in sec. 63, Maxwell considered changes in the magnetic flux through the circuit, \(\int \mathbf{A} \cdot d\mathbf{l}\), our eq. (82), to infer the strength of his vector \(\mathbf{E}\). The part of the line integral over the sliding bar changes at rate,\(^{41}\)

\[
a \frac{dA_x}{dz} \frac{dz}{dt},
\]  

as indicated in the first equation on p. 485. Because the length of the circuit in \(z\) is increasing, the line integral also changes at rate

\[
\frac{dz}{dt}[A_z(x = 0) - A_z(x = a)] = -\frac{dz}{dt} \frac{dA_z}{dx} a,
\]  

as given in the second equation on p. 485. Hence, the total rate of change of magnetic flux, given in the third and fourth equations on p. 485, is\(^{42}\)

\[
\frac{d\Phi_m}{dt} = a v_z \left( \frac{dA_x}{dz} - \frac{dA_z}{dx} \right) = a v_z B_y = -c \mathbf{E} = -c \oint \mathbf{E} \cdot d\mathbf{l}.  
\]  

Maxwell considered that eq. (92) describes a contribution to the electromotive force \(\mathbf{E}\) beyond that in eq. (89), which additional contribution would be localized to the component \(E_x\) along the sliding bar (of length \(a\)), \(i.e., \oint \mathbf{E} \cdot d\mathbf{l} = a E_x\). Hence, he concluded in his eq. (36) that

\[
E_x = -\frac{v_z B_y}{c} \left( P = -\mu \beta \frac{dz}{dt} \right), \quad i.e., \quad \mathbf{E} = \frac{v}{c} \times \mathbf{B},
\]  

is the part of \(\mathbf{E}\) due to the motion of the sliding bar.

\(^{41}\)In eqs.(90)-(93), the quantities \(A_x, A_z\) and \(B_y\) are evaluated at the location of the sliding bar.

\(^{42}\)On p. 485, the equations of Magnetic Force (8) should read: the equations of Magnetic Force (B).
Finally, in sec. 65, Maxwell stated that the total electromotive force on a moving conductor is his eq. (D),

\[ \mathbf{E} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi \]

recalling his eq. (35), our eq. (89). Again, we have written \( \partial \mathbf{A}/\partial t \) where Maxwell wrote \( d\mathbf{A}/dt \).

43 Note that Maxwell’s argument in sec. 65 does not address the mechanical force on the sliding bar, \( I \mathbf{a} \times \mathbf{B}/c \), which is the subject of most present discussions of this example.

A.4 In Maxwell’s Treatise [3, 4, 5, 6]

Maxwell’s discussion of the “Lorentz” force law in Arts. 598-601 of his Treatise has been reviewed in sec. 1 above. However, Maxwell’s presentation in his earlier papers of this topic (and of other aspects of his novel vision of electrodynamics) is perhaps superior to that in his Treatise.44

B Appendix: From Faraday’s Law to the Lorentz Transformation of the Electromagnetic Fields

This section is based on [60] (1979).

B.1 Force on a Moving Circuit

For a circuit at rest, the integral form of Faraday’s law can be written as

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{1}{c} \frac{d\Phi_B}{dt} \left( = -\frac{1}{c} \frac{d}{dt} \oint \mathbf{A} \cdot d\mathbf{l} \right) , \]

where \( \mathcal{E} \) is the electromotive force in the circuit, \( \mathbf{E} \) is the electric field, \( d\mathbf{l} \) is an element of length along the circuit, \( d\mathbf{S} \) is an element of area of a surface bounded by the current loop.

43 Note that convective....

44 On p. 300 of Whittaker’s History of the Theories of Aether and Electricity [59], one reads about Maxwell: In 1871 he returned to Cambridge as Professor of Experimental Physics; and two years later published his Treatise on Electricity and Magnetism. In this celebrated work is comprehended almost every branch of electric and magnetic theory; but the intention of the writer was to discuss the whole as far as possible from a single point of view, namely, that of Faraday; so that little or no account was given of the hypotheses which had been propounded in the two preceding decades by the great German electricians. So far as Maxwell’s purpose was to disseminate the ideas of Faraday, it was undoubtedly fulfilled; but the Treatise was less successful when considered as the exposition of its author’s own views. The doctrines peculiar to Maxwell—the existence of displacement-currents, and of electromagnetic vibrations identical with light were not introduced in the first volume, or in the first half of the second volume; and the account which was given of them was scarcely more complete, and was perhaps less attractive, than that which had been furnished in the original memoirs.
that generates magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \), \( \mathbf{A} \) is the vector potential (Faraday’s electrotonic state\(^{45}\)), and \( \Phi_B = \int \mathbf{B} \cdot d\mathbf{S} \) is the magnetic flux through the circuit.\(^{46,47}\)

We now consider a circuit that moves with velocity \( \mathbf{v} \) in the lab. An inference from Galilean relativity (which implies Newton’s First Law\(^ {48}\)) is that an observer moving with velocity \( \mathbf{v} \) measures fields \( \mathbf{E}' \) and \( \mathbf{B}' \), and the integral form of Faraday’s law for this observer would be

\[
\mathcal{E}' = \oint \mathbf{E}' \cdot d\mathbf{l}' = \oint \mathbf{E}' \cdot d\mathbf{l} = -\frac{1}{c'} \frac{d}{dt} \int \mathbf{B}' \cdot d\mathbf{S}' = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B}' \cdot d\mathbf{S}, \tag{97}
\]

where the integrals are performed over the circuit in the moving frame, where it is at rest. That is, according to Galilean relativity, the moving observer measures length and time to be

\(^{45}\)Faraday introduced his electrotonic state in Art. 60 of [53].

\(^{46}\)Equation (95) also implies the relation

\[
\mathbf{E}_{\text{induced}} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \tag{96}
\]

for the electric field induced by a changing magnetic field (due to a changing current in the circuit).

\(^{47}\)Faraday’s Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [41]: the electromotive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit, as a summary of Faraday’s comments in [32].

Maxwell did not give the mathematical form (95) in [41]), but he did give (the equivalent of) eq. (96) on p. 64 (\( \alpha_2 = -(1/4\pi) \partial \Phi_0 / \partial t \), etc.). Then, on p. 66 he stated this equation in words as: Law VI. The electromotive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction.

Maxwell deduced (via an energy argument!) the differential form of Faraday’s law, \( \nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B}/\partial t \), in eq. (54), p. 290, of [19] (although he used \( \mu \mathbf{H} \) rather than \( \mathbf{B} \) for the magnetic field as the latter was only invented by W. Thomson in 1871, eq. (r), p. 401 of [45]). Surprisingly, Maxwell did not give this differential form either in [1] or in his Treatise [6].

Digression: On p. 64 of [41], Maxwell deduced that the electric field induced by changing currents at a point at rest in the lab is that given in eq. (96) above. Then, he stated that for a moving (charged) “particle” with velocity \( \mathbf{v} \), the field it experiences should be computed using the convective derivative, \( -c \mathbf{E}_{\text{on moving charge}} = D\mathbf{A}/Dt = \partial \mathbf{A}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{A} \).

While we now consider electric charge to be a phenomenon separate from the electromagnetic field, Maxwell considered charge (density) to be an aspect of “displacement” in the æther, \( \rho = \mathbf{v} \cdot \mathbf{D} \). In this view, it seems natural to suppose that a moving charge samples an electromagnetic field in a manner analogous to a particle moving through a fluid, where the convective derivative describes the time dependence of, for example, pressure and density its experiences.

Later, in eq. (D), sec. 65, p. 485, of [1] and eq. (B), Art. 598, p. 239 (and also eq. (10), Art. 599, p. 241) of [6], Maxwell realized that the force on a moving charge should include the term \( \mathbf{v}/c \times \mathbf{B} \) (and not \( \mathbf{H} \)), and managed to arrive at the correct “Lorentz” force law despite his use of the convective derivative (in eq. (2) of Art. 598 of [6]). However, Helmholtz, eq. (54), p. 309 of [27], argued that the term \( \mathbf{v}/c \times \mathbf{B} \) should be accompanied by the additional term \( -\nabla (\mathbf{A} \cdot \mathbf{v}/c) \), which claim was seconded on p. 12 of [28] (1884) and on p. 273 of [29] (1888). These claims may have had the effect that Maxwell’s derivation of the force on a moving “particle,” when corrected/clarified, was not considered to yield the Lorentz force law. For example, when J.J. Thomson edited the 3rd edition of Maxwell’s Treatise he added a comment, p. 260 of [6], casting doubt Maxwell’s analysis of the force on a moving charge. This is unfortunate in that Lorentz, eq. (V), sec. 12 of [13] and eq. (23) of [14], wrote the force law as \( \mathbf{F} = q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H}) \), which is not correct, although this was clarified only in 1944 by experiments [58] on the motion of high-energy particles penetrating magnetized steel.

\(^{48}\)Law I. Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon; p. 83 of [61].
the same as for an observer at rest, and the constant \( c \), which is determined by experiments performed at rest in any frame, has the same value in any (inertial) frame.

We next suppose that \( B' = B \), i.e., observers in the lab frame and in the moving frame assign the same value to the magnetic field. We now consider how an observer at rest in the lab frame might describe the moving observer’s calculation, of \((d/dt) \int B' \cdot dS\) for a circuit at rest in his frame, as

\[
\int \frac{DB}{Dt} \cdot dS, \tag{98}
\]

over the circuit at time \( t \) in the lab frame. Of course, \( DB/Dt \) does not equal \( \partial B/\partial t \) as it must incorporate effects of the motion of the circuit in the lab frame.\(^{49}\)

Referring to the figure below, which is in the lab frame, and where the direction of a surface element \( dS \) is related to the line element \( dl \) by the right-hand rule,

we can express the time derivative of the magnetic flux through the moving circuit in lab-frame quantities as,

\[
\frac{d}{dt} \int_{\text{moving circuit}} B \cdot dS \approx \frac{1}{dt} \left[ \int_{t+dt} B_{t+dt} \cdot dS_{t+dt} - \int_t B_t \cdot dS_t \right]. \tag{99}
\]

We have, approximately, that

\[
B_{t+dt} = B_t + \frac{\partial B_t}{\partial t} dt, \tag{100}
\]

so

\[
\frac{d}{dt} \int_{\text{moving circuit}} B \cdot dS \approx \int_{t+dt} \frac{\partial B_t}{\partial t} \cdot dS_{t+dt} + \frac{1}{dt} \left[ \int_{t+dt} B_t \cdot dS_{t+dt} - \int_t B_t \cdot dS_t \right]. \tag{101}
\]

We now play a famous trick, and consider the integral over the entire surface of the volume swept out by the circuit during time interval \( dt \), taking \( dS \) to be directed out of this volume,

\[
\int_{\text{entire surface}} B_t \cdot dS = \int_{t+dt} B_t \cdot dS_{t+dt} - \int_t B_t \cdot dS_t + \int_{\text{side}} B_t \cdot dS_{\text{side}}. \tag{102}
\]

By Gauss’ theorem,

\[
\int B_t \cdot dS = \int \nabla \cdot B_t \, d\text{Vol} = \int (\nabla \cdot B_t) \, v \, dt \cdot dS_t, \tag{103}
\]

\(^{49}\)Such issues occur frequently in fluid dynamics.
and an area element on the “sides” of the surface is related by \( dS_{\text{side}} = d\mathbf{l} \times \mathbf{v} \, dt \), such that
\[
\int_{\text{side}} \mathbf{B}_t \cdot dS_{\text{side}} = \oint \mathbf{B}_t \cdot d\mathbf{l} \times \mathbf{v} \, dt = -dt \oint \mathbf{B}_t \times \mathbf{v} \cdot d\mathbf{l} = dt \int \nabla \times (\mathbf{B}_t \times \mathbf{v}) \cdot dS_t. \tag{104}
\]

We can now rewrite eq. (102) as We now play a famous trick, and consider the integral over the entire surface of the volume swept out by the circuit during time interval \( dt \), taking \( dS \) to be directed out of this volume,
\[
\int_{t+dt} \mathbf{B}_t \cdot dS_{t+dt} - \int_t \mathbf{B}_t \cdot dS_t = dt \int (\nabla \cdot \mathbf{B}_t) \mathbf{v} \cdot dS_t + dt \int \nabla \times (\mathbf{B}_t \times \mathbf{v}) \cdot dS_t. \tag{105}
\]
Then, recalling eq. (101), and taking the limit as \( dt \to 0 \), we have that
\[
\frac{d}{dt} \int_{\text{moving circuit}} \mathbf{B} \cdot dS = \int \frac{\partial \mathbf{B}}{\partial t} \cdot dS + \int (\nabla \cdot \mathbf{B}) \mathbf{v} \cdot dS + \int \nabla \times (\mathbf{B} \times \mathbf{v}) \cdot dS = \int \frac{DB}{Dt} \cdot dS, \tag{106}
\]
where
\[
\frac{DB}{Dt} = \frac{\partial \mathbf{B}}{\partial t} + (\nabla \cdot \mathbf{B}) \mathbf{v} + \nabla \times (\mathbf{B} \times \mathbf{v}) = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} \tag{107}
\]
is the convective derivative.

Faraday’s Law for the moving circuit, eq. (97), can now be written as
\[
\mathcal{E}' = \oint_{\text{moving circuit}} \mathbf{E}' \cdot d\mathbf{l} = \int_{\text{moving circuit}} \nabla \times \mathbf{E}' \cdot dS = -\frac{1}{c} \int_{\text{fixed circuit}} \frac{DB}{Dt} \cdot dS
\]
\[
= -\frac{1}{c} \int_{\text{fixed circuit}} \left[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) \right] \cdot dS, \tag{108}
\]
noting that \( \nabla \cdot \mathbf{B} = 0 \). This holds at any fixed time \( t \), so we infer that
\[
\nabla \times (\mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}) = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}, \tag{109}
\]
and hence,
\[
\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}. \tag{110}
\]
That is, the force on the moving circuit is given by the Lorentz force law.

This argument, which did not use potentials, but did involve a convective derivative, is perhaps what Maxwell’s Art. 598 of [6] could/should have been. Instead, this argument may have been first given in sec. 86, p. 398 of [26] (1904). See also sec. 9-3, p 160 of [25].

### B.2 Magnetic Field According to a Moving Observer

The preceding section was based on the assumption that the magnetic field is the same for an observer on the moving circuit as for one at rest in the lab. However, we could consider a moving, mathematical loop (not associated with a physical electric current) in a region
of vacuum, away from the sources of laboratory fields $\mathbf{E}$ and $\mathbf{B}$. Instead of emphasizing Faraday’s Law, and supposing that $\mathbf{B}' = \mathbf{B}$ according to a moving observer, we could emphasize Maxwell’s extension of Ampère’s Law, which in vacuum and in the lab frame reads,$^{50}$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad \int_{\text{fixed loop}} \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint_{\text{fixed loop}} \mathbf{B} \cdot d\mathbf{l} = \frac{1}{c} \frac{d}{dt} \int_{\text{fixed loop}} \mathbf{E} \cdot d\mathbf{S},$$

(112)

and suppose that for a moving loop, $\mathbf{E}' = \mathbf{E}$. Then, an argument parallel to that of the preceding section would imply that the magnetic field according to the moving observer is

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E},$$

(113)

where the change of sign compared to eq. (110) is due to the difference in signs between the Faraday’s Law and Maxwell version of Ampère’s law.

### B.3 Both $\mathbf{E}$ and $\mathbf{B}$ are Different for Observers in the Lab and in a Moving Frame

The previous two sections have assumed that either $\mathbf{E}$ or $\mathbf{B}$ is the same for observers in the lab and in a moving frame. But, it is much more plausible that both $\mathbf{E}$ and $\mathbf{B}$ have different values in different frames (for the same physical configuration).

If eqs. (110) and (113) were the correct general relations for the fields $\mathbf{E}'$ and $\mathbf{B}'$ according to an observer with velocity $\mathbf{v}$ in the lab, we would expect that the transformation from fields in the moving frame to the lab frame, which latter has velocity $\mathbf{v}' = -\mathbf{v}$ with respect to the former, would be obtained by exchanging primed and unprimed quantities,

$$\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B'},$$

(114)

$$\mathbf{B} = \mathbf{B}' + \frac{\mathbf{v}}{c} \times \mathbf{E'}.$$  

(115)

We could then check for consistency of these transformations by, form example, starting from eq. (115), slightly rearranged, and then using eq. (114), also slightly rearranged,

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B'} \right)$$

$$\left( 1 - \frac{v^2}{c^2} \right) \mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} - \left( \frac{\mathbf{v}}{c} \cdot \mathbf{B'} \right) \frac{\mathbf{v}}{c}.$$  

(116)

Similarly, we would find,

$$\left( 1 - \frac{v^2}{c^2} \right) \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \left( \frac{\mathbf{v}}{c} \cdot \mathbf{E'} \right) \frac{\mathbf{v}}{c}.$$  

(117)

---

$^{50}$Ampère’s law in the form

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{free}},$$

(111)

is not due to Ampère himself, but was first given by Maxwell in eq. (9), p. 171, of [18] (1861).
Thus our inferences in secs. B.1-2 for the transformations of the fields to a moving frame lead to inconsistencies at order $v^2/c^2$ (although they are valid at order $v/c$).

A partial remedy would be to “split” the factor $1 - v^2/c^2$ between the transformations and their inverses,

$$E' = \frac{1}{\sqrt{1 - v^2/c^2}} \left( E + \frac{v}{c} \times B \right) + ?, \quad E = \frac{1}{\sqrt{1 - v^2/c^2}} \left( E' - \frac{v}{c} \times B' \right) + ?, \quad (118)$$

$$B' = \frac{1}{\sqrt{1 - v^2/c^2}} \left( B - \frac{v}{c} \times E \right) + ?, \quad B = \frac{1}{\sqrt{1 - v^2/c^2}} \left( B' + \frac{v}{c} \times E' \right) + ?, \quad (119)$$

but it is less obvious how to deal with the “extra” terms $(v/c \cdot E)/v/c$ and $(v/c \cdot B)/v/c$ in eqs. (116)-(117). An inspired “guess” would be to include these forms in the transformations, but with an as-yet-undetermined coefficient $\alpha$. Introducing the notation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad (120)$$

we consider the transformations,

$$E' = \gamma \left( E + \frac{v}{c} \times B \right) - \alpha \left( \frac{v}{c} \cdot E \right) \frac{v}{c}, \quad (121)$$

$$B' = \gamma \left( B - \frac{v}{c} \times E \right) - \alpha \left( \frac{v}{c} \cdot B \right) \frac{v}{c}. \quad (122)$$

The inverse transformations are again obtained by swapping primed and unprimed quantities, and changing $v$ to $-v$,

$$E = \gamma \left( E' - \frac{v}{c} \times B' \right) - \alpha \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}, \quad (123)$$

$$B = \gamma \left( B' + \frac{v}{c} \times E' \right) - \alpha \left( \frac{v}{c} \cdot B' \right) \frac{v}{c}. \quad (124)$$

To determine $\alpha$, we rearrange eq. (123)-(124),

$$E' = \frac{E}{\gamma} + \frac{v}{c} \times B' + \frac{\alpha}{\gamma} \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}, \quad (125)$$

$$B' = \frac{B}{\gamma} - \frac{v}{c} \times E' + \frac{\alpha}{\gamma} \left( \frac{v}{c} \cdot B' \right) \frac{v}{c}, \quad (126)$$

and then use eq. (126) in (125) to find,

$$\frac{E'}{\gamma^2} = \frac{1}{\gamma} \left( E + \frac{v}{c} \times B \right) + \left( \frac{\alpha}{\gamma} - 1 \right) \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}. \quad (127)$$

We also have from eq. (121) that

$$\frac{v}{c} \cdot E' = \left( \gamma - \frac{v^2}{c^2} \right) \frac{v}{c} \cdot E, \quad (128)$$

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so that eq. (127) can be rewritten as
\[
E' = \gamma \left( E + \frac{v}{c} \times B \right) + \gamma^2 \left( \frac{\alpha}{\gamma} - 1 \right) \left( \gamma - \frac{\alpha^2}{c^2} \right) \left( \frac{v}{c} \cdot E' \right) \frac{v}{c}.
\] (129)

This should be the same as eq. (123), which gives a quadratic equation for \( \alpha \),
\[
\alpha = \gamma^2 \left( \frac{\alpha}{\gamma} - 1 \right) \left( \gamma - \frac{\alpha^2}{c^2} \right), \quad \alpha^2 - 2\gamma \frac{c^2}{v^2} \alpha + \gamma^2 \frac{c^2}{v^2} = 0, \quad \alpha = \frac{c^2}{v^2} (\gamma \pm 1).
\] (130)

For the transformations to have the trivial form when \( v = 0 \), we take the negative root, \( \alpha = (c^2/v^2)(\gamma - 1) \).

The self-consistent transformations of the electromagnetic fields from the lab frame to a moving frame, deduced from use of Faraday’s and Ampère’s Laws (as formulated by Maxwell), are
\[
E' = \gamma \left( E + \frac{v}{c} \times B \right) - (\gamma - 1)(\dot{v} \cdot E) \dot{v}, \quad B' = \gamma \left( B - \frac{v}{c} \times E \right) - (\gamma - 1)(\dot{v} \cdot B) \dot{v};
\] (131)
\[
E = \gamma \left( E' - \frac{v}{c} \times B' \right) - (\gamma - 1)(\dot{v} \cdot E') \dot{v}, \quad B = \gamma \left( B' + \frac{v}{c} \times E' \right) - (\gamma - 1)(\dot{v} \cdot B') \dot{v};
\] (132)
as deduced by Einstein, sec. 6 of [2].\textsuperscript{51,52} Of course, the present analysis does not yield the insight that there also is a transformation of spacetime coordinates between the two frames.

If we decompose the field vectors into components parallel and perpendicular to velocity \( v, E = E_\parallel + E_\perp \) where \( E_\parallel = (\dot{v} \cdot E) \dot{v} \), then
\[
E_\parallel' = E_\parallel, \quad E_\perp' = \gamma \left( E_\perp + \frac{v}{c} \times B \right), \quad B_\parallel' = B_\parallel, \quad B_\perp' = \gamma \left( B_\perp - \frac{v}{c} \times E \right), \quad (134)
\]
\[
E_\parallel' = E_\parallel', \quad E_\perp' = \gamma \left( E_\perp' - \frac{v}{c} \times B' \right), \quad B_\parallel' = B_\parallel', \quad B_\perp' = \gamma \left( B_\perp' + \frac{v}{c} \times E' \right). \quad (135)
\]

The low-velocity approximations to these transformations are
\[
E' \approx E + \frac{v}{c} \times B, \quad B' \approx B_\perp - \frac{v}{c} \times E,
\] (136)
\[
E \approx E' - \frac{v}{c} \times B', \quad B \approx B_\perp' + \frac{v}{c} \times E', \quad (137)
\]
as previously found in secs. B.1-2.

\textsuperscript{51}Einstein’s derivation was based on the assumptions that Maxwell’s equations have the same form in both the lab frame and a moving (inertial) frame. In particular, he used Faraday’s and Ampère’s Laws in empty space,
\[
\nabla \times E = -\partial B/\partial ct, \quad \nabla \times B = \partial E/\partial ct, \quad \text{and} \quad \nabla' \times E' = -\partial B'/\partial ct', \quad \nabla' \times B' = \partial E'/\partial ct'.
\] (133)

\textsuperscript{52}The transformation of the fields \( D \) and \( H \) was given by Lorentz (1904) in eq. (6), p. 812, of [62], where \( k = \gamma \) (Lorentz’ eq. (3)) and \( l = 1 \) (p. 824).
B.4 Comments

This Appendix shows that arguments using convective (time) derivatives can, with considerable effort, lead to the full, relativistic transformations of fields from the lab frame to a moving frame, although most straightforward use of the convective derivative only yields the low-velocity transformation.

B.4.1 Use of Potentials to Compute the Fields

As also remarked in footnote 47 above, Maxwell made an argument based on potentials, rather than fields, in Art. 598 of [6], and while he arrived at the correct low-velocity approximation to the electric field in a moving frame via discussion of the convective derivative of the vector potential, there was an ambiguity as to whether his symbol Ψ was the electrical scalar potential, as he claimed it to be. It is felicitous that there is no ambiguity if one considers the fields rather than the potentials (sec. 5.1 above), as perhaps first done by Abraham, sec. 86, p. 398 of [26] (1904).

We elaborate on this topic by deducing how the transform of potentials for low-velocity, and the resulting transform of the fields if they are then deduced from the transformed potentials.

In any (inertial) frame the fields \( \mathbf{E} \) and \( \mathbf{B} \) are related the scalar potential \( \Psi \) and the vector potential \( \mathbf{A} \) (in some gauge) by

\[
\mathbf{E} = -\nabla \Psi - \frac{\partial \mathbf{A}}{\partial ct}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A},
\]

provided the derivatives are taken with respect to the coordinates of that frame.

The electromagnetic potentials comprise a 4-vector \( (V, A) \), so the Lorentz transformations of the potentials from the lab frame to the primed frame that has velocity \( \mathbf{v} \) with respect to the lab frame are

\[
\Psi' = \gamma \left( \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) \approx \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A},
\]

\[
\mathbf{A}' = \mathbf{A} + (\gamma - 1)(\mathbf{A} \cdot \hat{v}) \hat{v} - \gamma \frac{\mathbf{v}}{c} \Psi \approx \mathbf{A} - \frac{\mathbf{v}}{c} \Psi,
\]

where the approximations hold for low velocity.

For the derivatives, we note that \( (\partial/\partial ct, -\nabla) \) is a 4-vector,\(^{53}\) so its transform is

\[
\frac{\partial}{\partial ct'} = \gamma \left( \frac{\partial}{\partial ct} - \frac{\mathbf{v}}{c} \cdot (-\nabla) \right) \approx \frac{\partial}{\partial ct} + \frac{\mathbf{v}}{c} \cdot \nabla, \quad \frac{\partial}{\partial t'} \approx \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,
\]

\[
-\nabla' = -\nabla + (\gamma - 1)(-\nabla \cdot \hat{v}) \hat{v} - \gamma \frac{\mathbf{v}}{c} \frac{\partial}{\partial ct} \approx -\nabla,
\]

where we neglect terms of order \( 1/c^2 \). Note that the low-velocity approximation to the time derivative in the moving frame is the convective derivative in terms of lab-frame quantities,

\(^{53}\)Strictly, \((\Psi, \mathbf{A})\) is a covariant 4-vector and \((\partial/\partial ct, -\nabla)\) is a contravariant 4-vector. These distinctions are unimportant in special relativity for inertial frames, but are significant when considering accelerated frames. See, for example, [31].
The fields in the moving frame can now be computed as

\[ \mathbf{E}' = -\nabla' \Psi' - \frac{\partial \mathbf{A}'}{\partial ct'} \approx -\nabla \left( \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) - \left( \frac{\partial}{\partial ct} + \frac{\mathbf{v}}{c} \cdot \nabla \right) \left( \mathbf{A} - \frac{\mathbf{v}}{c} \mathbf{V} \right) \]

\[ \approx -\Psi - \frac{\partial \mathbf{A}}{\partial ct} + \nabla \left( \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) - \left( \frac{\mathbf{v}}{c} \cdot \nabla \right) \mathbf{A} \]

\[ = \mathbf{E} + \left( \frac{\mathbf{v}}{c} \cdot \nabla \right) \mathbf{A} + \frac{\mathbf{v}}{c} \times (\nabla \times \mathbf{A}) - \left( \frac{\mathbf{v}}{c} \cdot \nabla \right) \mathbf{A} = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad (143) \]

\[ \mathbf{B}' = \nabla' \times \mathbf{A}' \approx \nabla \times \left( \mathbf{A} - \frac{\mathbf{v}}{c} \mathbf{V} \right) = \mathbf{B} + \nabla \times \left( \frac{\mathbf{v}}{c} (-\mathbf{V}) \right) = \mathbf{B} - \frac{\mathbf{v}}{c} \times (-\nabla \mathbf{V}) \]

\[ = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} - \frac{\mathbf{v}}{c} \times \frac{\partial \mathbf{A}}{\partial ct} \approx \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}. \quad (144) \]

Thus, using the potentials and the various low-velocity Lorentz transformations, we recover the low-velocity forms (136)-(137) for the electromagnetic fields. \(^{54}\)

### B.4.2 Lorentz Force

Another felicitous result is that the low-velocity transform of the electric field from lab frame to a moving frame has the same form as the lab-frame Lorentz force (per unit charge), \( \mathbf{F}/q = \mathbf{E} + \mathbf{v}/c \times \mathbf{B} \), for arbitrary velocity. Of course, neither Maxwell (1873) nor Lorentz (1892) were aware of this happy result of the Lorentz transformation of 4-force (Minkowski force).

### C Appendix: J.J. Thomson (1880)

While Arts. 599 and 769-770 of Maxwell’s *Treatise* are consistent with the low-velocity limit of special relativity, this was not evident at the time, when electromagnetism was generally interpreted in an æther theory. For example, in his first research paper, J.J. Thomson \(^{[38]}\) used Arts. 598-599 of Maxwell’s *Treatise* to reach a “peculiar” conclusion as to the speed of light in a dielectric medium that has velocity \( \mathbf{v} \) with respect to the frame of the æther.

In the present section, quantities in the ether frame will be unprimed, while a quantity in the frame of the moving dielectric will be denoted with a ‘. \(^{55}\)

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\(^{54}\)Hence, it seems to this author that Maxwell’s correct expression in sec. (65) of \([1]\) and Art. 598 of \([6]\), for the low-velocity field experienced by a moving circuit could well have been deduced by a valid argument, despite the doubts cast on this by Helmholtz, and Thomson.

Thomson felt that his objection was validated by the example of a rotating, conducting sphere is in uniform external magnetic field, in his note on p. 260 of \([6]\). However, this example involves an accelerated frame, for which the magnetic field is considered to be the same by a rotating observer and one at rest (see, for example, \([31]\)).

If we had supposed that the magnetic field, and the vector potential were that same in the lab frame and in the moving frame, then eq. (143) above would read \( \mathbf{E}' \approx \mathbf{E} + \mathbf{v}/c \times \mathbf{B} + (\mathbf{v} \cdot \nabla) \mathbf{A}/c. \) So, while the assumption that \( \mathbf{B}' = \mathbf{B} \) permits one to deduce the correct, low-velocity electric-field transformation via consideration of Faraday’s Law for moving circuit (as in sec. B.1 above), use of this assumption in an argument based on potentials does lead to an erroneous result.

\(^{55}\)The dielectric medium could be vacuum.
Thomson begins with Maxwell’s eq. (7) and notes that $\mathbf{B} = \nabla \times \mathbf{A}$ where the vector potential obeys $\nabla \cdot \mathbf{A} = 0$.

Thomson’s eqs. (1)-(3) correspond to relating the electric displacement field in the moving frame by

$$\mathbf{D}' = \varepsilon_0 (\mathbf{E}' + \mathbf{P}')$$

where $\mathbf{P}'$ is the electric polarization field in the moving frame. His eq. (4) is more properly then

$$\nabla' \cdot \mathbf{D}' = 0.$$  

Thomson’s goal is to deduce a wave equation for $\mathbf{D}'$, and to infer from this the speed of light in the moving frame. To this end, he takes the curl of eq. (145), assuming that $\nabla' \times \mathbf{P}' = 0$ and that the dielectric medium is in uniform motion such that derivatives of the velocity $\mathbf{v}$ are zero,

$$\nabla' \times \mathbf{D}' = \varepsilon_0 \nabla' \times \mathbf{E}' = \nabla' \times \mathbf{E} + \mathbf{v}(\nabla' \cdot \mathbf{B}) - \left( \frac{\mathbf{v}}{c} \cdot \nabla' \right) \mathbf{B}\text{.}$$  

Thomson then supposes that the effect of taking derivatives with respect to spacetime coordinates $x, y, z$ and $t$ is the same in the ether frame and in the moving frame. That is, he assumes that Galilean relativity relates these coordinates in the two frames. Note that this assumption was not needed in the interpretation of Maxwell’s Arts. 599 and 769-770 (although Maxwell did use this assumption in his Arts. 600-601).

With this tacit assumption of Galilean relativity for $(x, y, z, t)$, eq. (147) can be written

$$\nabla \times \mathbf{D}' = \nabla \times \mathbf{E} + \mathbf{v}(\nabla \cdot \mathbf{B}) - \left( \frac{\mathbf{v}}{c} \cdot \nabla \right) \mathbf{B} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} - \left( \frac{\mathbf{v}}{c} \cdot \nabla \right) \mathbf{B},$$

in that $\nabla \cdot \mathbf{B} = 0$ (while in special relativity, $\nabla' \cdot \mathbf{B} \neq 0$).

It was not appreciated in 1880, and perhaps not until the work of LeBellac and Levy-Leblond in 1973 [10], that Maxwell’s equations are not consistent with Galilean relativity. When Maxwell’s eq. (7) applies, the appropriate modifications to Maxwell’s equations to be compatible with Galilean relativity are those of so-called magnetic Galilean relativity (sec. 2.3 of [10]),

$$\nabla \cdot \mathbf{D}_m = 4\pi \rho_m, \quad \nabla \cdot \mathbf{B}_m = 0, \quad \nabla \times \mathbf{E}_m = -\frac{1}{c} \frac{\partial \mathbf{B}_m}{\partial t}, \quad \nabla \times \mathbf{H}_m = \frac{4\pi}{c} \mathbf{J}_m \text{ (149)}$$

where $\rho$ and $\mathbf{J}$ and the volume densities of “free” charge and currents, and there is no “displacement current” and no electromagnetic waves. While the velocity $c$ has a value equal to the speed of light in vacuum it is to be deduced from static experiments and is not related to (nonexistent) wave propagation in Galilean relativity.

Thomson considered a nonmagnetic dielectric medium in which $\mathbf{B} = \mathbf{H}$, with no free charge or current densities, and supposed that in this case

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t},$$

(150)
whereas in a consistent Galilean view of this case, $\nabla \times \mathbf{B} = 0$. He then took the curl of eq. (148), writing (his eq. (10))

$$\nabla^2 \mathbf{D}' = \frac{1}{c^2} \frac{\partial \mathbf{D}}{\partial t^2} - \left( \frac{\mathbf{v}}{c^2} \cdot \nabla \right) \frac{\partial \mathbf{D}}{\partial t},$$  \hspace{1cm} (151)

In the magnetic Galilean relativity of [10] this would be just $\nabla^2 \mathbf{D}' = 0$, corresponding to instantaneous propagation of electromagnetic effects.

At this point Thomson seems to have assumed that $\mathbf{D}' = \mathbf{D}$ even though his argument began with Maxwell’s relation (7) in which $\mathbf{E}' \neq \mathbf{E}$. Assuming a wavefunction $\mathbf{D}' = \mathbf{D} = D_0 e^{i(kx - \omega t)}$ and $\mathbf{v} = v \hat{\mathbf{x}}$, the wave equation (151) leads to the dispersion relation

$$\omega^2 - vk\omega - k^2 c^2 = 0, \quad \omega' = \frac{\omega}{k} = \frac{1}{2} \left( v \pm 2c \sqrt{1 + \frac{v^2}{8c^2}} \right).$$ \hspace{1cm} (152)

Only the positive root could make sense, leading to\(^{56}\)

$$v' \approx c + \frac{v}{2} \quad (v \ll c),$$ \hspace{1cm} (153)

which was interpreted as the speed of the waves in the moving dielectric (even if that dielectric were vacuum).

While this result makes no sense from a “modern” perspective, it illustrates how naïve assumptions about relativity and Maxwell’s equations can lead to “peculiar” conclusions.

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\(^{56}\)For sound waves we expect $v' = c - v$. That the sign in eq. (153) is “wrong” seems to have gone unnoticed.

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