Radiation in the Near Zone of a Center-Fed Linear Antenna

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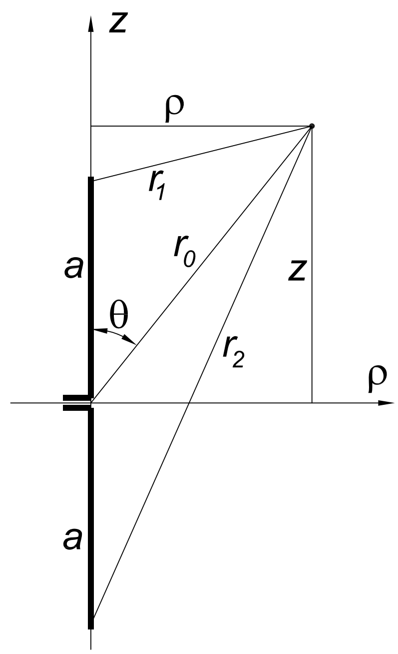
(June 21, 2004)

1 Problem

The electromagnetic fields far from any antenna can be conveniently described as the sum of the radiation fields of a series of oscillating point multipoles, of which the leading term is a dipole in many cases of practical interest. The form of the fields associated with the nth multipole is independent of the details of the physical layout of the antenna (other than that the layout determines the magnitudes of the multipole moments). However, close the to antenna the electromagnetic fields include quasistatic components as well as radiation terms. A well-known argument due to Hertz [1, 2] gives the fields in the near and far zone of an ideal point dipole. In this and two companion notes [3, 4] we explore examples in which analytic expression can be given for the near and far zone fields of antennas of finite dimensions.

Here, the task is to describe the electromagnetic fields, and the Poynting vector [5], produced by oscillating currents of angular frequency ω that flow along a pair of conductors of length a each, that are fed by, say, a coaxial cable at their common ends, as shown in the figure below.



The goal is to locate where radiated power originates on the antenna.

2 Solution

This problem has a long, and to me, somewhat unsatisfactory history.

From a quantum perspective, a solution may be elusive. The uncertainty principle tells us that photons radiated into a broad angular pattern cannot be localized to much less than a wavelength.[1] Nonetheless, it is compelling to seek a classical answer to the question of where radiated power is radiated from.

A brief review that discusses some of the difficulties in classical approaches to this problem is given in [6]; a more extensive review is [7].

While Hertz's original discussion of point dipole radiators [1] gave expressions for the electromagnetic fields at all distances from the point source, calculations of radiation from finite-size antennas before 1920 only gave expressions for the fields in the radiation zone, *i.e.*, more than several wavelengths distant from the source. This type of calculation is based on a plausible model of the distribution of current along the antenna [8], as illustrated in sec. 9.4A of [9]. The far-field radiation pattern is based on calculation of the Poynting vector. The spirit of this approach is that if the Poynting vector could also be calculated in the near zone, its (time-average) strength at the surface of the antenna conductors could be taken as describing the distribution of radiation from the surface of the antenna.

For an antenna whose conductors lie along the z-axis, it is natural to use a cylindrical coordinate system (ρ, ϕ, z) when close to the antenna. Then, the flow of energy away from the antenna should be described by the ρ component of the Poynting vector,

$$S_\rho = \frac{c}{4\pi} E_z B_\phi, \tag{1}$$

in Gaussian units. We are led to ascribe the power $dP(z)$ of the radiation emitted from a segment of length dz of the antenna as

$$dP = 2\pi\rho \; dz \; S_\rho = \frac{c}{4\pi} dz \; E_z(2\pi\rho B_\phi). \tag{2}$$

According to Ampère's law, the azimuthal magnetic field, B_ϕ close to the antenna is related to the current $I(z,t)$ along the antenna by

$$2\pi\rho B_\phi = \frac{4\pi}{c} I. \tag{3}$$

Thus we can calculate the power emitted at position z along the linear antenna as

$$\frac{dP(z,t)}{dz} = E_z(\rho = 0, z, t) I(z,t). \tag{4}$$

This prescription was apparently first pointed out by Brillouin in 1922 [10], but it is hardly surprising if we recall that Poynting's derivation [5] began by noting that the rate per unit volume at which the sources lose power to the fields $-\mathbf{E} \cdot \mathbf{J}$, where $\mathbf{J}$ is the current density.

[1] In the Heisenberg uncertainty relation, $\Delta z \Delta P_z \gtrsim \hbar$, we have $\Delta P_z \approx P = \hbar k$ for a radiation pattern with "vertical" angular spread ≈ 1 radian, as is the case for dipole antennas. Then, $\Delta z \gtrsim 1/k = \lambda/2\pi$.

Hence, if we can calculate the field E_z at the surface of the conductors of the antenna when these carry current I, we can then say that the radiation is emitted from the antenna according to eq. (1) or (4).

An immediate conceptual problem is that if for simplicity of calculation, we approximate the conductors as perfect conductors, then the tangential component of the electric field must vanish. For the geometry described above, this would imply that $E_z = 0$, and hence that the radiation (as described by the Poynting vector) cannot come from the conductors of the antenna itself!

Of course, the antenna is not a power source, but is fed from an appropriate generator by a transmission line, perhaps a coaxial cable. The logic of the preceding paragraphs seems to be impelling us to the vision that the radiation from a good or perfectly conducting antenna is emitted at the feed point to the antenna.[2] This would be little different from Hertz's analysis of an idealized point dipole radiator.

As Sommerfeld reminds us (p. 130 of [12]), "conductors are nonconductors of energy". Rather than thinking of the conductors of an antenna as sources of radiation, it may be better to consider them as a kind of inverse wave guide, which "tells" the radiation where not to go. In contrast to the role of conductors in transmission lines, where waves propagate parallel to (and largely outside of) their surfaces, the waves from an antenna tend to propagate away from the conductors preferentially at right angles to the conductors.

As implied in the solutions given in sec. 2.6, the directionality of a center-fed linear antenna is improved by lengthening the antenna to beyond a wavelength. In this case, the current distribution along the antenna includes (standing wave) nodes. The antenna could be constructed of segments with gaps at the positions of the current nodes with little change in performance. The outer segments of such an antenna would not be directly powered by the feed line to the center point. This reinforces the view that the conductors of the antenna serve to redirect the waves "created" at the central feed point, rather than to create the wave themselves.

It may be useful to compare radiation by an antenna with the scattering of a plane wave by a conducting sphere [13], where it is more straightforward to make analytic calculations in which the total electromagnetic fields satisfy perfect-conductor boundary conditions. Then lines of the Poynting vector do not originate or terminate on the conductor but flow past it. However, if one examines the Poynting vector due to the incident wave, the scattered wave, and the cross terms of these two waves, then lines of the various Poynting vectors do intercept the conductor, and may be interpreted as describing absorption and emission of energy by/from the surface of the conductor.

Similarly, an antenna could be regarded as a scatterer of the TEM wave that propagates towards the feedpoint in a coaxial feed cable. The total electromagnetic fields satisfy (to a good approximation) perfect-conductor boundary conditions, and the total Poynting vector never intercepts the conductors of the antenna. However, if we can separate the total fields into a part that is emitted from the end of the feed cable, and another part that is scattered (*i.e.*, emitted) by the conductors of the antenna, then the latter part can be ascribed to a current distribution in those conductors. If we can somehow "guess" the form of those

[2]This view appears generally unpopular with antenna engineers, and is little discussed in the literature. An exception is [11].

current distributions, we can proceed without the benefit of a full solution.

It was first pointed out by Kliatzkin in 1927 [14] (in Russian; for an English version, see [15]), that the near fields due to simple current distributions in linear antennas can be deduced analytically via the retarded potentials. The solution given below follows this argument. See also [16, 17], sec. 8.11 of [18], sec. 9.25 of [19], sec. 5.2 of [6], chap. 5 of [20] (which has a very extensive bibliography), and sec. 12.03 of [23].

A possibly astonishing feature of the solution is that although the derivation appears to begin with the premise of perfect conductors, nonetheless a nonzero tangential electric field is eventually deduced. This behavior is less unexpected if we realize that we are actually calculating only part of the total fields, namely that part which is "scattered" by the antenna. Then, eqs. (1) or (4) can be interpeted as providing a prescription as to where the radiation is emitted along the antenna. This prescription is not complete, since it does not explain how the power is transmitted from the feed point at the center of the antenna to the distributed source of the radiation along the antenna.

The ambiguities in the classic treatments of linear antennas provided a motivation for Schelkunoff [19, 24, 25, 26] to consider biconical antennas [4], in the analysis of which the tangential component of the electric field can be held to zero at the surface of the conductors, thereby perhaps avoiding many of the shortcomings of the following discussion.

Another important advance in antenna methodology was the integral-equation approach of Hallén [27] which successfully implements the perfect conductor boundary condition in a manner that is well suited for numerical computation (although Hallén's original work was analytic). This approach forms the basis of contemporary NEC (Numerical Electromagnetic Codes), an example of which is given in sec. 2.8.

2.1 Voltage and Current Distribution Along a Thin, Straight, Perfectly Conducting Wire

It is much easier to calculate the potentials and fields if the current distribution is known. In a typical engineering problem involving antennas only the drive voltage $V_0(t)$ is specified, and the resulting current distribution must be calculated along with the fields. However, in the approximation of a thin, perfectly conducting wire, the form of the current distribution can be deduced from general considerations, first given by Pocklington in 1897 [8].

Throughout this problem we consider the currents and voltages to have angular frequency ω, and we write their time dependence at $e^{-i\omega t}$. We work in the Lorentz gauge, and in Gaussian units, so the scalar potential V and the vector potential $\mathbf{A}$ are related by

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t} = ikV, \tag{5}$$

where c is the speed of light and $k = \omega/c$ is the wave number.

We take the portion of the wire of interest to lie along the z axis. The vector potential at the wire is dominated by the contribution from the nearby currents that flow along the z axis. Thus, the only significant component of the vector potential the wire is its z component, and eq. (5) becomes

$$\frac{\partial A_z}{\partial z} = ikV, \tag{6}$$

Although real wires have finite resistance, this resistance is typically small compared to the radiation resistance (to be found below). Hence, it is a good approximation to consider the wires to be perfect conductors in antenna problems. In this approximation, the z component of the electric field vanishes at the wire, and we have

$$E_z(\text{along the wire}) = 0 = -\frac{\partial V}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{\partial V}{\partial z} + ikA_z. \tag{7}$$

Thus, on the wire

$$\frac{\partial V}{\partial z} = ikA_z. \tag{8}$$

Combining eqs. (6) and (8) we obtain

$$\frac{\partial^2 V}{\partial z^2} = -k^2 V, \qquad \frac{\partial^2 A_z}{\partial z^2} = -k^2 A_z. \tag{9}$$

Hence, both the voltage and the vector potential vary sinusoidally (with kz) on a perfectly conducting wire (along the z direction). We have not yet assumed the wire to be thin.

In the case of a thin wire, the vector potential at a point on the wire is very large, and essentially due to the current $I(z)$ at that point. So, for a thin wire we can also conclude that the current distribution is a sinusoidal function (of kz).

This last conclusion need not hold, for example, at the feed point of the antenna, where the wires can make a sharp 90° bend, but it should be a good approximation over the bulk of the wire.

In the present problem of a center-fed linear antenna of length $2a$, the current must vanish at the ends of the antenna, $I(\pm a) = 0$ (whether or not the wire is a perfect conductor), which implies that the current forms a standing wave along the wire. The current must also be symmetric about $z = 0$ (rather than antisymmetric, since the opposing currents in the two wires of the feed line become in-phase currents in the two arms of the antenna). The requirement of sinusoidal dependence on kz then leads us to postulate the form[3]

$$I(z,t) = I_0 \frac{\sin[k(a - |z|)] \cos \omega t}{\sin ka}. \tag{10}$$

The relation between the voltage difference V_0 across the feed point of the antenna and the peak current I_0 at the feed point is not specified by the preceding argument.

The form (10) was deduced under the assumption of a perfectly conducting wire. However, the current distribution in a wire with finite conductivity is not expected to be very different, given the simplicity of eq. (10).

2.2 The Retarded Potential

The electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, outside the wire can both be deduced from the vector potential $\mathbf{A}$, according to

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{11}$$

[3]This form is not, however, unique. For example, the merits of the form $I \propto (\cos kz - \cos ka) \cos \omega t$, which is sinusoidal in kz, is symmetric about $z = 0$ and vanishes at $z = \pm a$, are explored in [20].

and the fourth Maxwell equation in free space,

$$\left[\frac{i}{kc}\frac{\partial \mathbf{E}}{\partial t} =\right] \mathbf{E} = \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B} \ \left[= \frac{i}{k}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \frac{i}{k}\boldsymbol{\nabla}^2\mathbf{A} = -\boldsymbol{\nabla}V + ik\mathbf{A} = -\boldsymbol{\nabla}V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right], \tag{12}$$

noting that the vector potential satisfies the wave equation

$$\boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \boldsymbol{\nabla}^2\mathbf{A} + k^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}, \tag{13}$$

where $\mathbf{J}$ is the current density in the wire.

The well-known solution to the wave equation (13) is the retarded potential, which in the present problem can be written

$$\begin{aligned} A_z(\mathbf{r},t) &= \frac{1}{c}\int_{-a}^{a} dz' \ \frac{I(z',t'=t-r/c)}{r} = \frac{I_0 e^{-i\omega t}}{c\sin ka}\int_{-a}^{a} dz' \ \sin\sin[k(a-|z'|)]\frac{e^{ikr}}{r} \\ &= \frac{I_0 e^{-i\omega t}}{2ic\sin ka}\left[\int_0^a dz'\left(\frac{e^{ik(a-z)}e^{ik(r+z-z')}}{r} - \frac{e^{-ik(a-z)}e^{ik(r-z+z')}}{r}\right)\right. \\ &\qquad \left. - \int_0^{-a} dz'\left(\frac{e^{ik(a+z)}e^{ik(r-z+z')}}{r} - \frac{e^{-ik(a+z)}e^{ik(r+z-z')}}{r}\right)\right]. \end{aligned} \tag{14}$$

It is now convenient to work in cylindrical coordinates (ρ, ϕ, z), so that for the observation point $\mathbf{r} = (\rho, 0, z)$,

$$r = \sqrt{\rho^2 + (z-z')^2}. \tag{15}$$

We make four changes of variables, corresponding to the four integrals in eq. (14):

$$\begin{array}{llll} s = k(r+z-z'), & t = k(r-z+z'), & u = k(r-z+z'), & v = k(r+z-z'), \\ \frac{dz'}{r} = -\frac{ds}{s}, & \frac{dz'}{r} = \frac{dt}{t}, & \frac{dz'}{r} = \frac{du}{u}, & \frac{dz'}{r} = -\frac{dv}{v}, \\ s_0 = k(r_0+z), & t_0 = k(r_0-z), & u_0 = k(r_0-z), & v_0 = k(r_0+z), \\ s_1 = k(r_1-a+z), & t_1 = k(r_1+a-z), & u_1 = k(r_2-a-z), & v_1 = k(r_2+a+z), \end{array} \tag{16}$$

where the distances

$$r_0 = \sqrt{\rho^2+z^2}, \qquad r_1 = \sqrt{\rho^2+(z-a)^2}, \qquad \text{and} \qquad r_2 = \sqrt{\rho^2+(z+a)^2} \tag{17}$$

are shown on the figure above. Then,

$$\begin{aligned} A_z(\mathbf{r},t) &= \frac{iI_0 e^{-i\omega t}}{2c\sin ka}\left[e^{ik(a-z)}\int_{s_0}^{s_1} ds\frac{e^{is}}{s} + e^{-ik(a-z)}\int_{t_0}^{t_1} dt\frac{e^{it}}{t}\right. \\ &\qquad \left. + e^{ik(a+z)}\int_{u_0}^{u_1} du\frac{e^{iu}}{u} + e^{-ik(a+z)}\int_{v_0}^{v_1} dv\frac{e^{iv}}{v}\right]. \end{aligned} \tag{18}$$

While the vector potential (18) is given in terms of exponential integrals, it turns out that the fields will involve only elementary functions (all hail Kliatzkin!).

2.3 The Electric and Magnetic Fields

The magnetic field, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, has only a ϕ component,

$$B_\phi = -\frac{\partial A_z}{\partial \rho} \,. \tag{19}$$

The dependence of A_z on ρ is entirely through the limits of integration. So, for example,

$$\frac{\partial}{\partial \rho}\int_{s_0}^{s_1} ds \frac{e^{is}}{s} = \frac{e^{is_1}}{s_1}\frac{\partial s_1}{\partial \rho} - \frac{e^{is_0}}{s_0}\frac{\partial s_0}{\partial \rho} = \frac{e^{is_1}}{s_1}\frac{k\rho}{r_1} - \frac{e^{is_0}}{s_0}\frac{k\rho}{r_0} \,. \tag{20}$$

Thus,

$$\begin{aligned} B_\phi(\mathbf{r},t) &= -\frac{i\rho I_0 e^{-i\omega t}}{2c\sin ka}\left[e^{ik(a-z)}\left(\frac{e^{ik(r_1-a+z)}}{r_1(r_1-a+z)} - \frac{e^{ik(r_0+z)}}{r_0(r_0+z)}\right)\right. \\ &\qquad +e^{-ik(a-z)}\left(\frac{e^{ik(r_1+a-z)}}{r_1(r_1+a-z)} - \frac{e^{ik(r_0-z)}}{r_0(r_0-z)}\right) \\ &\qquad +e^{ik(a+z)}\left(\frac{e^{ik(r_2-a-z)}}{r_2(r_2-a-z)} - \frac{e^{ik(r_0-z)}}{r_0(r_0-z)}\right) \\ &\qquad \left. +e^{-ik(a+z)}\left(\frac{e^{ik(r_2+a+z)}}{r_2(r_2+a+z)} - \frac{e^{ik(r_0+z)}}{r_0(r_0+z)}\right)\right] \\ &= -\frac{i\rho I_0 e^{-i\omega t}}{2c\sin ka}\left[\frac{e^{ikr_1}}{r_1(r_1-a+z)} - \frac{e^{ikr_0}(\cos ka + i\sin ka)}{r_0(r_0+z)}\right. \\ &\qquad +\frac{e^{ikr_1}}{r_1(r_1+a-z)} - \frac{e^{ikr_0}(\cos ka - i\sin ka)}{r_0(r_0-z)} \\ &\qquad +\frac{e^{ikr_2}}{r_2(r_2-a-z)} - \frac{e^{ikr_0}(\cos ka + i\sin ka)}{r_0(r_0-z)} \\ &\qquad \left. +\frac{e^{ikr_2}}{r_2(r_2+a+z)} - \frac{e^{ikr_0}(\cos ka - i\sin ka)}{r_0(r_0+z)}\right] . \end{aligned} \tag{21}$$

This expression can be simplified by noting, for example,

$$\frac{1}{r_1-a+z} + \frac{1}{r_1+a-z} = \frac{2r_1}{r_1^2-(z-a)^2} = \frac{2r_1}{\rho^2} \,. \tag{22}$$

Finally,

$$B_\phi(\mathbf{r},t) = -\frac{iI_0 e^{-i\omega t}}{c\rho\sin ka}\left[e^{ikr_1} + e^{ikr_2} - 2e^{ikr_0}\cos ka\right] . \tag{23}$$

Taking the real part, we have

$$B_\phi(\mathbf{r},t) = \frac{I_0}{c\rho\sin ka}\left[\sin(kr_1-\omega t) + \sin(kr_2-\omega t) - 2\cos ka\sin(kr_0-\omega t)\right] . \tag{24}$$

We obtain the electric field from the magnetic field (23) via eq. (12). Hence,

$$E_z(\mathbf{r},t) = \frac{i}{k\rho}\frac{\partial(\rho B_\phi)}{\partial \rho} = \frac{iI_0 e^{-i\omega t}}{c\sin ka}\left[\frac{e^{ikr_1}}{r_1} + \frac{e^{ikr_2}}{r_2} - 2\frac{e^{ikr_0}}{r_0}\cos ka\right] , \tag{25}$$

the real part of which is

$$E_z(\mathbf{r},t) = -\frac{I_0}{c\sin ka}\left[\frac{\sin(kr_1-\omega t)}{r_1} + \frac{\sin(kr_2-\omega t)}{r_2} - 2\cos ka\frac{\sin(kr_0-\omega t)}{r_0}\right]. \qquad (26)$$

Similarly,

$$E_\rho(\mathbf{r},t) = -\frac{i}{k}\frac{\partial(\rho B_\phi)}{\partial z} = -\frac{iI_0 e^{-i\omega t}}{c\rho\sin ka}\left[\frac{(z-a)e^{ikr_1}}{r_1} + \frac{(z+a)e^{ikr_2}}{r_2} - 2\frac{ze^{ikr_0}}{r_0}\cos ka\right], \qquad (27)$$

the real part of which is

$$E_\rho(\mathbf{r},t) = \frac{I_0}{c\rho\sin ka}\left[(z-a)\frac{\sin(kr_1-\omega t)}{r_1} + (z+a)\frac{\sin(kr_2-\omega t)}{r_2} - 2z\cos ka\frac{\sin(kr_0-\omega t)}{r_0}\right]. \qquad (28)$$

On the z-axis ($\rho = 0$) we have $r_0 = |z|$, $r_1 = |z-a|$ and $r_2 = |z+a|$, and eqs. (24), (26) and (28) reduce to

$$B_\phi(\rho\approx 0,\phi,|z|<a,t) = \frac{2I_0}{c\rho\sin ka}\sin[k(a-|z|)]\cos\omega t, \qquad (29)$$

$$B_\phi(0,\phi,|z|>a,t) = 0, \qquad (30)$$

$$E_\rho(\rho\approx 0,\phi,|z|<a,t) = \frac{|z|}{z}\frac{2I_0}{c\rho\sin ka}\cos[k(a-|z|)]\sin\omega t, \qquad (31)$$

$$E_\rho(0,\phi,|z|>a,t) = 0, \qquad (32)$$

$$\begin{aligned} E_z(0,\phi,|z|<a,t) &= \frac{2I_0}{c\sin ka(a^2-z^2)}\left\{a\left[ka\cos ka\frac{\sin kz}{kz} - \sin ka\cos kz\right]\cos\omega t \right. \qquad (33)\\ &\quad \left. + \left[z\sin ka\sin kz - \left(a+|z|-\frac{a^2}{|z|}\right)\cos ka\cos kz\right]\sin\omega t\right\}, \end{aligned}$$

$$\begin{aligned} E_z(0,\phi,|z|>a,t) &= \frac{2aI_0}{c\sin ka(z^2-a^2)}\left[\sin ka\cos(k|z|-\omega t)\right. \\ &\quad \left. -\frac{a}{|z|}\cos ka\sin(k|z|-\omega t)\right]. \qquad (34) \end{aligned}$$

The expressions (29)-(30) for B_ϕ at the wire obey Ampère's law, $2\pi\rho B_\phi = 4\pi I(z,t)/c$, for small ρ. The charge density $\sigma(z,t)$ on the wire can be deduced from the current distribution via the equation of continuity,

$$\frac{\partial\sigma}{\partial t} = -\frac{\partial I}{\partial z} = \frac{|z|}{z}kI_0\frac{\cos[k(a-|z|)]}{\sin ka}\cos\omega t. \qquad (35)$$

This integrates to give the charge distribution

$$\sigma(|z|<a,t) = \frac{|z|}{z}\frac{I_0}{c}\cos[k(a-|z|)]\sin\omega t. \qquad (36)$$

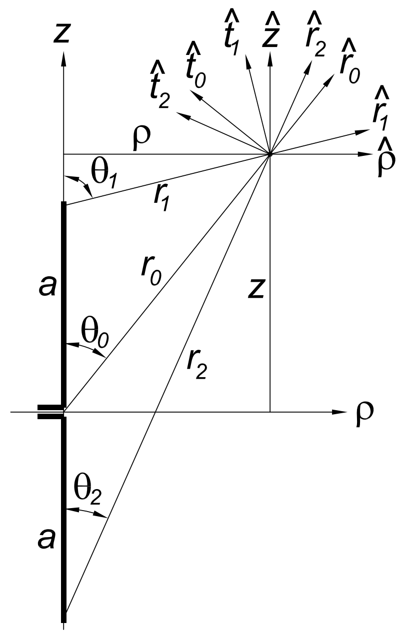
Then, Gauss' law applied to a cylinder of small radius ρ and axial extent dz tells us that the radial electric field near the wire should be

$$E_\rho = \frac{2\sigma}{\rho} = \frac{|z|}{z}\frac{2I_0}{c\rho}\frac{\cos[k(a-|z|)]}{\sin ka}\sin\omega t, \qquad (37)$$

in agreement with eq. (31).

We remain somewhat surprised by the nonzero value for eq. (33), which seems inconsistent with the initial assumption of perfect conductors for $|z| < a$ that led to the hypothesis (10) for the current distribution in the antenna.

An alternative form for the electric field can be given [22] by re-expressing the unit vector $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\rho}}$ in terms of unit vectors $\hat{\mathbf{r}}_0$, $\hat{\mathbf{t}}_0$, $\hat{\mathbf{r}}_1$, $\hat{\mathbf{t}}_1$, $\hat{\mathbf{r}}_2$ and $\hat{\mathbf{t}}_2$ as shown in the figure below.



We have,

$$\hat{\mathbf{z}} = \frac{z}{r_0}\hat{\mathbf{r}}_0 + \frac{\rho}{r_0}\hat{\mathbf{t}}_0 = \frac{z-a}{r_1}\hat{\mathbf{r}}_1 + \frac{\rho}{r_1}\hat{\mathbf{t}}_1 = \frac{z+a}{r_2}\hat{\mathbf{r}}_2 + \frac{\rho}{r_2}\hat{\mathbf{t}}_2, \tag{38}$$

$$\hat{\boldsymbol{\rho}} = \frac{\rho}{r_0}\hat{\mathbf{r}}_0 - \frac{z}{r_0}\hat{\mathbf{t}}_0 = \frac{\rho}{r_1}\hat{\mathbf{r}}_1 - \frac{z-a}{r_1}\hat{\mathbf{t}}_1 = \frac{\rho}{r_2}\hat{\mathbf{r}}_2 - \frac{z+a}{r_2}\hat{\mathbf{t}}_2. \tag{39}$$

Then for points not on the z-axis ($\rho \neq 0$) eqs. (25) and (27) take on the appealing forms

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \frac{iI_0 e^{-i\omega t}}{c\rho \sin ka}\left[e^{i(kr_1-\omega t)}\hat{\mathbf{t}}_1 + e^{i(kr_2-\omega t)}\hat{\mathbf{t}}_2 - 2\cos ka\, e^{i(kr_0-\omega t)}\hat{\mathbf{t}}_0\right] \\ &= \frac{iI_0 e^{-i\omega t}}{c\sin ka}\left[\frac{e^{i(kr_1-\omega t)}}{r_1\sin\theta_1}\hat{\mathbf{t}}_1 + \frac{e^{i(kr_2-\omega t)}}{r_1\sin\theta_2}\hat{\mathbf{t}}_2 - 2\cos ka\frac{e^{i(kr_0-\omega t)}}{r_1\sin\theta_0}\hat{\mathbf{t}}_0\right]. \end{aligned} \tag{40}$$

These forms apparently give many people the impression that the radiation from a dipole antenna comes from 3 points: its center and its two tips. Furthermore, these forms do not hold on the z-axis, and mask that fact that the electric field has a nonzero tangential component along that axis.[4]

[4]Use of eqs. (38)-(39) in eqs. (25) and (27) leads to factors of ρ/ρ which can justifiably be set equal to 1 only if $\rho \neq 0$.

2.4 The Poynting Vector Close to the Antenna

The Poynting vector, which describes the flow of energy in the electromagnetic field, is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}(\hat{\boldsymbol{\rho}}E_z B_\phi + \hat{\mathbf{z}}E_\rho B_\phi), \tag{41}$$

The time average of the Poynting vector close to the wire follows from eqs. (29), (31) and (33) as

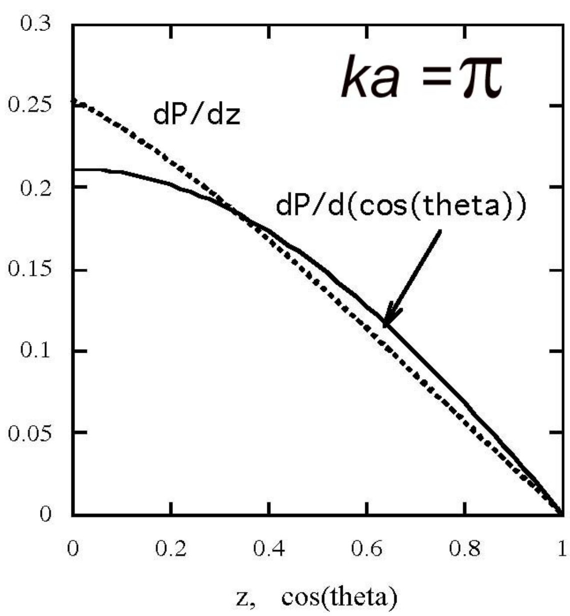
$$\langle S_z(\rho \approx 0, |z| < a, t)\rangle = 0, \tag{42}$$

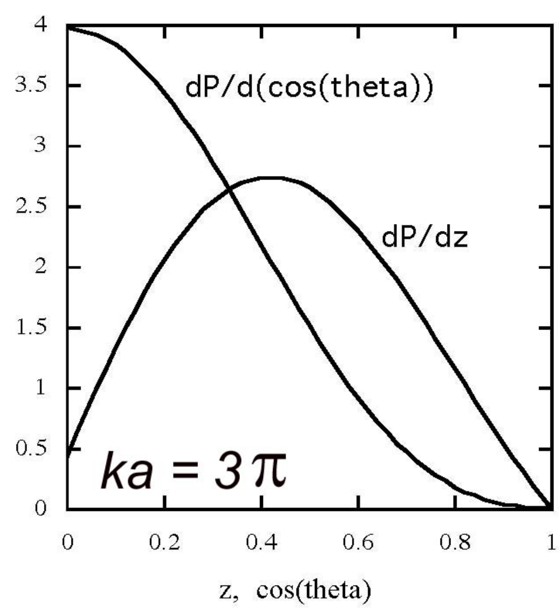
$$\langle S_\rho(\rho \approx 0, |z| < a, t)\rangle = \frac{aI_0^2}{2\pi c\rho(a^2 - z^2)}\frac{\sin[k(a-|z|)]}{\sin^2 ka}\left[\sin ka\cos kz - ka\cos ka\frac{\sin kz}{kz}\right]. \tag{43}$$

The time-average power $dP(z)$ radiated by a segment of length dz is

$$dP = 2\pi\rho\ dz\,\langle S_\rho\rangle = \frac{aI_0^2\ dz}{c(a^2 - z^2)}\frac{\sin[k(a-|z|)]}{\sin^2 ka}\left[\sin ka\cos kz - ka\cos ka\frac{\sin kz}{kz}\right]. \tag{44}$$

As anticipated in the introduction to the solution, the spatial dependence of the radiated power is the product of the spatial distribution of the current and another factor. Despite the factor $a^2 - z^2$ in the denominator of eq. (44), this expression vanishes as $z \to a$, and we do not predict a large amount of radiation from the ends ("tips") of the antenna.[5] This is illustrated in the figures below for two values of the length of the antenna. Also shown for later reference are the far-field radiation patterns according to eq. (62).





The total (time-average) radiated power can be calculated by integrating eq. (44),

$$P = \frac{2I_0^2}{c}\int_0^a \frac{dz}{a(1-(z/a)^2)}\frac{\sin[k(a-|z|)]}{\sin^2 ka}\left[\sin ka\cos kz - ka\cos ka\frac{\sin kz}{kz}\right]$$

[5]Equation (44) quickly provides the solution to problem 12-6 of [23], on replacing $1/c$ by $\mu_0 c/4\pi$ when converting to MKSA units.

$$= \frac{2I_0^2}{c}\int_0^1 \frac{d\cos\theta}{1-\cos^2\theta}\frac{\sin ka\cos(ka\cos\theta)-\cos ka\sin(ka\cos\theta)}{\sin^2 ka} \times\left[\sin ka\cos(ka\cos\theta)-\cos ka\frac{\sin(ka\cos\theta)}{\cos\theta}\right], \tag{45}$$

with the change of variable $z = a\cos\theta$. The mathematical parameter θ does not necessarily have an interpretation as an angle associated with the far-field radiation pattern. However, the spirit of Poynting is that the lines of the vector field $\mathbf{S}$ trace the flow of energy from the near zone to the far zone, and hence there should be a one-to-one mapping of energy flow from points in the near zone to angles in the far zone. It is therefore agreeable that eq.(45) is identical to eq. (63) for the far-field radiated power when $ka = (2n+1)\pi/2$, and we conclude that for this case the radiation (*i.e.*, lines of Poynting vector $\mathbf{S}$) at far-field angle θ emerged from point $z = a\cos\theta$ on the antenna. This simple mapping does not hold in general. For example, when $ka = n\pi$ the integrand of eq (45) differs from that of eq. (63) by a factor of $1/\cos\theta$, as illustrated in the figures on the previous page. Nonetheless, it can be verified numerically that the integrals (45) and (63) are identical for any value of ka.[6]

An additional interpretation of eq. (44) is that the radiated power corresponds to a radiation resistance per unit length, $dR(z)/dz$, that varies along the antenna, such that

$$\frac{1}{2}I_0R_{\rm rad} = \frac{1}{2}I_0\int\frac{dR(z)}{dz}\,dz = \int\frac{dP}{dz}dz, \tag{46}$$

so that

$$\frac{dR(z)}{dz} = \frac{2a}{c(a^2-z^2)}\frac{\sin[k(a-|z|)]}{\sin^2 ka}\left[\cos kz - ka\cot ka\frac{\sin kz}{kz}\right]. \tag{47}$$

However, the physical meaning of a spatially distributed radiation resistance $dR(z)/dz$ is not well explained by our brief discussion. See, for example, sec. 5.2 of [6] where $dR(z)/dz$ is considered to be the real part of a complex impedance. The total radiation resistance $R_{\rm rad}$ (which could be expressed in terms of exponential integrals) has greater operational meaning as a lumped circuit element with which the power source interacts.

Close to the antenna, the z component of the Poynting vector follows from eqs. (29) and (31) as

$$S_z(\rho\approx 0, |z|<a, t) = \frac{|z|}{z}\frac{I_0^2}{4\pi c\rho^2}\frac{\sin 2[k(a-|z|)]}{\sin^2 ka}\sin 2\omega t. \tag{48}$$

The time average of this is zero, so our solution does not include a net flow of energy from the central feed point to points along the antenna, as is required if energy is being radiated from the antenna according to eq. (44). This suggests that the present solution should be augmented by one that involves traveling waves of the Poynting vector along the z-axis in the near zone. Further remarks on this are made in sec. 2.8.

[6] Thanks to J.D. Jackson for pointing this out, and for providing the figures.

2.5 The Short, Center-Fed Linear Dipole Antenna ($ka \ll 1$)

For a short, center-fed dipole antenna ($ka \ll 1$), the fields (29), (31) and (33) near the wire become

$$B_\phi(\rho \approx 0, \phi, |z| < a, t) = \frac{2I_0}{ca\rho}(a - |z|) \cos \omega t, \tag{49}$$

$$E_\rho(\rho \approx 0, \phi, |z| < a, t) = -\frac{|z|}{z} \frac{2I_0}{cka\rho} \sin \omega t, \tag{50}$$

$$E_z(0, \phi, |z| < a, t) = \frac{2I_0 k^2 a}{3c} \cos \omega t + \frac{2I_0}{ck\,|z|\,(a + |z|)} \sin \omega t, \tag{51}$$

and eq. (44) for the time-average power emitted at position z on the antenna reduces to

$$\frac{dP(ka \ll 1)}{dz} = \frac{I_0^2}{3c} k^2 (a - |z|). \tag{52}$$

Our interpretation is that the distribution of radiated power along the antenna peaks at the center and falls off linearly with distance from the center, thereby having the same functional form as does the current distribution. The total power radiated is

$$P(ka \ll 1) = \int_{-a}^{a} \frac{dP}{dz} dz = \frac{I_0^2}{3c} k^2 a^2, \tag{53}$$

in agreement with the usual analysis based on the far fields associated with the current distribution $k(a - |z|)$.

2.6 Far Fields

For completeness, we record the electromagnetic fields in the far zone, and the consequent far-field radiation pattern.

Referring to the figure on p. 1, we see that in the far zone the exponential factors in the fields (23), (25) and (27) can be approximated by

$$r_1 = r_0 - a \cos \theta, \qquad r_2 = r_0 + a \cos \theta, \qquad e^{ikr_1} + e^{ikr_2} = 2e^{ikr_0} \cos(ka \cos \theta), \tag{54}$$

while the factors outside the exponentials can be written

$$\frac{1}{\rho} = \frac{1}{r_0 \sin \theta}, \qquad \frac{1}{r_1} \approx \frac{1}{r_2} \approx \frac{1}{r_0}, \qquad \frac{z - a}{r_1} \approx \frac{z + a}{r_2} \approx \frac{z}{r_0} = \cos \theta. \tag{55}$$

where angle θ is measured in a spherical coordinate system (r, θ, ϕ). The electric field components in spherical coordinates are related to those in cylindrical coordinates by

$$E_r = E_z \cos \theta + E_\rho \sin \theta, \qquad E_\theta = -E_z \sin \theta + E_\rho \cos \theta. \tag{56}$$

Thus, in the far zone the fields become

$$B_\phi(\mathbf{r}, t) = -\frac{2iI_0}{c} \frac{e^{i(kr_0 - \omega t)}}{r_0} \left[\frac{\cos(ka \cos \theta) - \cos ka}{\sin ka \sin \theta} \right], \tag{57}$$

$$E_z(\mathbf{r},t) = \frac{2iI_0}{c}\frac{e^{i(kr_0-\omega t)}}{r_0}\frac{\cos(ka\cos\theta)-\cos ka}{\sin ka}, \qquad (58)$$

$$E_\rho(\mathbf{r},t) = -\frac{2iI_0}{c}\frac{e^{i(kr_0-\omega t)}}{r_0}\cos\theta\left[\frac{\cos(ka\cos\theta)-\cos ka}{\sin ka\sin\theta}\right], \qquad (59)$$

$$E_r(\mathbf{r},t) = 0, \qquad (60)$$

$$E_\theta(\mathbf{r},t) = -\frac{2iI_0}{c}\frac{e^{i(kr_0-\omega t)}}{r_0}\left[\frac{\cos(ka\cos\theta)-\cos ka}{\sin ka\sin\theta}\right] = B_\phi. \qquad (61)$$

The time-average radiation power in the far zone is

$$\frac{dP}{d\Omega} = \frac{cr_0^2}{8\pi}Re(E_\theta^\star B_\phi) = \frac{I_0^2}{2\pi c}\left[\frac{\cos(ka\cos\theta)-\cos ka}{\sin ka\sin\theta}\right]^2. \qquad (62)$$

The total (time-average) radiated power can be written

$$P = 4\pi\int_0^1 \frac{dP}{d\Omega}d\cos\theta = \frac{2I_0^2}{c\sin^2 ka}\int_0^1 \frac{[\cos(ka\cos\theta)-\cos ka]^2}{1-\cos^2\theta}d\cos\theta, \qquad (63)$$

which can be expressed in terms of a cosine integral when $ka = n\pi/2$.

For completeness, we note that for a short linear antenna eq. (62) reduces to

$$\frac{dP}{d\Omega} = \frac{cr_0^2}{8\pi}Re(E_\theta^\star B_\phi) = \frac{I_0^2}{2\pi c}k^2a^2\sin\theta \qquad (ka \ll 1), \qquad (64)$$

and the total, time-average radiated power is

$$P = \frac{I_0^2}{3c}k^2a^2 = \frac{I_0^2}{2}R_{\rm rad}, \qquad \textit{with} \qquad R_{\rm rad} = \frac{2k^2a^2}{3c} = 20k^2a^2 \textit{ Ohms}. \qquad (65)$$

2.7 The Poynting Vector in the Near Zone in Spherical Coordinates

The electric field components in spherical coordinates are from eqs. (25) and (27),

$$E_r(\mathbf{r},t) = \frac{z}{r_0}E_z + \frac{\rho}{r_0}E_\rho = \frac{iaI_0e^{-i\omega t}}{cr_0}\left[\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right], \qquad (66)$$

$$\begin{aligned} E_\theta(\mathbf{r},t) &= -\frac{\rho}{r_0}E_z + \frac{z}{r_0}E_\rho \\ &= -\frac{iI_0e^{-i\omega t}}{cr_0^2\sin ka\sin\theta}\left[\frac{(r_0^2-az)e^{ikr_1}}{r_1} + \frac{(r_0^2+az)e^{ikr_2}}{r_2} - 2r_0e^{ikr_0}\cos ka\right]. \qquad (67) \end{aligned}$$

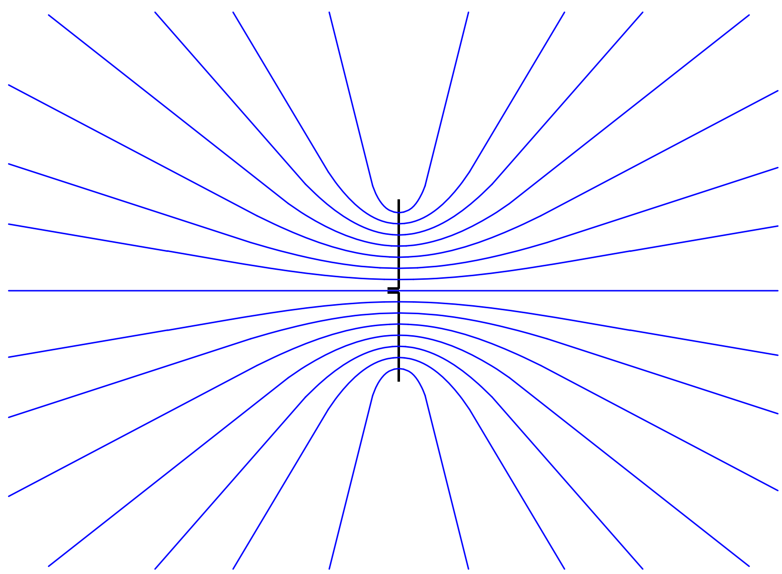
These forms go over to eqs. (60)-(61) in the far zone.

The time-average components of the Poynting vector are

$$
\begin{aligned}
\langle S_r \rangle &= \frac{c}{8\pi} Re(E_\theta^\star B_\phi) \\
&= \frac{I_0^2}{4\pi c r_0^2 \sin^2 ka \sin^2\theta}\left[\left(\frac{r_0 - a\cos\theta}{r_1} + \frac{r_0 + a\cos\theta}{r_2}\right)\cos^2 k\frac{r_2 - r_1}{2} + 2\cos^2 ka \right. \\
&\qquad - \left(1 + \frac{r_0 - a\cos\theta}{r_1}\right)\cos ka \cos k(r_1 - r_0) \\
&\qquad \left. - \left(1 + \frac{r_0 + a\cos\theta}{r_2}\right)\cos ka \cos k(r_2 - r_0)\right], \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
\langle S_\theta \rangle &= \frac{c}{8\pi} Re(E_r^\star B_\phi) \\
&= \frac{a I_0^2}{4\pi c r_0^2 \sin^2 ka \sin\theta}\left[\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\cos^2 k\frac{r_2 - r_1}{2} \right. \\
&\qquad \left. + \frac{\cos ka}{r_1}\cos k(r_1 - r_0) - \frac{\cos ka}{r_2}\cos k(r_2 - r_0)\right]. \qquad (69)
\end{aligned}
$$

At a point $0 < z < a$ on the antenna, $r_1 = a - z$, $r_0 = z$, $r_2 = a + z$, and eq. (68) indicates that $\langle S_r \rangle = \langle S_z \rangle = 0$, in agreement with eq. (42). Note also that $\langle S_\theta(\theta = 90^\circ) \rangle = 0$. No net energy is transfered from the upper half to the lower half space. The lines of the Poynting flux $\mathbf{S}$ emerge from the antenna at right angles to the z-axis, and bend in the near zone until they become purely radial in the far zone, as sketched in the figure below.



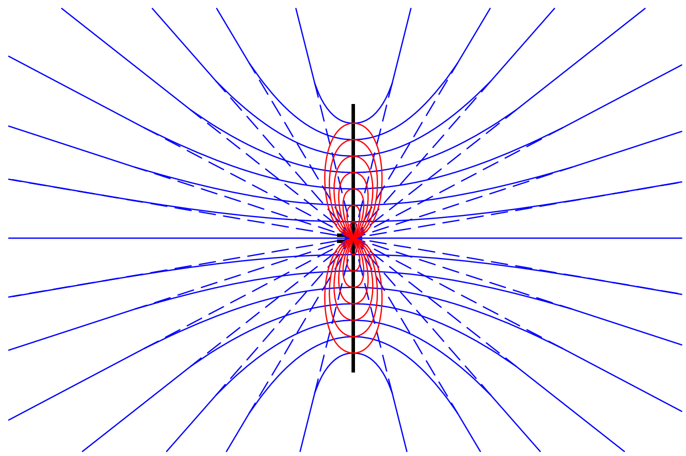
This figure indicates that the problem of a linear antenna can be regarded as a limiting case of a prolate spheroid, and that a solution might usefully be presented in prolate spheroidal coordinates [28].

2.8 Comments

We are left with the question of how the power flows from the feed point out to the points on the (perfectly conducting) antenna, after which the power can be said to flow along the lines of Poynting vector shown above.

We infer that there is an additional set of near-zone electromagnetic fields that describe the "missing" flow of energy. These fields include a tangential component of the electric field along the antenna that is the equal and opposite of eq. (33), so that the total tangential electric field is zero at the conductors.

The additional magnetic field near the antenna must combine with the negative of eq. (33) to give a Poynting vector near the antenna that is the negative of eq. (43), possibly as sketched by the red lines in the figure below. This requires an additional magnetic field close to the antenna that is exactly the same as eq. (29). Our logic is that the total magnetic field close to the antenna is twice that given by eq. (29).



Ampère's law then implies that the current in the antenna is twice that assumed in eq. (10). Since the parameter I_0 in eq. (10) was arbitrary, this is not an immediate problem. However, if we consider that the antenna is driven by a specified external voltage difference $V_0 e^{-i\omega t}$ at the central feed point, our doubled current would imply that the antenna impedance is half the usual value.

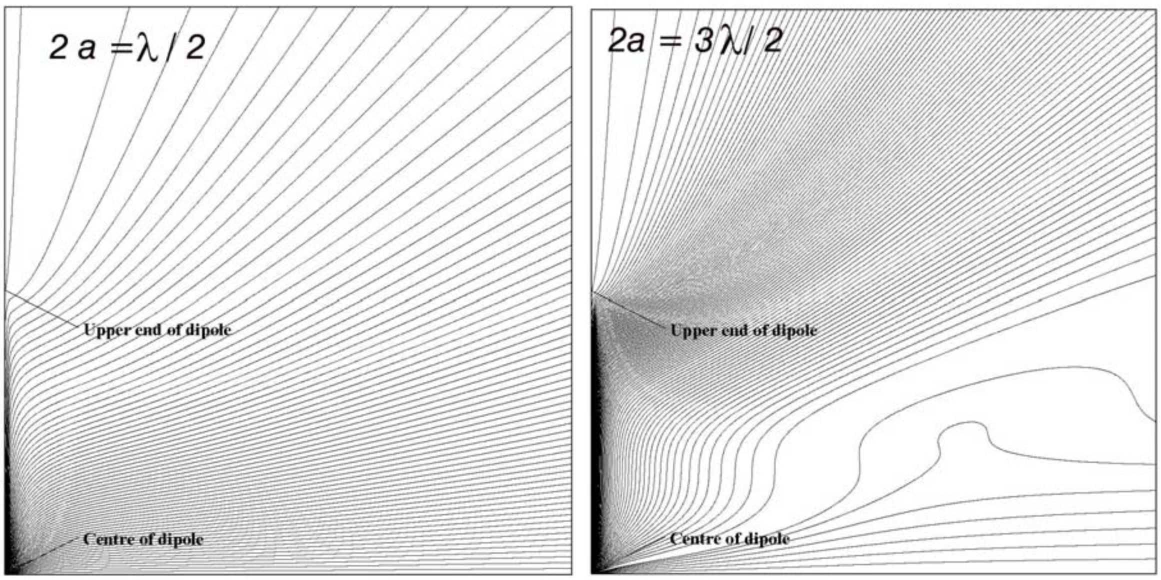
Is this an actual contradiction with known engineering behavior? Is the current in a linear antenna actually known from measurements, or is it entirely a theoretical construct based on models such as that presented in this note? The far-zone radiated power remains the same after the addition of the "missing" near zone fields, so the load on the source due to radiation is unchanged. The load due to Joule heating in an antenna made from real conductors would be multiplied by four if my speculations were correct, but this load is usually negligible compared to the radiation load.

Besides the additional current in the antenna, there will need to be additional charges and currents in the feed region (where the feedline couples to the linear antenna). If these were known, then we could again use the method of retarded potentials to calculate the fields whose associated Poynting vector corresponds to the red lines in the sketch above.

This description is getting somewhat baroque. It seems simpler to this author to adopt the view that the radiation from a linear antenna emerges from the small gap between the two halves of the antenna. The conductors of the antenna (beyond the central feed region) play an important role in directing this radiation, but not in creating it.

We close with two examples of numerical computations of the time-average Poynting vector for linear dipole antennas of total length $\lambda/2$ and $3\lambda/2$, kindly provided by Alan

Boswell (alan.boswell@blueyonder.co.uk). The figures below show quarter sections of the lines of Poynting vector, with the feedpoint of the antenna at the lower left corner of each plot. The computation is a so-called method-of-moments implementation [29, 30] of the integral-equation approach of Hallén [27], which solves an integral equation first written down by Pocklington [8] for both the current distribution and electromagnetic fields, subject to perfect-conductor boundary conditions at the surface of the antenna. We see that the lines of Poynting flux emanate from the feed point, and are parallel to the conductor of the antenna when close to it, then bending over to take on the far-zone pattern described in eq. (62). Apparently, plots of this type were first systematically produced by Meinke and Landstorfer [31].



It appears to this author that the results of the numerical computation are very satisfactory, which indicates that numerical computations now play a prominent role in providing understanding of antennas.

This note has been about the idealization of an antenna made of infinitely thin wires. Calculations for antennas made from thin but finite wires by Hallén [27] and by King [20] have shown that the current distribution in this case, for fields that obey perfect-conductor boundary conditions at the surface of the wire, are very close to the form (10), and that the radiation resistance is also very close to that which follows from eq. (62). Hence, it appears that there is actually little merit in pursuing the imperfections of the analysis of infinitely thin wires, since a somewhat naive use of the current distribution (10) happens to lead to results of practical significance.

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