Wave Amplification in a Magnetic Medium

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1 Problem

One way to prepare an optically active medium is to turn on a strong DC magnetic field at right angles to a static magnetic field that has initially aligned the dipoles of a magnetic medium. Then the dipoles will precess about the direction of the strong magnetic field, before eventually relaxing into alignment with that field. During those intervals while the dipoles \( \mathbf{m} \) are antialigned with the initial static field, they are in a state of high energy \( U = -\mathbf{m} \cdot \mathbf{B} \). When in this state, the medium can give up energy to a probe electromagnetic wave (with magnetic field along the direction of the strong DC field), thereby amplifying it.

Deduce the equations of motion for the magnetization \( \mathbf{M} = N\mathbf{m} \) of a medium that consists of \( N \) permanent dipoles \( \mathbf{m} \) (with angular momentum \( \mathbf{L} = \Gamma \mathbf{m} \)) per unit volume when the medium is immersed in a magnetic field \( \mathbf{B} \). Consider the specific example of a static magnetic field \( \mathbf{B}_0 = B_{0x} \hat{x} + B_{0y} \hat{y} \) where \( B_{0x} \ll B_{0y} \), and an oscillatory field \( B_y e^{-i\omega t} \hat{y} \). You may suppose that \( M \ll B_x \) and \( M \ll B_y \).

A measure of the ability of the medium to amplify a probe wave is the frequency-dependent index of refraction \( n(\omega) = \sqrt{\mu} \), where \( \mu \) is the magnetic susceptibility related by \( \mathbf{B} = \mu \mathbf{H} = \mathbf{H} + 4\pi \mathbf{M} \) (in Gaussian units, and in a medium of dielectric constant \( \epsilon = 1 \)). In the present example, the wave field has magnetic field along the \( y \) axis, so that you can write

\[
B_y(\omega) = \mu H_y(\omega) = H_y \left( 1 + 4\pi \frac{M_y}{H_y} \right),
\]

Since we assume that \( M \ll B_y \), we also have \( M_y \ll H_y \), and the index of refraction is given by

\[
n(\omega) = \sqrt{\mu} \approx 1 + 2\pi \frac{M_y}{H_y}.
\]

If the medium is to exchange energy with a wave, there must be additional processes occurring. For index of refraction to include absorption (or amplification), it suffices to suppose that there is a kind of damping mechanism that aligns the magnetic dipoles with the static magnetic field. A phenomenological form for this is

\[
\frac{d\mathbf{m}}{dt} = \gamma (\mathbf{m} \times \mathbf{B}) \times \mathbf{m} \approx -\gamma m(\hat{m} - \hat{y}),
\]

where \( \gamma \) is the damping factor, and the approximation notes that the static field is largely along the \( y \) axis. Include this damping in the equations of motion, solve for the oscillatory behavior of \( M_y \propto e^{-i\omega t} \) assuming the damping is slow so that \( \gamma \ll \Gamma B_x \), and then calculate the index \( n(\omega) \). Show that when \( M_x \) has precessed to be opposite to \( B_{0x} \), the index of refraction implies amplification of a traveling wave of \( H_y \) (and \( M_y \)).
2 Solution

The merits of an oscillatory magnetic field transverse to a static magnet field in the study of individual magnetic moments were emphasized by Rabi [1]. Bloch [2] extended this approach to magnetic media, but it was perhaps Dicke [3] who realized that the optically active medium thereby created could lead to “super-radiance”, i.e., to laser beams.

When a magnetic dipole \( \mathbf{m} \) is subject to a magnetic field \( \mathbf{B} \) it experiences a torque \( \mathbf{m} \times \mathbf{B} \) that precesses the angular momentum \( \mathbf{L} = \mathbf{m}/\Gamma \), where \( \Gamma = m/L \) is the gyromagnetic ratio of the dipole. If the magnetic dipoles are electrons, then \( \Gamma = e/2m_e c \approx 10^7 \) Hz/ gauss, where \( e \) and \( m_e \) are the charge and mass of the electron, and \( c \) is the speed of light. Thus,

\[
\mathbf{m} \times \mathbf{B} = \frac{d\mathbf{L}}{dt} = \frac{1}{\Gamma} \frac{d\mathbf{m}}{dt}.
\]

The precession frequency is \( \Gamma B \approx 10^7 B \) for \( B \) in gauss. We will consider magnetic fields \( B_y(t) \) of optical frequencies, \( \approx 10^{15} \) Hz, so the precession will be very slow compared to the wave frequency.

The equation of motion of a single moment, including the damping (3) of the moment to alignment with the static magnetic field that is predominantly along the \( y \) axis, is

\[
\frac{d\mathbf{m}}{dt} = \Gamma \mathbf{m} \times \mathbf{B} - \gamma \mathbf{m}(\hat{\mathbf{m}} - \hat{\mathbf{y}}).
\]

The equation of motion for the magnetization \( \mathbf{M} = N\mathbf{m} \) is therefore

\[
\frac{d\mathbf{M}}{dt} + \gamma \mathbf{M} = \Gamma \mathbf{M} \times \mathbf{B} + \gamma \mathbf{M} \hat{\mathbf{y}}.
\]

For a magnetic field \( B_{0x} \hat{\mathbf{x}} + (B_{0y} + B_y(t)) \hat{\mathbf{y}} = (H_x + 4\pi M_x) \hat{\mathbf{x}} + (H_y + 4\pi M_y) \hat{\mathbf{y}} \), the components of eq. (6) are

\[
\frac{dM_x}{dt} + \gamma M_x = -\Gamma M_z B_y, \quad (7)
\]
\[
\frac{dM_y}{dt} + \gamma M_y = \Gamma M_z B_x + \gamma M, \quad (8)
\]
\[
\frac{dM_z}{dt} + \gamma M_z = \Gamma (M_x B_y - M_y B_x). \quad (9)
\]

The desired physical picture is that the magnetization \( \mathbf{M} \) precesses around the \( y \) axis (subject to the “slow” damping \( \gamma \)), with the oscillatory magnetization \( M_y \) being only a small perturbation about this dominant motion. From eqs. (7) and (9) we see that this is a good approximation so long as \( M_y B_x \ll M_x B_y \). We choose \( B_{0x} \) to be small compared to \( B_{0y} \), and prepare the medium in an initial state with \( M_y \ll M_x \). The latter might be accomplished, for example, by starting with \( B_{0y} = 0 \) so the dipoles line up with \( B_{0x} \), and then turning on the field \( B_y \) quickly; if the damping time is long compared to the precession period, then there is a useful interval during which the desired behavior obtains.

We are principally interested in the behavior of \( M_y \) for use in calculating the index of refraction, so we take the derivative of eq. (8), noting that \( M \) is constant since the medium
is comprised of permanent dipoles, and insert eq. (9) to find
\[
\frac{d^2 M_y}{dt^2} + \gamma \frac{dM_y}{dt} = \Gamma \frac{dM_z}{dt} B_x = \Gamma B_x [\Gamma (M_x B_y - M_y B_x) - \gamma M_z]
\]
\[
= \Gamma^2 B_x (M_x H_y - M_y H_x) - \gamma \left( \frac{dM_y}{dt} + \gamma M_y - \gamma M \right).
\]

Assuming that \(M \ll B_x\), then \(H_x \approx B_x\) and we may approximate \(\Gamma^2 B_x H_x \equiv \omega_0^2\) as being constant (\(\omega_0 \approx \Gamma B_x\)). Then,
\[
\frac{d^2 M_y}{dt^2} + 2\gamma \frac{dM_y}{dt} + (\gamma^2 + \omega_0^2) M_y = \Gamma^2 B_x H_x \frac{M_x}{H_x} H_y + \gamma^2 M = \omega_0^2 \frac{M_x}{H_x} H_y + \gamma^2 M.
\]

The term \(\gamma^2 M\) leads to a constant component \(M_y = \gamma^2 M / (\gamma^2 + \omega_0^2)\), which we can ignore since we assume that the damping constant \(\gamma\) is small compared to the frequency \(\omega_0 \approx \Gamma B_x\).

Our main interest is the behavior of the system when a wave is present, \(H_y = H_{0y} e^{-i\omega t}\) and \(M_y = M_{0y} e^{-i\omega t}\), at frequency \(\omega \gg \omega_0\), in which case we can regard \(M_x\) as effectively constant over a few cycles of the high frequency wave. Inserting this hypothesis in eq. (11), we find that the high-frequency part of \(M_y\) obeys
\[
M_y = \frac{M_x}{H_x} \frac{\omega_0^2 H_y}{\omega_0^2 - \omega^2 + \gamma^2 - 2i\gamma \omega}.
\]

Recall that we need \(M_y H_x \ll M_x H_y\) for the dominant behavior of the magnetization to be precession about the \(y\) axis. From eq. (12) we see that this would not hold for frequency \(\omega\) close to \(\omega_0\) (since we assume that \(\gamma \ll \omega_0\)). But we consider \(\omega\) of optical frequencies, so \(\omega \gg \omega_0\) for any reasonable value of \(B_x\), as noted previously.

The index of refraction for a wave propagating in the \(z\) direction with magnetic field along the \(y\) axis is therefore
\[
n(\omega) = \sqrt{\mu} \approx 1 + 2\pi \frac{M_y}{H_y} = 1 + 2\pi \frac{M_x}{H_x} \frac{\omega_0^2 (\omega_0^2 - \omega^2 + \gamma^2 + 2i\gamma \omega)}{(\omega_0^2 - \omega^2 + \gamma^2)^2 + 4\gamma^2 \omega^2}.
\]

In particular, during the part of the precession cycle when the magnetization \(M_x\) is antialigned with \(B_x \approx H_x\), \(Im(n) < 0\), and a propagating wave \(H_{0y} e^{i\omega(nz/c-t)}\) is amplified during its passage through the medium.

It appears difficult to realize the desired precession of \(M\) about the \(y\) axis as suggested, since \(B_y\) would have to reach full strength in less than the damping time \(1/\gamma\), and no actual laser has (I believe) been built utilizing a magnetic medium. The interest of this problem is in providing a classical viewpoint of how wave amplification is possible in principle by preparing a medium in an optically active state.

References
