

# Electromagnetic Field Angular Momentum of a Charge at Rest in a Uniform Magnetic Field

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## 1 Problem

Discuss the electromagnetic field momentum of a charge at rest in a uniform magnetic field.<sup>1</sup> Comment also on “hidden” angular momentum.

## 2 Solution

The electromagnetic field momentum can be written for static fields in three equivalent forms for stationary systems,<sup>2</sup>

$$\mathbf{P}_{\text{EM}} = \int \mathbf{p}_{\text{EM}} d\text{Vol} = \mathbf{P}_{\text{M}} = \mathbf{P}_{\text{P}} = \mathbf{P}_{\text{F}}, \quad (1)$$

where

$$\mathbf{P}_{\text{M}} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol}, \quad \mathbf{P}_{\text{P}} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \quad \mathbf{P}_{\text{F}} = \int \frac{V \mathbf{J}}{c^2} d\text{Vol}, \quad (2)$$

in Gaussian units, where  $\rho$  is the electric charge density,  $\mathbf{A}$  is the magnetic vector potential (where  $\nabla \cdot \mathbf{A} = 0$  for static fields in both the Coulomb and Lorenz gauges),  $\mathbf{E}$  is the electric field,  $\mathbf{B}$  is the magnetic field,  $V$  is the electric scalar potential, and  $\mathbf{J}$  is the electric current density.<sup>3</sup> The first form is due to Faraday [4] and Maxwell [5]; the second form is due to Poynting [6] as elaborated upon by J.J. Thomson [7]<sup>4</sup> and Poincaré [12]; and the third form was introduced by Furry [13].<sup>5</sup>

While the three integrals  $\mathbf{P}_{\text{M}}$ ,  $\mathbf{P}_{\text{P}}$  and  $\mathbf{P}_{\text{F}}$  are equal for stationary systems, only the integrand  $\mathbf{p}_{\text{P}} = \mathbf{E} \times \mathbf{B}/4\pi c$  (and not  $\mathbf{p}_{\text{M}} = \rho \mathbf{A}/c$  or  $\mathbf{p}_{\text{F}} = V \mathbf{J}/c^2$ ) has the interpretation of the volume density of electromagnetic field momentum in a “Maxwellian” field-theory view.

This suggests that the field angular momentum (about the origin) can be computed (for static fields) via the expressions

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \mathbf{p}_{\text{EM}} d\text{Vol} = \mathbf{L}_{\text{M}} = \mathbf{L}_{\text{P}} = \mathbf{L}_{\text{F}}, \quad (3)$$

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<sup>1</sup>Aspects of this problem have been considered in [1].

<sup>2</sup>See, for example, [2].

<sup>3</sup>For discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence, see, for example, [3].

<sup>4</sup>Thomson noted the equivalence of  $\mathbf{P}_{\text{M}}$  and  $\mathbf{P}_{\text{P}}$  for a special case in [8, 9], but their general equivalence for stationary systems went unremarked until independent papers by Trammel [10] and Calkin [11].

<sup>5</sup>The form of Furry was largely anticipated by Coleman and van Vleck [14], as discussed in Appendix C.

where

$$\mathbf{L}_M = \int \mathbf{r} \times \frac{\varrho \mathbf{A}}{c} d\text{Vol}, \quad \mathbf{L}_P = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \quad \mathbf{L}_F = \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol}. \quad (4)$$

Although it is natural in “Maxwellian” field theory to regard  $\mathbf{l}_P = \mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$  (and not  $\mathbf{l}_M = \mathbf{r} \times \varrho \mathbf{A}/c$  or  $\mathbf{l}_F = \mathbf{r} \times V\mathbf{J}/c^2$ ) as the volume density of electromagnetic field angular momentum, we will conclude that the form  $\mathbf{L}_M$  is the most reliable for computation of field angular momentum, that form  $\mathbf{L}_F$  should not be used, and that the form  $\mathbf{L}_P$  is valid only for fields that fall off sufficiently quickly at large distances (which is not the case for an infinite solenoid).

## 2.1 Infinite Solenoid

Consider an infinite solenoid of radius  $R$  about the  $z$  axis with magnetic field  $\mathbf{B} = B \hat{\mathbf{z}}$  inside and essentially zero field outside. The surface current density is  $\mathbf{K} = cB \hat{\boldsymbol{\phi}}/4\pi$  in cylindrical coordinates  $(\rho, \phi, z)$ , while the surface charge density  $\sigma$  is zero.<sup>6</sup> The vector potential can be written as

$$\mathbf{A}(\rho < R) = \frac{B\rho}{2} \hat{\boldsymbol{\phi}}, \quad \mathbf{A}(\rho > R) = \frac{BR^2}{2\rho} \hat{\boldsymbol{\phi}}, \quad (5)$$

### 2.1.1 A Mechanical Argument for Field Angular Momentum

If we suppose the infinite solenoid to be the limit of a long, but finite solenoid, we could imagine bringing charge  $q$  to, say, distance  $a$  from the axis at the center of the solenoid by starting it on the axis but well outside the solenoid, moving it along the axis to the center of the solenoid, and finally moving it radially to distance  $a$ .<sup>7</sup> It is reasonable to assume that the field angular momentum is initially zero in this case.

The charge  $q$  experiences no  $\mathbf{v}_q \times \mathbf{B}$  force as it is brought to the center of the solenoid along its axis (and the electrically neutral, azimuthal currents  $\mathbf{K}$  in the solenoid also experience no force,  $\sigma \mathbf{E}_q + \mathbf{K} \times \mathbf{B}_q/c = 0$ ), so the total angular momentum remains zero during this process. It seems reasonable to infer that the field angular momentum remains zero so long as the charge is on the axis of the solenoid. But as the charge is moved radially, after having reached the center of the solenoid, at some low velocity  $v \hat{\boldsymbol{\rho}}$ , it experiences a torque,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \rho \hat{\boldsymbol{\rho}} \times \left( \frac{qv \hat{\boldsymbol{\rho}}}{c} \times B \hat{\mathbf{z}} \right) = -\frac{qB}{c} \rho \frac{d\rho}{dt} \hat{\mathbf{z}} = -\frac{qB}{2c} \frac{d\rho^2}{dt} \hat{\mathbf{z}}. \quad (6)$$

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<sup>6</sup>If the solenoid is based on electrical currents flowing in a wire of resistivity  $\varrho$  there must be an electric field  $\mathbf{E} = \varrho \mathbf{K}$  inside the wire, which is shaped by an appropriate nonzero surface charge density along the wire (see, for example, [15]). We avoid such complications here by supposing the solenoid current to be generated by two counterpropagating flows of noninteracting charges of opposite signs inside nonconducting tubes [16, 17], or by two counterrotating cylinders of fixed, opposite surface charge density [18]. This assumption has the additional advantage that the initial mechanical angular momentum of the solenoid currents is zero.

<sup>7</sup>This argument follows sec. IV of [19] and p. 35 of [20]. See also [23, 24, 25]. The spirit of the argument can be traced to early experiments by Lodge [26] and by Einstein/de Haas [27], and underlies the Feynman disk paradox [22].

so long as  $\rho < R$ , where the magnetic field is nonzero. To keep the charge moving on a straight line to its desired final position, an equal and opposite external torque must be applied, such that

$$\boldsymbol{\tau}_{\text{ext}} = \frac{qB}{2c} \frac{d\rho^2}{dt} \hat{\mathbf{z}} = \frac{d\mathbf{L}}{dt}, \quad (7)$$

When the charge ends at rest at its desired position, with zero mechanical angular momentum,<sup>8</sup> the nonzero angular momentum of the system must be stored in the electromagnetic field,

$$\mathbf{L}_{\text{EM}} = \int \frac{d\mathbf{L}}{dt} dt = \frac{qB}{2c} \hat{\mathbf{z}} \int_{\rho < R} \frac{d\rho^2}{dt} \hat{\mathbf{z}} = \frac{qB}{2c} \hat{\mathbf{z}} \begin{cases} a^2 & (a < R), \\ R^2 & (a > R). \end{cases} \quad (8)$$

### 2.1.2 Maxwell Field Angular Momentum

For an electric charge  $q$  at distance  $a$  from the solenoid axis, the field angular momentum based on the Maxwell form  $\mathbf{P}_M$  of the field momentum is

$$\mathbf{L}_M = \int \mathbf{r} \times \frac{\rho \mathbf{A}}{c} d\text{Vol} = \frac{qB}{2c} \hat{\mathbf{z}} \begin{cases} a^2 & (a < R), \\ R^2 & (a > R), \end{cases} \quad (9)$$

which agrees with (8).

### 2.1.3 Poynting Field Angular Momentum

For charge  $q$  at  $(x, y, z) = (a, 0, 0)$  the electric field is  $\mathbf{E}(x, y, z) = q \mathbf{d}/d^3 = q[(x-a)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]/[(x-a)^2 + y^2 + z^2]^{3/2}$ , so the field angular momentum based on the Poynting form  $\mathbf{P}_P$  of the field momentum is (prob. 8.14 of [28])<sup>9</sup>

$$\begin{aligned} \mathbf{L}_P &= \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \frac{qB}{4\pi c} \int (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \times \left( \frac{(x-a)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{[(x-a)^2 + y^2 + z^2]^{3/2}} \times \hat{\mathbf{z}} \right) d\text{Vol} \\ &= -\frac{qB}{4\pi c} \hat{\mathbf{z}} \int \frac{x^2 + y^2 - xa}{[(x-a)^2 + y^2 + z^2]^{3/2}} dx dy dz = -\frac{qB}{2\pi c} \hat{\mathbf{z}} \int \frac{x^2 + y^2 - xa}{(x-a)^2 + y^2} dx dy \\ &= -\frac{qB}{2\pi c} \hat{\mathbf{z}} \int_0^R \rho d\rho \int_0^{2\pi} d\phi \frac{\rho^2 - a\rho \cos \phi}{\rho^2 - 2a\rho \cos \phi + a^2} = -\frac{qB}{2\pi c} \hat{\mathbf{z}} \int_0^R \rho d\rho \int_0^{2\pi} d\phi \frac{1 - \frac{a}{\rho} \cos \phi}{1 - 2\frac{a}{\rho} \cos \phi + \frac{a^2}{\rho^2}}. \end{aligned} \quad (10)$$

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<sup>8</sup>There is no obvious mechanical angular momentum in the final configuration, but in principle there could be “hidden” mechanical angular momentum, as discussed in secs. 2.4.2-2.4.3. The agreement between eqs. (8) and (9) suggests that there is no “hidden” mechanical angular momentum, but the disagreement between eqs. (8) and (11)-(12) and (15) casts doubt on this. We eventually conclude that the field angular momentum is given by eqs. (8)-(9), and so there is no “hidden” mechanical angular momentum in the system for any position of the charge.

<sup>9</sup>The angular momentum  $\mathbf{L}_P$  for  $a > R$  is also evaluated in sec. J of [29], where it is noted that this does not equal  $\mathbf{L}_M$ , with the conclusion (in endnote 19) that the latter form must be “in error”.

For  $a > R \geq \rho$  the remaining integral is (Dwight 859.113 and 859.122)

$$\mathbf{L}_P(a > R) = -\frac{qB}{c} \hat{\mathbf{z}} \int_0^R \rho d\rho \left( \frac{\rho^2}{a^2 - \rho^2} - \frac{\rho^2}{a^2 - \rho^2} \right) = 0. \quad (11)$$

For  $a < R$  the part of the integral for  $a > \rho$  vanishes as in eq. (11), and the remaining integral is

$$\mathbf{L}_P(a < R) = -\frac{qB}{c} \hat{\mathbf{z}} \int_a^R \rho d\rho \left( \frac{\rho^2}{\rho^2 - a^2} - \frac{a^2}{\rho^2 - a^2} \right) = -\frac{qB(R^2 - a^2)}{2c} \hat{\mathbf{z}}. \quad (12)$$

It is perhaps surprising that the Poynting form (11)-(12) does not agree with the Maxwell form (9) or with the argument leading to eq. (8), in view of general arguments as given, for example, in sec. II of [13].<sup>10,11</sup>

As a check, for a charge at the origin inside a long solenoid that extends over  $|z| < l$ ,

$$\begin{aligned} \mathbf{L}_P &= \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \frac{qB}{4\pi c} \int (\rho \hat{\boldsymbol{\rho}} + z \hat{\mathbf{z}}) \times \left( \frac{\rho \hat{\boldsymbol{\rho}} + z \hat{\mathbf{z}}}{(\rho^2 + z^2)^{3/2}} \times \hat{\mathbf{z}} \right) d\text{Vol} \\ &\approx -\frac{qB}{c} \int_0^R \rho d\rho \int_0^l dz \frac{\rho^2}{(\rho^2 + z^2)^{3/2}} \hat{\mathbf{z}} = -\frac{qBl}{c} \int_0^R \frac{\rho d\rho}{(\rho^2 + l^2)^{1/2}} \hat{\mathbf{z}} = -\frac{qBl(\sqrt{l^2 + R^2} - l)}{c} \hat{\mathbf{z}} \approx -\frac{qBR^2}{2c} \hat{\mathbf{z}}. \end{aligned} \quad (13)$$

#### 2.1.4 Furry Field Angular Momentum

For the field angular momentum based on the Furry form  $\mathbf{P}_F$  of the field momentum, we content ourselves with the case of a charge on the solenoid axis at  $z = 0$ , for a solenoid of length  $2l$ . Then, the electric scalar potential for charge  $q$  is<sup>12</sup>

$$V = \frac{q}{r} = \frac{q}{(\rho^2 + z^2)^{1/2}}, \quad (14)$$

so the Furry angular momentum for the solenoid plus charge is

$$\begin{aligned} \mathbf{L}_F &= \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol} = \frac{qB}{4\pi c} \int (R \hat{\boldsymbol{\rho}} + z \hat{\mathbf{z}}) \times \frac{\hat{\boldsymbol{\phi}}}{(R^2 + z^2)^{1/2}} d\text{Area} \\ &= \frac{qBR^2}{c} \int_0^l \frac{dz}{\sqrt{R^2 + z^2}} \hat{\mathbf{z}} = \frac{qBR^2}{c} \sinh^{-1} \frac{l}{R} \hat{\mathbf{z}}, \end{aligned} \quad (15)$$

which diverges as the solenoid length  $l$  grows large.

<sup>10</sup>It seems not to have been noticed in [19, 20] that the Poynting form of field angular momentum does not lead to the result (8) found there.

<sup>11</sup>The discrepancy between the Maxwell and Poynting forms for the field angular momentum associated with an infinite solenoid appears in a variant [21] of Feynman's disk paradox [22].

<sup>12</sup>The form (14) applies everywhere only if the conductor of the solenoid does not "shield" the electric field of the charge. This is consistent with the model of the solenoid currents as due to counterrotating, opposite charges in nonconducting tubes, or to a pair of counterrotating, oppositely charged cylinders.

### 2.1.5 Comments

We are belatedly reminded to review the derivation of eqs. (1) in, for example, [2], where it was noted on the bottom of p. 3 that the various forms for the field momentum are equivalent only if the fields fall off sufficiently fast at large distances. While the magnetic field of the infinite solenoid does not fall off with  $z$ , it happens that field momenta  $\mathbf{P}_M = \mathbf{P}_P = \mathbf{P}_F$ , at least when the charge is on the axis.<sup>13</sup>

Furthermore, it was shown in sec. 2.2 of [2] that field angular momentum  $\mathbf{L}_M = \mathbf{L}_P$  when the fields fall off sufficiently quickly at large distance. It appears that the field of an infinite solenoid does not fall off fast enough for this to hold.<sup>14</sup>

This disagreement between the Furry form of field angular momentum and eqs. (8)-(9) is an example of a general result given in Appendix B. There is also disagreement between the computations of angular momentum considered here and the subtle issue of “hidden” angular momentum, as discussed in sec. 2.4 below.

## 2.2 Magnetic Field Uniform within a Sphere of Radius $R$

As a calculable example of a system with bounded charge and current distributions, for which the fields falls off at least as fast as  $1/r^2$ , consider surface currents  $\mathbf{K} = 3B \sin \theta \hat{\phi}/8\pi c$  on a sphere of radius  $R$ , centered on the origin in a spherical coordinate system  $(r, \theta, \phi)$ . These generate the magnetic field<sup>15</sup>

$$\mathbf{B}(r < R) = B \hat{\mathbf{z}}, \quad \mathbf{B}(r > R) = \frac{BR^3}{2r^3}(3 \cos \theta \hat{\mathbf{r}} - \hat{\mathbf{z}}), \quad (17)$$

which is uniform inside the sphere and the field of a dipole of magnetic moment  $BR^3 \hat{\mathbf{z}}/2$  outside it. The vector potential can be written as

$$\mathbf{A}(r < R) = \frac{Br \sin \theta}{2} \hat{\phi}, \quad \mathbf{A}(r > R) = \frac{BR^3 \sin \theta}{2r^2} \hat{\phi}. \quad (18)$$

For an electric charge  $q$  at the origin, where the vector potential is zero, the field angular momentum based on the Maxwell form of the field momentum is

$$\mathbf{L}_M = \int \mathbf{r} \times \frac{\rho \mathbf{A}}{c} d\text{Vol} = 0, \quad (19)$$

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<sup>13</sup>The field momentum  $\mathbf{P}_M$  is zero for a charge at rest on the solenoid axis, the Poynting form of the field momentum is

$$\mathbf{P}_P = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \frac{qB}{4\pi c} \int \frac{\rho \hat{\rho} + z \hat{\mathbf{z}}}{(\rho^2 + z^2)^{3/2}} \times \hat{\mathbf{z}} d\text{Vol} = 0, \quad (16)$$

and the Furry form of the field momentum is  $\mathbf{P}_F = \int V \mathbf{J} d\text{Vol}/c^2 = 0$  for a charge on the axis.

See Appendix A for discussion of the case when the charge is not on the solenoid axis.

<sup>14</sup>Another view is that for a finite solenoid, the integral of  $\mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}$  inside the solenoid is canceled by the integral outside it for the case of a charge on the axis, but for an infinite solenoid the integral outside the solenoid is set to zero, leaving an unphysical, nonzero value (13) for the field angular momentum  $\mathbf{L}_P$ .

<sup>15</sup>See, for example, [24] and sec. II of [30].

The electric field is  $\mathbf{E} = q\hat{\mathbf{r}}/r^2$ , so the field angular momentum based on the Poynting form of the field momentum is

$$\begin{aligned}
\mathbf{L}_P &= \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} \\
&= \frac{qB}{4\pi c} \int_{r < R} \mathbf{r} \times \left( \frac{\hat{\mathbf{r}}}{r^2} \times \hat{\mathbf{z}} \right) d\text{Vol} + \frac{qBR^3}{8\pi c} \int_{r > R} \mathbf{r} \times \left( \frac{\hat{\mathbf{r}}}{r^2} \times \frac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} \right) d\text{Vol} \\
&= -\frac{qB}{2c} \int_0^R r^2 dr \int_{-1}^1 d\cos\theta \frac{\sin^2\theta}{r} \hat{\mathbf{z}} + \frac{qBR^3}{4c} \int_R^\infty r^2 dr \int_{-1}^1 d\cos\theta \frac{\sin^2\theta}{r^4} \hat{\mathbf{z}} \\
&= -\frac{qBR^2}{3c} \hat{\mathbf{z}} + \frac{qBR^2}{3c} \hat{\mathbf{z}} = 0.
\end{aligned} \tag{20}$$

The electric scalar potential is  $V = q/r$ , so the field angular momentum based on the Furry form of the field momentum is

$$\begin{aligned}
\mathbf{L}_F &= \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol} = \frac{3qB}{8\pi c} \int R\hat{\mathbf{r}} \times \frac{\sin\theta\hat{\phi}}{R} d\text{Area} \\
&= \frac{3qBR^2}{4c} \int_{-1}^1 d\cos\theta \sin^2\theta \hat{\mathbf{z}} = \frac{qBR^2}{c} \hat{\mathbf{z}}.
\end{aligned} \tag{21}$$

The forms  $\mathbf{L}_M$  and  $\mathbf{L}_P$  give the same result, as predicted in sec. 2.2 of [2],<sup>16</sup> where no argument was made as to whether the angular momentum computed from the Furry form,  $\mathbf{L}_F$ , of the field momentum would agree with the common value based on the Maxwell and Poynting forms (for examples with fields that fall off rapidly enough at large distances).<sup>17</sup>

### 2.3 Charge Not on the Axis of a Uniform Magnetic Field

This example encourages us to suppose that it is consistent to compute field angular momentum in static fields that fall off quickly enough with distance according to either of the forms

$$\mathbf{L}_M = \int \mathbf{r} \times \frac{\rho\mathbf{A}}{c} d\text{Vol}, \quad \text{or} \quad \mathbf{L}_P = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \tag{22}$$

while the Furry form should not be used. However, the Furry form does play a role, discussed in sec. 2.4.

It is tempting to favor the Maxwell form,  $\mathbf{L}_M$ , which appears simple to evaluate, as only the vector potential is needed. However, the vector potential in a region of uniform magnetic field is not unique:  $\mathbf{A} = (a_x + by)\hat{\mathbf{x}} + [a_y + (B + b)x]\hat{\mathbf{y}}$  leads to  $\mathbf{B} = B\hat{\mathbf{z}}$  for any values of

<sup>16</sup>The equivalence of  $\mathbf{L}_M$  and  $\mathbf{L}_P$  for well enough behaved fields was first argued by Trammel at the end of [10].

<sup>17</sup>Due to the additional factor of  $\mathbf{r}$  in the integrands for field angular momentum, compared to those of field momentum, the requirements on the falloff of the fields at larger distances are greater in the case of field angular momentum for the equivalence of different forms to hold. In particular, an infinite solenoid is well enough behaved for computations of field momentum via various forms, but not for computations of field angular momentum. For the latter, the Maxwell form  $\mathbf{L}_M$  appears to be the most robust.

the constants  $a_x$ ,  $a_y$  and  $b$ .<sup>18</sup> To make a computation of angular momentum via the vector potential it seems advisable to calculate the latter from the (static) current density  $\mathbf{J}$ ,

$$\mathbf{A} = \int \frac{\mathbf{J}}{cr} d\text{Vol}, \quad (23)$$

rather than using other forms that corresponds to the same magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$ . Then, the vector potential of a finite solenoid vanishes on the solenoid axis, and we infer that the angular momentum is zero for a charge at rest on the axis of a finite solenoid. Hence, use of the Poynting form (10) is unreliable if one uses the field of an infinite solenoid as an approximation for that of a finite solenoid.

The vector potential for a long solenoid is approximately given by eq. (5), so it appears that the (Maxwell) field angular momentum of charge  $q$  at rest at distance  $a$  from the solenoid axis (and  $|z_q| \ll l$ , where solenoid extends over the range  $-l/2 < z < l/2$ ), is approximately that given in eq. (9),

$$\mathbf{L}_M(a < R) \approx \frac{qBa^2}{2c} \hat{\mathbf{z}}, \quad \mathbf{L}_M(a > R) \approx \frac{qBR^2}{2c} \hat{\mathbf{z}}. \quad (24)$$

The field angular momentum is nonzero even when the charge is outside the solenoid where the magnetic field is very weak. This indicates that there exists some interaction between the charge and the magnetic field outside the solenoid.<sup>19</sup>

## 2.4 “Hidden” Momentum and Angular Momentum

### 2.4.1 Thomson’s Momentum Paradox

As part of his considerations of momentum in the electromagnetic field, J.J. Thomson (1904) discussed the example of an electric charge external to a short solenoid that can be represented as a magnetic dipole  $\mathbf{m}$  [8, 9]. He deduced that this system has electromagnetic field momentum,

$$\mathbf{P}_{EM} = \frac{\mathbf{E} \times \mathbf{m}}{c}, \quad (25)$$

where  $\mathbf{E}$  is the (approximately constant) electric field of the charge at the location of the solenoid, by computing both  $\mathbf{P}_M$  and  $\mathbf{P}_P$  (and approximating  $\mathbf{B}$  as zero outside the solenoid). He then argued on p. 348 of [8] that if the solenoid were demagnetized the stored electromagnetic momentum (25) should appear as mechanical momentum, but the form  $\mathbf{P}_M$  suggests that the momentum is associated with the solenoid, which should therefore move, while the form  $\mathbf{P}_P$  suggests that the momentum is associated with the charge, which should therefore move.

Thomson left this paradox unresolved.

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<sup>18</sup>The vector potential  $\mathbf{A} = Br_{\perp} \hat{\phi}/2$  corresponds to  $a_x = 0 = a_y$ ,  $b = -B/2$ .

<sup>19</sup>Such effects were considered by Lodge in 1889 [26]. For example, if there is initially zero current in the solenoid, an azimuthal electric field is induced outside the solenoid as the current rises, which imparts a mechanical angular momentum to a charge outside the solenoid with sign opposite to the field angular momentum of eq. (24), such that the total angular momentum remains zero.

The existence of this classical interaction between a long solenoid and a charge outside should make it less surprising that there is also an interesting quantum interaction in this case, as noted by Aharonov and Bohm [33, 34].

### 2.4.2 “Hidden” Mechanical Momentum

A resolution to this paradox emerged in 1967 when Shockley introduced the notion of “hidden” mechanical momentum [18], a consequence of which is that Thomson’s charge plus solenoid contains an unnoticed (“hidden”) amount of mechanical momentum that happens to be equal and opposite to the field momentum (25). So, if the current in the solenoid drops to zero, no mechanical momentum appears in the system and nothing moves.

An explanation of “hidden” mechanical momentum was given by Coleman and van Vleck [14] who argued (see Appendix C) that for stationary systems

$$\mathbf{P}_{\text{hidden}} = - \int \frac{V\mathbf{J}}{c^2} d\text{Vol} = -\mathbf{P}_F = -\mathbf{P}_{\text{EM}}, \quad (26)$$

such that  $\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{EM}} = \mathbf{P}_{\text{hidden}} + \mathbf{P}_{\text{EM}} = 0$ .

### 2.4.3 The Density of “Hidden” Mechanical Momentum

The success of eq. (26) in resolving Thomson’s paradox suggests that we identify the density of “hidden” mechanical momentum as

$$\mathbf{p}_{\text{hidden}} \stackrel{?}{=} - \frac{V\mathbf{J}}{c^2}. \quad (27)$$

which is an aspect of the moving charges that constitute the current density  $\mathbf{J}$ . Then, classical mechanics implies that associated with the density (27) a density of mechanical angular momentum with respect to the origin,

$$\mathbf{l}_{\text{hidden}} \stackrel{?}{=} - \mathbf{r} \times \frac{V\mathbf{J}}{c^2}, \quad (28)$$

such that the total “hidden” mechanical angular momentum is

$$\mathbf{L}_{\text{hidden}} \stackrel{?}{=} \int \mathbf{l}_{\text{hidden}} d\text{Vol} = - \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol} = -\mathbf{L}_F, \quad (29)$$

recalling eq. (4).

However, we encounter a difficulty with this prescription for the case of a charge plus solenoid (all “at rest”) in that sec. 2.1, particularly footnote 7, leads to the conclusion that there is no “hidden” mechanical angular momentum in this system.<sup>20</sup> This difficulty suggests that eq. (27) is not actually the density of “hidden” mechanical momentum.

We recall that Aharonov, Peale and Vaidman proposed a relation between “hidden” mechanical momentum in stationary systems and the stress tensor, in eqs. (10)-(12) of [32]. A generalization of this proposal that applies to nonstationary systems has been given in

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<sup>20</sup>In sec. 2.1 we found  $\mathbf{L}_F$  to be nonzero while the field angular momentum  $\mathbf{L}_{\text{EM}} = \mathbf{L}_P = \mathbf{L}_M$  is zero. Then we have

$$\mathbf{L}_{\text{total}} = \mathbf{L}_{\text{EM}} + \mathbf{L}_{\text{hidden}} \neq 0, \quad (30)$$

in contradiction to the argument of sec. 2.1.1 that  $\mathbf{L}_{\text{total}} = 0$  when the charge is on the solenoid axis.



[31], where a subsystem with a specified volume can interact with the rest of the system via bulk forces as well as contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\text{hidden}} \equiv - \int \frac{f^0}{c} (\mathbf{r} - \mathbf{r}_{\text{cm}}) d\text{Vol} = \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{r} - \mathbf{r}_{\text{cm}}) (\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \quad (31)$$

where<sup>21</sup>

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu} = \partial_\nu T^{\mu\nu}, \quad \partial_\nu = \left( \frac{\partial}{\partial ct}, \nabla \right), \quad (32)$$

is the 4-force density exerted on the subsystem by the rest of the system, with  $T_{\mu\nu}$  being the stress-energy-momentum 4-tensor of the subsystem,  $\mathbf{P}$  is the total momentum of the subsystem,  $M = U/c^2$  is its total “mass”,  $U$  is its total energy,  $\mathbf{r}_{\text{cm}}$  is its center of mass/energy,  $\mathbf{v}_{\text{cm}} = d\mathbf{r}_{\text{cm}}/dt$ ,  $\mathbf{p}$  is its momentum density,  $\rho = u/c^2$  is its “mass” density,  $u$  is its energy density,  $\mathbf{v}_b$  is the velocity (field) of its boundary.<sup>22</sup>

If a system is partitioned into subsystems that are “mechanical” and “electromagnetic, these subsystems have no boundary (except at infinity), so that for a system with all matter within a finite region, the “hidden” momenta of the two subsystems each has the form<sup>23</sup>

$$\mathbf{P}_{\text{hidden}} = \mathbf{P} - M\mathbf{v}_{\text{cm}}. \quad (33)$$

Note that the electromagnetic subsystem can have “hidden” momentum according to eq. (32), and for a stationary system, where  $\mathbf{v}_{\text{cm,EM}} = 0$ , the field momentum is to be called a “hidden” momentum. In contrast, the field momentum of a plane electromagnetic wave is not a “hidden” momentum.<sup>24</sup>

For an isolated, stationary system the center of mass/energy is at rest,  $\mathbf{r}_{\text{cm}}$  is constant, and the total momentum is zero [14]. According to eq. (31), such a system (as a whole) has no “hidden” momentum. But if the system is partitioned into (stationary) mechanical and electromagnetic subsystems, each can have nonzero “hidden” momentum, with the constraint that

$$\mathbf{P}_{\text{total}} = 0 = \mathbf{P}_{\text{hidden}}^{\text{mech}} + \mathbf{P}_{\text{hidden}}^{\text{EM}} \quad (\text{stationary}). \quad (34)$$

According to eqs. (31)-(32),

$$\mathbf{P}_{\text{hidden}}^{\text{EM}} = \mathbf{P}_{\text{EM}} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} \quad (\text{stationary}), \quad (35)$$

but the equivalent statement for the mechanical subsystem,  $\mathbf{P}_{\text{hidden,mech}} = \mathbf{P}_{\text{mech}}$ , is not immediately informative.<sup>25</sup> Instead, we consider the first form of eq. (31),

$$\mathbf{P}_{\text{hidden}}^{\text{mech}} = -\frac{1}{c} \int f_{\text{mech}}^0 (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{mech}}) d\text{Vol} \quad (\text{stationary}). \quad (36)$$

<sup>21</sup>We use the conventions of the Appendix to [35].

<sup>22</sup>Important qualifications are that the values of  $\mathbf{p}$  and  $\rho$  at the boundary are the limit of those just inside the boundary, and that the volume integral of  $f^0$  does not include possible delta-functions at the boundary.

<sup>23</sup>The form (33) was proposed for the “hidden” momentum of a nonstationary mechanical subsystem in eq. (78) of [30], but field momentum was taken to be “overt” there.

<sup>24</sup>See sec. 3.2 of [31].

<sup>25</sup>Many people appear to assume that  $\mathbf{P}_{\text{mech}} = 0$  for a stationary mechanical subsystem.

As noted in eq. (10) of [32], for a mechanical system (stationary or not) that interacts with an electromagnetic field described by the tensor  $\mathcal{F}^{\mu\nu}$ , the 4-force density on the mechanical subsystem due to the electromagnetic subsystem is the Lorentz force,

$$f_{\text{mech}}^\mu (= \partial_\nu T_{\text{mech}}^{\mu\nu}) = \mathcal{F}^{\mu\nu} J_\nu = \left( \frac{\mathbf{J} \cdot \mathbf{E}}{c}, \rho \mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B} \right), \quad f_0^{\text{mech}} = \frac{\mathbf{J} \cdot \mathbf{E}}{c}, \quad (37)$$

where the current density 4-vector is  $J_\mu = (c\rho, -\mathbf{J})$ . Thus the “hidden” momentum of the mechanical subsystem is<sup>26</sup>

$$\mathbf{P}_{\text{hidden}}^{\text{mech}} = - \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{mech}}) d\text{Vol}. \quad (38)$$

For a stationary system,  $\mathbf{E} = -\nabla V$  and  $\nabla \cdot \mathbf{J} = \partial^j J_j = 0$ , so

$$-\mathbf{J} \cdot \mathbf{E} = J_j \partial^j V = \partial^j (J_j V) - V \partial^j J_j = \partial^j (J_j V), \quad (39)$$

$$\begin{aligned} -\mathbf{J} \cdot \mathbf{E} r_i &= r_i J_j \partial^j V = \partial^j (r_i J_j V) - V \partial^j (r_i J_j) = \partial^j (r_i J_j V) - V J_j \partial^j r_i - V r_i \partial^j J_j \\ &= \partial^j (r_i J_j V) - V J_i, \end{aligned} \quad (40)$$

and

$$P_{\text{hidden},i}^{\text{mech}} = - \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} (r_i - r_{\text{cm},i}^{\text{mech}}) d\text{Vol} = - \int \frac{V J_i}{c^2} d\text{Vol} + \frac{1}{c^2} \int \partial^j [(r_i - r_{\text{cm},i}^{\text{mech}}) J_j V] d\text{Vol}. \quad (41)$$

The latter volume integral becomes a surface integral at infinity, so is zero for any bounded current density. Then,

$$\mathbf{P}_{\text{hidden}}^{\text{mech}} = - \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{mech}}) d\text{Vol} = - \int \frac{V \mathbf{J}}{c^2} d\text{Vol} = -\mathbf{P}_{\text{EM}} = -\mathbf{P}_{\text{hidden}}^{\text{EM}} \quad (\text{stationary}). \quad (42)$$

Thus, Thomson’s paradox of sec. 2.4.1 is resolved by the existence of “hidden” mechanical momentum according to eq. (31).

*Note that the argument leading to eq. (42) indicates that for stationary systems the “hidden” mechanical momentum does not depend on the location of  $\mathbf{r}_{\text{cm}}^{\text{mech}}$  (which could be varied arbitrarily by changing the charge of mass ratio of the various particles in the system),*

$$\mathbf{P}_{\text{hidden}}^{\text{mech}} = - \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} = - \int \frac{V \mathbf{J}}{c^2} d\text{Vol} = -\mathbf{P}_{\text{EM}} = -\mathbf{P}_{\text{hidden}}^{\text{EM}} \quad (\text{stationary}). \quad (43)$$

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<sup>26</sup>The “hidden” momentum for the electromagnetic subsystem (stationary or not) follows from  $f_{\text{EM}}^0 = \partial_\mu T_{\text{EM}}^{0\mu}$ , where  $T_{\text{EM}}^{0\mu} = (u_{\text{EM}}, c\mathbf{P}_{\text{EM}}) = (u_{\text{EM}}, \mathbf{S}/c)$ , and  $\mathbf{S}$  is the Poynting vector. Then,  $f_{\text{EM}}^0 = \partial u_{\text{EM}}/\partial ct + \nabla \cdot \mathbf{S}/c = \partial u_{\text{EM}}/\partial ct - \partial u_{\text{EM}}/\partial ct - \mathbf{J} \cdot \mathbf{E}/c = -\mathbf{J} \cdot \mathbf{E}/c$ , and  $\mathbf{P}_{\text{hidden}}^{\text{EM}} = \int \mathbf{J} \cdot \mathbf{E} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{EM}}) d\text{Vol}/c^2$  for any isolated system that is partitioned into electromagnetic and mechanical subsystems. If  $\int \mathbf{J} \cdot \mathbf{E} d\text{Vol} = 0$  (as will be seen below to hold for stationary systems), or  $\mathbf{r}_{\text{cm}}^{\text{EM}} = \mathbf{r}_{\text{cm}}^{\text{mech}}$ , then  $\mathbf{P}_{\text{hidden}}^{\text{EM}} = -\mathbf{P}_{\text{hidden}}^{\text{mech}}$ .

Note that for “free” fields, where  $\mathbf{J} = 0$  everywhere, the “hidden” electromagnetic momentum vanishes. That is, in general  $\mathbf{P}_{\text{hidden}}^{\text{EM}} \neq \mathbf{P}_{\text{EM}}$  (for example, see [36]), although they are equal for stationary systems.

Hence, we now have a fourth form for the field momentum of a finite, stationary system, which we call the Aharonov form,

$$\mathbf{P}_{\text{EM}} = \mathbf{P}_A = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} \quad (\text{stationary}). \quad (44)$$

The success of the form (42) for the “hidden” momentum of the mechanical subsystem suggests that we consider the merits of defining the density of “hidden” mechanical momentum in a stationary system to be

$$\mathbf{P}_{\text{hidden}}^{\text{mech}} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c^2} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{mech}}) \quad (\text{stationary}). \quad (45)$$

#### 2.4.4 “Hidden” Angular Momentum

According to eq. (45), the density of “hidden” mechanical angular momentum for the mechanical subsystem of a stationary system is<sup>27</sup>

$$\begin{aligned} \mathbf{I}_{\text{hidden}}^{\text{mech}} &= \mathbf{r} \times \mathbf{P}_{\text{hidden}}^{\text{mech}} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} \times (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{mech}}) = \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} \times \mathbf{r}_{\text{cm}}^{\text{mech}} \\ &= \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{mech}}) \times \mathbf{r}_{\text{cm}}^{\text{mech}} = \mathbf{r}_{\text{cm}}^{\text{mech}} \times \mathbf{P}_{\text{hidden}}^{\text{mech}} \quad (\text{stationary}). \end{aligned} \quad (46)$$

In this view, the total “hidden” mechanical angular momentum for a stationary system is

$$\mathbf{L}_{\text{hidden}}^{\text{mech}} = \int \mathbf{I}_{\text{hidden}}^{\text{mech}} d\text{Vol} = \mathbf{r}_{\text{cm}}^{\text{mech}} \times \int \mathbf{P}_{\text{hidden}}^{\text{mech}} d\text{Vol} = \mathbf{r}_{\text{cm}}^{\text{mech}} \times \mathbf{P}_{\text{hidden}}^{\text{mech}}, \quad (47)$$

where  $\mathbf{r}_{\text{cm}}^{\text{mech}}$  is the center of mass/energy of the mechanical subsystem.<sup>28,29,30</sup>

For the example of a solenoid plus charge at rest, where the mass of the solenoid is much larger than the mass of the charge,  $\mathbf{r}_{\text{cm}}^{\text{mech}}$  is approximately at the center of the solenoid, which we take to be the origin. That is,  $\mathbf{r}_{\text{cm}}^{\text{mech}} \approx 0$ , and so  $\mathbf{L}_{\text{hidden}}^{\text{mech}} \approx 0$ , according to eq. (47), independent of the location of the charge. This agrees with the inference in footnote 7 of

<sup>27</sup>If we inferred from eq. (43) that the density of “hidden” mechanical angular momentum were  $\mathbf{r} \times -(\mathbf{J} \cdot \mathbf{E}) \mathbf{r}/c^2 = 0$ , there would be no “hidden” mechanical angular momentum in any stationary system.

<sup>28</sup>The result (47) is similar to that of eq. (26) of [23]. As the result is zero when measured about the center of mass/energy of the mechanical subsystem, we could say that the “hidden” mechanical angular momentum is “orbital” rather than “intrinsic”, where the latter is with respect to the center of mass/energy.

<sup>29</sup>The existence of nonzero “hidden” mechanical angular momentum when  $\mathbf{r}_{\text{cm}}^{\text{mech}}$  is not at the origin is the key to the resolution of Mansuripur’s paradox [37].

<sup>30</sup>For example, the “hidden” mechanical angular momentum of a charge  $q$  of mass  $M_q$  at distance  $\mathbf{r}_q$  from a point, Ampèrian magnetic dipole  $\mathbf{m}$  of mass  $M_m$ , all at rest, is, about the location of moment  $\mathbf{m}$ ,

$$\mathbf{L}_{\text{hidden}}^{\text{mech}} = \frac{M_q}{M_q + M_m} \mathbf{r}_q \times \left( \frac{\mathbf{m}}{c} \times \frac{-q\mathbf{r}_q}{r_q^3} \right), \quad \text{as [13]} \quad \mathbf{P}_{\text{hidden}}^{\text{mech}} = -\mathbf{P}_{\text{EM}} = \frac{\mathbf{m} \times \mathbf{E}_q}{c}. \quad (48)$$

sec. 2.1.1 that the “hidden” mechanical angular momentum of a charge + solenoid is zero for any location of the charge.<sup>31</sup>

Hence, it appears that the definition (31) of “hidden” momentum of a (sub)system leads to a satisfactory identification of its density as

$$\mathbf{P}_{\text{hidden}}^{\text{subsystem}} = -\frac{f_{\text{subsystem}}^0}{c} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{subsystem}}) = -\frac{\partial_{\mu} T_{\text{subsystem}}^{0\mu}}{c} (\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{subsystem}}), \quad (49)$$

which can be written as eq. (45) for the mechanical subsystem of a stationary system.<sup>32</sup>

## 2.5 Quasistatic Systems

The above results were established using three conditions of stationary electromagnetic systems,  $\mathbf{E} = -\nabla V$ ,  $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c$  and  $\nabla \cdot \mathbf{J} = 0$ . Then, the electromagnetic momentum, which is equal and opposite to the “hidden” mechanical momentum, is of order  $1/c^2$ , which is very small compared to typical mechanical momenta. Only in isolated, stationary systems, for which  $\mathbf{P}_{\text{total}} = 0$ , are these tiny momenta relatively prominent.

In nonstationary systems, the electromagnetic and “hidden” mechanical momentum can have terms of order  $1/c^3$  (and higher). For **quasistatic** systems with all velocities small compared to  $c$  (and all accelerations small compared to  $c^2/R$  where  $R$  is the characteristic size of the system) these higher-order terms will be negligible, and it is a good approximation to consider the electromagnetic and “hidden” mechanical momentum only at order  $1/c^2$ , for which the results of secs. 2.1-2.4 can be used.<sup>33,34,35</sup>

While the field momentum (= “hidden” field momentum) of a quasistatic system can be computed using any of four volume densities, it turns out that the field angular momentum can only be computed in general using two of those densities, of which  $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$  is often considered to be “the” density of field angular momentum,

$$\mathbf{L}^{\text{EM}} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \mathbf{r} \times \frac{\rho \mathbf{A}}{c} d\text{Vol} \neq \int \mathbf{r} \times \frac{V \mathbf{J}}{c^2} d\text{Vol} \neq \mathbf{r}_{\text{cm}}^{\text{EM}} \times \mathbf{P}^{\text{EM}} = \mathbf{L}_{\text{hidden}}^{\text{EM}}. \quad (50)$$

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<sup>31</sup>It remains that the charge-to-mass ratio of the particles in a stationary system could be altered to change the location of  $\mathbf{r}_{\text{cm}}^{\text{mech}}$  without any change to the field momentum, the “hidden” mechanical momentum or the field angular momentum, but the “hidden” mechanical angular momentum would change (even in models of electrical-current filaments as due to counterflows of opposite charges where the charge-to-mass ratio is the same along any particular filament).

<sup>32</sup>It is noteworthy that eq. (49) indicates that the local density of “hidden” momentum in any subsystem is directed radially from the center of mass/energy of that subsystem.

<sup>33</sup>The departure of the electric field  $\mathbf{E} = -\nabla V - \partial \mathbf{A} / \partial ct$  from the form  $\mathbf{E} = -\nabla V$  for stationary systems is of order  $v/c$ , which has negligible effect on the electromagnetic and “hidden” mechanical momentum in the quasistatic approximation.

Similarly, motion of charges leads to nonzero time dependence of the electric field,  $\partial \mathbf{E} / \partial t$ , and of the charge density,  $\partial \rho / \partial t$ , at order  $v/c$ . The resulting “corrections” to the Maxwell equation  $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c + \partial \mathbf{E} / \partial ct$  and  $\nabla \cdot \mathbf{J} = -\partial \rho / \partial t$  due to the time derivatives are at one order higher in  $1/c$ , and so negligible effect on the electromagnetic and “hidden” mechanical momentum in the quasistatic approximation.

<sup>34</sup>See also [39].

<sup>35</sup>The quasistatic approximation consider here is looser than the so-called Darwin approximation [38] in which terms of order  $1/c^2$  in the electric field are considered, leading to terms of order  $1/c^4$  in field momentum and “hidden” momentum. The present quasistatic approximation encompasses the approximation of the fields by their instantaneous values, as well as effects of induction and of the displacement current for low frequencies.

The equality of the first two forms in eq. (50) was first demonstrated in [10]. That these forms differ from  $\int \mathbf{r} \times \frac{\nabla \mathbf{J}}{c^2} d\text{Vol}$  is shown in Appendix B. For a solenoid + single charge, the fourth form,  $\mathbf{L}_{\text{hidden}}^{\text{EM}}$ , equals the first two forms only when the charge is on the solenoid axis.<sup>36</sup>

## Appendix A: Field Momentum for an Infinite Solenoid plus Charge

Although there is a general theorem that  $\mathbf{P}_M = \mathbf{P}_P = \mathbf{P}_F$  for bounded, stationary systems [2], we illustrate this equality happens to hold for an infinite solenoid plus charge at rest.

For an electric charge  $q$  at distance  $a$  from the axis of an infinite solenoid of radius  $R$ , the field momentum based on the Maxwell form  $\mathbf{P}_M$  is, recalling eq. (5),

$$\mathbf{P}_M = \int \frac{\rho \mathbf{A}}{c} d\text{Vol} = \frac{qB}{2c} \hat{\phi} \begin{cases} a & (a < R), \\ \frac{R^2}{a} & (a > R). \end{cases} \quad (51)$$

For charge  $q$  at  $(x, y, z) = (a, 0, 0)$  the electric field is  $\mathbf{E}(x, y, z) = q \mathbf{d}/d^3 = q[(x-a)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]/[(x-a)^2 + y^2 + z^2]^{3/2}$ , so the field momentum based on the Poynting form  $\mathbf{P}_P$  is (prob. 8.14 of [28])

$$\begin{aligned} \mathbf{P}_P &= \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \frac{qB}{4\pi c} \int \frac{(x-a)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{[(x-a)^2 + y^2 + z^2]^{3/2}} \times \hat{\mathbf{z}} d\text{Vol} \\ &= \frac{qB}{4\pi c} \int \frac{y\hat{\mathbf{x}} - (x-a)\hat{\mathbf{y}}}{[(x-a)^2 + y^2 + z^2]^{3/2}} dx dy dz = \frac{qB}{2\pi c} \hat{\mathbf{y}} \int \frac{a-x}{(x-a)^2 + y^2} dx dy \\ &= \frac{qB}{2\pi c} \hat{\mathbf{y}} \int_0^R \rho d\rho \int_0^{2\pi} d\phi \frac{a - \rho \cos \phi}{\rho^2 - 2a\rho \cos \phi + a^2} = \frac{qB}{2\pi c} \hat{\mathbf{y}} \int_0^R d\rho \int_0^{2\pi} d\phi \frac{\frac{a}{\rho} - \cos \phi}{1 - 2\frac{a}{\rho} \cos \phi + \frac{a^2}{\rho^2}}. \end{aligned} \quad (52)$$

For  $a > R \geq \rho$  the remaining integral is (Dwight 859.113, 859.122, 141.1 and 143.1)

$$\mathbf{P}_P(a > R) = \frac{qB}{ac} \hat{\mathbf{y}} \int_0^R d\rho \frac{a^2 \rho - \rho^3}{a^2 - \rho^2} = \frac{qBR^2}{2ac} \hat{\mathbf{y}}. \quad (53)$$

For  $a < R$  the part of the integral for  $a > \rho$  has the form of eq. (53) but with  $R \rightarrow a$ , and the remaining integral is

$$\frac{qB}{c} \hat{\mathbf{y}} \int_a^R d\rho \left( \frac{a\rho}{\rho^2 - a^2} - \frac{a\rho}{\rho^2 - a^2} \right) = 0. \quad (54)$$

That is,

$$\mathbf{P}_P(a < R) = \frac{qBa}{2c} \hat{\mathbf{y}}, \quad (55)$$

and since  $\hat{\mathbf{y}} = \hat{\phi}$  at  $(a, 0, 0)$ ,  $\mathbf{P}_P = \mathbf{P}_M$  for an infinite solenoid (although  $\mathbf{L}_P \neq \mathbf{L}_M$ ).

For the field momentum based on the Furry form  $\mathbf{P}_F$  we consider a solenoid of half length  $l$ . Then, the electric scalar potential for charge  $q$  at  $(a, 0, 0)$  is<sup>37</sup>

$$V = \frac{q}{r} = \frac{q}{(\rho^2 - 2a\rho \cos \phi + a^2 + z^2)^{1/2}}, \quad (56)$$

<sup>36</sup>In computing  $\mathbf{r}_{\text{cm}}^{\text{EM}} = \int U^{\text{EM}} \mathbf{r} d\text{Vol} / \int U^{\text{EM}} d\text{Vol}$ , the field energy  $U^{\text{EM}}$  does not include the electromagnetic self energy of charges. For the case of a solenoid + single charge (at rest) the field energy is just the magnetic energy, whose center is the center of the solenoid, *i.e.*, the origin, so  $\mathbf{r}_{\text{cm}}^{\text{EM}} = 0$  and  $\mathbf{L}_{\text{hidden}}^{\text{EM}} = 0$  for any location of the charge.

<sup>37</sup>The form (56) applies everywhere only if the conductor of the solenoid does not “shield” the electric field of the charge. This is consistent with the model of the solenoid currents as due to counterrotating, opposite charges in nonconducting tubes, or to a pair of counterrotating, oppositely charged cylinders.

so the Furry momentum for the solenoid plus charge is

$$\begin{aligned}
\mathbf{P}_F &= \int \frac{V\mathbf{J}}{c^2} d\text{Vol} = \frac{qB}{4\pi c} \int \frac{1}{(R^2 + z^2)^{1/2}} d\text{Area} \hat{\phi} = \frac{qB}{4\pi c} \int_0^{2\pi} d\phi \int_0^l dz \frac{-\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi}{\sqrt{R^2 - 2aR \cos \phi + a^2 + z^2}} \\
&= \frac{qB}{4\pi c} \hat{\mathbf{y}} \int_0^{2\pi} d\phi \int_0^l dz \frac{\cos \phi}{\sqrt{R^2 - 2aR \cos \phi + a^2 + z^2}} \\
&= \frac{qB}{2\pi c} \hat{\mathbf{y}} \int_0^{2\pi} d\phi \cos \phi (R^2 - 2aR \cos \phi + a^2) \sinh^{-1} \frac{l}{\sqrt{R^2 - 2aR \cos \phi + a^2}}. \tag{57}
\end{aligned}$$

While it may seem unlikely that this result is finite and independent of the half length  $l$  of the solenoid as this grows large, there is a general theorem that  $\mathbf{P}_M = \mathbf{P}_P = \mathbf{P}_F$  for bounded, stationary systems. Then, since the result (51) holds for the charge at radius  $a$  small compared to the half length  $l$  of the solenoid, we infer that  $\mathbf{P}_M = \mathbf{P}_P = \mathbf{P}_F$  also holds for an infinite solenoid.<sup>38</sup>

## Appendix B: General Relations between $\mathbf{L}_P$ , $\mathbf{L}_F$ and $\mathbf{L}_M$

This Appendix is due to V. Hnizdo.

For stationary systems where  $\mathbf{E} = -\nabla V$ , *i.e.*,  $E_i = -\partial_i V$ , and  $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c$ ,

$$\begin{aligned}
[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &= \epsilon_{ijk} r_j \epsilon_{klm} (-\partial_l V) B_m = -\epsilon_{ijk} \epsilon_{klm} r_j [\partial_l (V B_m) - V \partial_l B_m] \\
&= -\epsilon_{ijk} \epsilon_{klm} [\partial_l (r_j V B_m) - V B_m \partial_l r_j] + \epsilon_{ijk} V r_j (\nabla \times \mathbf{B})_k \\
&\rightarrow \epsilon_{ijk} \epsilon_{klm} V B_m \delta_{jl} + \frac{4\pi}{c} \epsilon_{ijk} V r_j J_k = -\epsilon_{ijk} \epsilon_{mjk} V B_m + \frac{4\pi V}{c} (\mathbf{r} \times \mathbf{J})_i \\
&= -\delta_{ij}^{mj} V B_m + \frac{4\pi}{c} (\mathbf{r} \times V\mathbf{J})_i = -2\delta_{im} V B_m + \frac{4\pi}{c} (\mathbf{r} \times V\mathbf{J})_i \\
&= -2V B_i + \frac{4\pi}{c} (\mathbf{r} \times V\mathbf{J})_i, \tag{59}
\end{aligned}$$

where in the third line we drop the full-differential term  $\partial_l (r_j V B_m)$ , anticipating that  $\int \partial_l (r_j V B_m) d\text{Vol} \rightarrow \oint r_j V B_m d\text{Area} \rightarrow 0$  for fields that fall off sufficiently quickly at large distances. Then, for bounded, stationary systems, where  $\mathbf{L}_P = \mathbf{L}_M$ ,

$$\mathbf{L}_P = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = - \int \frac{V\mathbf{B}}{2\pi c} d\text{Vol} + \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol} = \mathbf{L}_F - \int \frac{V\mathbf{B}}{2\pi c} d\text{Vol}. \tag{60}$$

That is, the Poynting and Furry forms of field angular momentum are not equal, in general.<sup>39</sup>

Following [10], we can also establish a relation between  $\mathbf{L}_P$  and  $\mathbf{L}_M$  by starting with the vector-calculus identity,

$$\nabla(\mathbf{E} \cdot \mathbf{A}) = (\mathbf{E} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{E} + \mathbf{E} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{E}), \tag{61}$$

where the last term vanishes for stationary systems where  $\nabla \times \mathbf{E} = 0$ . Then, since  $\mathbf{B} = \nabla \times \mathbf{A}$ , we have that

$$\mathbf{E} \times \mathbf{B} = \nabla(\mathbf{E} \cdot \mathbf{A}) - (\mathbf{E} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{E}. \tag{62}$$

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<sup>38</sup>For  $a = 0$ , *i.e.*, for the charge on the axis,

$$\mathbf{P}_F(a = 0) = \frac{qBR^2}{2\pi c} \sinh^{-1} \frac{l}{R} \hat{\mathbf{y}} \int_0^{2\pi} d\phi \cos \phi = 0, \tag{58}$$

which does equal  $\mathbf{P}_M$  and  $\mathbf{P}_P$  for any length  $l$  in this special case.

<sup>39</sup>For stationary fields that fall off sufficiently quickly at large distance,  $\int V\mathbf{B} d\text{Vol} = \int \mathbf{E} \times \mathbf{A} d\text{Vol}$ .

Now,

$$(\mathbf{E} \cdot \nabla)A_i = E_j \partial_j A_i = \partial_j (E_j A_i) - A_i \partial_j E_j = \partial_j (E_j A_i) - 4\pi\rho A_i, \quad (63)$$

using the first Maxwell equation. Similarly, in the Coulomb gauge, where  $\nabla \cdot \mathbf{A} = 0$ ,

$$(\mathbf{A} \cdot \nabla)E_i = A_j \partial_j E_i = \partial_j (A_j E_i) - E_i \partial_j A_j = \partial_j (A_j E_j). \quad (64)$$

Thus,

$$(\mathbf{E} \times \mathbf{B})_i = 4\pi\rho A_i + \partial_i(\mathbf{E} \cdot \mathbf{A}) - \partial_j(E_j A_i) - \partial_j(A_j E_j). \quad (65)$$

When integrating eq. (65) over all space, the terms beginning with derivatives can be integrated parts, and vanish for fields that fall off suitably quickly. Hence,

$$\mathbf{P}_P = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol} = \mathbf{P}_M, \quad (66)$$

Likewise, in the integral of  $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ , we use eq. (65), and again, integration by parts of the terms involving derivatives indicates that these terms vanish (for fields that fall off suitably quickly), and

$$\mathbf{L}_P = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \mathbf{r} \times \frac{\rho \mathbf{A}}{c} d\text{Vol} = \mathbf{L}_M. \quad (67)$$

## Appendix C: The Argument of Coleman and Van Vleck

An explanation of “hidden” mechanical momentum was given by Coleman and van Vleck [14], who remarked in footnote 9 that an electric charge  $q$  of rest mass  $m$  and velocity  $\mathbf{v}$  that moves in an electric field  $\mathbf{E} = -\nabla V - \partial \mathbf{A} / \partial ct$  has the conserved energy<sup>40</sup>

$$U = \gamma mc^2 + qV, \quad (68)$$

where  $\gamma = 1/\sqrt{1 - v^2/c^2}$ . If in addition the moving charge is part of a current density  $\mathbf{J}$  that is stationary (which implies that the “conductor” which supports the current density is “at rest”), then  $\mathbf{A}$  is constant in time,  $\mathbf{E} = -\nabla V$ , and

$$\int \mathbf{J} d\text{Vol} = 0, \quad (69)$$

and the current density at a point  $\mathbf{x}$  can be written as

$$\mathbf{J}(\mathbf{x}) = n(\mathbf{x})q\mathbf{v}(\mathbf{x}), \quad (70)$$

where  $n(\mathbf{x})$  is the volume number density of like charges  $q$  in a small region around  $\mathbf{x}$ . The mechanical momentum associated with the current density  $\mathbf{J}$  is

$$\mathbf{P}_{\text{mech}} = \int n(\mathbf{x})\gamma(\mathbf{x})m\mathbf{v}(\mathbf{x}) d\text{Vol} = \frac{m}{q} \int n(\mathbf{x})\gamma(\mathbf{x})q\mathbf{v}(\mathbf{x}) d\text{Vol} = \frac{m}{q} \int \gamma(\mathbf{x})\mathbf{J}(\mathbf{x}) d\text{Vol} \quad (71)$$

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<sup>40</sup>The energy conservation (68) ignores gravity, and supposes that the only force that can act on an electric charge is the Lorentz force  $q(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$ . Furthermore, the scalar potential  $V$  is the microscopic potential, which is not necessarily the same as the potential due to charges external to the current density.

The arguments of this section apply to the model of the current density in the solenoid as due to counterflowing, noninteracting, opposite charges with potential  $V$  due to external charges, but for the model of counterrotating cylinders of charge, the potential includes a microscopic electric potential that keeps the charges fixed with respect to the rotating cylinders in the presence of the external electric field. This issue was noted in [32], where it was remarked that there is a “hidden” mechanical momentum due to the stresses in the rotating cylinders.

If the electric scalar potential  $V$  were zero everywhere, then  $\mathbf{E} = 0$ , the speed  $v$  of the charges could not vary with position,  $\gamma(\mathbf{x}) = \gamma_0$ , a constant, and the energy of each charge is just  $U = \gamma_0 mc^2$ . Thus,

$$\mathbf{P}_{\text{mech}}(V = 0) = \frac{\gamma_0 m}{q} \int \mathbf{J}(\mathbf{x}) d\text{Vol} = 0. \quad (72)$$

When a (stationary) electric field is present, the scalar potential varies with position and energy conservation (68) implies that

$$\gamma_0 mc^2 = \gamma(\mathbf{x}) mc^2 + qV(\mathbf{x}), \quad (73)$$

such that

$$\begin{aligned} \mathbf{P}_{\text{mech}} &= \frac{m}{q} \int n(\mathbf{x}) \gamma(\mathbf{x}) q \mathbf{v}(\mathbf{x}) d\text{Vol} = \frac{m}{q} \int n(\mathbf{x}) \left( \gamma_0 - \frac{qV}{mc^2} \right) q \mathbf{v}(\mathbf{x}) d\text{Vol} \\ &= \frac{\gamma_0 m}{q} \int \mathbf{J} d\text{Vol} - \int \frac{V \mathbf{J}}{c^2} d\text{Vol} = - \int \frac{V \mathbf{J}}{c^2} d\text{Vol} \equiv \mathbf{P}_{\text{hidden}}. \end{aligned} \quad (74)$$

The result (74) is the “hidden” mechanical momentum postulated by Shockley.

Furthermore, comparing with eq. (1), which applies for stationary systems, we have that in such systems,<sup>41</sup>

$$\mathbf{P}_{\text{EM}} = -\mathbf{P}_{\text{hidden}}, \quad \mathbf{P}_{\text{total}} = \mathbf{P}_{\text{EM}} + \mathbf{P}_{\text{hidden}} = 0. \quad (75)$$

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<sup>41</sup>The Furry form  $\mathbf{P}_F$  is identical to the “hidden” mechanical momentum (74) except for the overall sign. The result of Furry was obtained by transforming the Maxwell form  $\mathbf{P}_F$  of field momentum, without any consideration of mechanical momentum. Furry’s paper [13] was published one year after the paper of Coleman and van Vleck [14], and although Furry referenced the latter paper he did not notice that his argument starting on p. 634 was anticipated in footnote 9 of [14].



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