

Torque Analyses of a Sliding Ladder

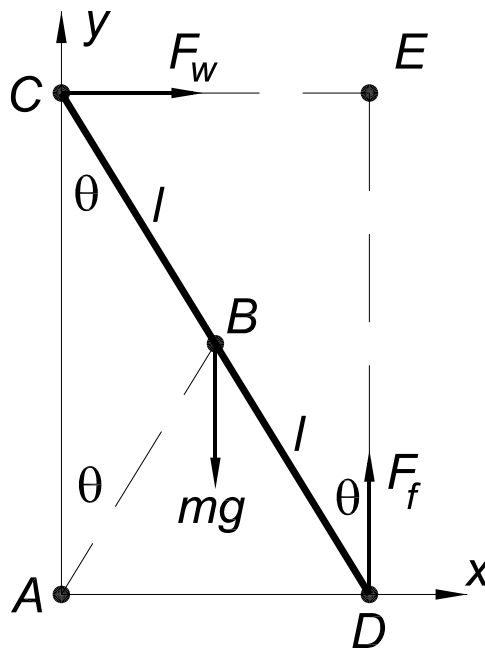
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1 Problem

The problem of a ladder that slides without friction while touching a floor and wall is often used to illustrate Lagrange's method for deducing the equation of motion of a mechanical system. Suppose the ladder has mass m , length $2l$, and makes angle θ to the vertical. Deduce the equation of motion via a torque analysis about each of the five points A , B , C , D and E as shown in the figure below. All of these points except A are accelerating in the lab frame.



The usual statement of the ladder problem is to ask at what angle θ does the ladder lose contact with the vertical wall if it starts from rest at $\theta = 0$ (and the bottom of the ladder is given a tiny horizontal velocity).

2 Solution¹

2.1 Lagrange's Method

We use θ as the single generalized coordinate. The center of mass of the ladder is at point B and moves in a circle of radius l about the fixed point A . The moment of inertia of the

¹A solution to this problem that avoids use of the equation of motion is given at <http://www.physics.harvard.edu/academics/undergrad/probweek/sol147.pdf>

ladder about its center of mass is

$$I_{\text{cm}} = \frac{ml^2}{3}. \quad (1)$$

The kinetic energy of the sliding ladder consists of the kinetic energy of the motion of the center of mass plus the kinetic energy of rotation about the center of mass,

$$T = \frac{mv_{\text{cm}}^2}{2} + \frac{I_{\text{cm}}\dot{\theta}^2}{2} = \frac{m(l\dot{\theta})^2}{2} + \frac{ml^2\dot{\theta}^2}{6} = \frac{2ml^2\dot{\theta}^2}{3}. \quad (2)$$

The gravitational potential energy of the ladder relative to the floor is

$$V = mgl \cos \theta. \quad (3)$$

The equation of motion of the ladder follows from Lagrange's equation,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}, \quad (4)$$

where the Lagrangian is $\mathcal{L} = T - V$. From eqs. (2)-(4) we find that

$$\ddot{\theta} = \frac{3g}{4l} \sin \theta. \quad (5)$$

The ladder loses contact with the vertical wall when (horizontal) contact force F_w vanishes. This contact force causes the horizontal acceleration of the center of mass,

$$F_w = ma_x. \quad (6)$$

The x and y coordinates of the center of mass (so long as the ladder remains in contact with the wall) are

$$x_{\text{cm}} = l \sin \theta, \quad y_{\text{cm}} = l \cos \theta, \quad (7)$$

so the acceleration of the center of mass has components

$$a_x = \ddot{x}_{\text{cm}} = l \cos \theta \ddot{\theta} - l \sin \theta \dot{\theta}^2, \quad a_y = \ddot{y}_{\text{cm}} = -l \sin \theta \ddot{\theta} - l \cos \theta \dot{\theta}^2. \quad (8)$$

The angular velocity $\dot{\theta}$ of the ladder follows from conservation of energy,

$$mgl = T + V, \quad (9)$$

so that²

$$\dot{\theta}^2 = \frac{3g}{2l}(1 - \cos \theta). \quad (10)$$

The ladder loses contact with the vertical wall when $F_w = ma_x$ vanishes, which occurs when $\cos \theta = 2/3$, using eqs. (5), (8) and (10).

²The equation of motion (5) can also be deduced by taking the time derivative of eq. (10).

2.2 Torque Analysis About Point A

A Newtonian approach to the equation of motion of the sliding ladder is based on a torque analysis about some point. If that point is at rest, the torques are due only to the forces that are apparent in the lab frame. However, if the point is accelerating, there are additional torques due to apparent forces associated with the use of an accelerated coordinate system.

First, we consider an analysis about point A , which is fixed in the lab frame.

The torque equation is

$$\frac{dL_A}{dt} = \tau_A = 2F_f l \sin \theta - mgl \sin \theta - 2F_w l \cos \theta, \quad (11)$$

where L_A is the angular momentum of the ladder about point A , and a torque is positive if its vector points along the $+z$ axis in a right-handed coordinate system.

The subtlety of the torque analysis about point A (and about point F) is that the rotation of the ladder is not a rigid body rotation about this point, so the angular momentum is not the product of the momentum of inertia about point A times the angular velocity $\dot{\theta}$.

In general, the angular momentum of a rigid body with respect to a point equals the angular momentum of the center of mass motion with respect to that point, plus the angular momentum of the body relative to the center of mass.

The angular momentum of the center of mass motion of the ladder relative to point A is $-ml^2 \dot{\theta}$, while the angular momentum of the ladder relative to the center of mass is $I_{\text{cm}} \dot{\theta} = ml^2 \dot{\theta}/3$, recalling eq. (1) and noting that the senses of these two rotations are opposite. Thus, the total angular momentum of the ladder about point A is

$$L_A = -ml^2 \dot{\theta} + \frac{ml^2 \dot{\theta}}{3} = -\frac{2ml^2 \dot{\theta}}{3}. \quad (12)$$

We note that the moment of inertia of the ladder about point A (as well as that about points C , D and E) follows from the parallel axis theorem,

$$I_A = I_C = I_D = I_E = I_{\text{cm}} + ml^2 = \frac{4}{3}ml^2. \quad (13)$$

But this moment of inertia is relevant only if the rotation of the center of mass about the point of reference and the rotation with respect to the center of mass are equal in magnitude and sign. In the present example this is true for points C and D but not points A and E .

As previously discussed, the horizontal force F_w equals horizontal acceleration of the center of mass, so that

$$F_w = ma_x = ml \cos \theta \ddot{\theta} - ml \sin \theta \dot{\theta}^2. \quad (14)$$

Similarly, the vertical force F_f is related to the vertical acceleration of the center of mass according to $F_f - mg = ma_y$, so that

$$F_f = mg + ma_y = mg - ml \sin \theta \ddot{\theta} - ml \cos \theta \dot{\theta}^2. \quad (15)$$

Combining the torque equation (11) with eqs. (12), (14) and (15) we find

$$\begin{aligned} -\frac{2}{3}ml^2 \ddot{\theta} &= 2mgl \sin \theta - 2ml^2 \sin^2 \theta \ddot{\theta} - 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &\quad - mgl \sin \theta - 2ml^2 \cos^2 \theta \ddot{\theta} - 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &= mgl \sin \theta - 2ml^2 \ddot{\theta}, \end{aligned} \quad (16)$$

which leads to the equation of motion (5).

2.3 Torque Analysis about Point B

Points B - E are accelerating, so the torque analyses about these points must include the effect of the “coordinate” force

$$\mathbf{F}_P = -m\mathbf{a}_P \quad (17)$$

that appears to act on the center of mass according to an observer at a point P that is accelerating with respect to the lab frame [1].

However, if the point of reference is the center of mass of the system, the “coordinate” force (17) causes no torque, so a torque analysis about the center of mass has the same form whether or not the center of mass is accelerating.

Point B is the center of mass of the ladder. The angular momentum about point B is therefore

$$L_B = I_{\text{cm}} \dot{\theta}, \quad (18)$$

and the torque equation is

$$\begin{aligned} \frac{dL_B}{dt} = \frac{ml^2 \ddot{\theta}}{3} &= \tau_B = F_f l \sin \theta - F_w l \cos \theta \\ &= mgl \sin \theta - ml^2 \sin^2 \theta \ddot{\theta} - ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &\quad - ml^2 \cos^2 \theta \ddot{\theta} + ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &= mgl \sin \theta - ml^2 \ddot{\theta}, \end{aligned} \quad (19)$$

which again leads to the equation of motion (5).

2.4 Torque Analysis About Point C

Point C has coordinates $(0, 2y_{\text{cm}})$ so the “coordinate” force associated with taking this point as our reference point for a torque analysis is

$$\mathbf{F}_C = (0, -2m\ddot{y}_{\text{cm}}) = (0, 2ml \sin \theta \ddot{\theta} + 2ml \cos \theta \dot{\theta}^2), \quad (20)$$

recalling eq. (8).

Since point C is located on the ladder, the angular momentum about point C is simply

$$L_C = I_C \dot{\theta} = \frac{4ml^2 \dot{\theta}}{3}, \quad (21)$$

recalling eq. (13). The torque analysis about point C is then

$$\begin{aligned} \frac{dL_C}{dt} = \frac{4ml^2 \ddot{\theta}}{3} &= \tau_C = -mgl \sin \theta + 2F_f l \sin \theta + F_{C,y} l \sin \theta \\ &= -mgl \sin \theta \\ &\quad + 2mgl \sin \theta - 2ml^2 \sin^2 \theta \ddot{\theta} - 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &\quad + 2ml^2 \sin^2 \theta \ddot{\theta} + 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &= mgl \sin \theta, \end{aligned} \quad (22)$$

which again leads to the equation of motion (5).

2.5 Torque Analysis About Point D

Point D has coordinates $(2x_{\text{cm}}, 0)$ so the “coordinate” force associated with taking this point as our reference point for a torque analysis is

$$\mathbf{F}_D = (-2m\ddot{x}_{\text{cm}}, 0) = (-2ml \cos \theta \ddot{\theta} + 2ml \sin \theta \dot{\theta}^2, 0), \quad (23)$$

recalling eq. (8).

Since point D is located on the ladder, the angular momentum about point C is simply

$$L_D = I_D \dot{\theta} = \frac{4ml^2 \dot{\theta}}{3}, \quad (24)$$

recalling eq. (13). The torque analysis about point D is then

$$\begin{aligned} \frac{dL_D}{dt} = \frac{4ml^2 \ddot{\theta}}{3} &= \tau_D = mgl \sin \theta - 2F_w l \cos \theta - F_{D,x} l \cos \theta \\ &= mgl \sin \theta \\ &\quad - 2ml^2 \cos^2 \theta \ddot{\theta} + 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &\quad + 2ml^2 \cos^2 \theta \ddot{\theta} - 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &= mgl \sin \theta, \end{aligned} \quad (25)$$

which again leads to the equation of motion (5).

2.6 Torque Analysis About Point E

The ordinary torques from the floor and wall about point E vanish, but we must still consider the torques due to gravity and to the “coordinate” forces.

Point E has coordinates $(2x_{\text{cm}}, 2y_{\text{cm}})$ so the “coordinate” force associated with taking this point as our reference point for a torque analysis is

$$\mathbf{F}_E = (-2m\ddot{x}_{\text{cm}}, -2m\ddot{y}_{\text{cm}}) = (-2ml \cos \theta \ddot{\theta} + 2ml \sin \theta \dot{\theta}^2, 2ml \sin \theta \ddot{\theta} + 2ml \cos \theta \dot{\theta}^2), \quad (26)$$

recalling eq. (8).

Point E is not on the ladder, so we calculate its angular momentum as the sum of the angular momentum of the center of mass relative to point E plus the angular momentum relative to the center of mass,

$$L_E = -ml^2 \dot{\theta} + \frac{ml^2 \dot{\theta}}{3} = -\frac{2ml^2 \dot{\theta}}{3}, \quad (27)$$

as for the analysis about point A . The torque analysis about point E is then

$$\begin{aligned} \frac{dL_E}{dt} = -\frac{2ml^2 \ddot{\theta}}{3} &= \tau_E = mgl \sin \theta + F_{E,x} l \cos \theta - F_{E,y} l \sin \theta \\ &= mgl \sin \theta \\ &\quad - 2ml^2 \cos^2 \theta \ddot{\theta} + 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &\quad - 2ml^2 \sin^2 \theta \ddot{\theta} - 2ml^2 \cos \theta \sin \theta \dot{\theta}^2 \\ &= mgl \sin \theta - 2ml^2 \ddot{\theta}, \end{aligned} \quad (28)$$

which again leads to the equation of motion (5).

2.7 Comments

We have deduced the equation of motion of the sliding ladder by six methods. Of these, Lagrange's method is perhaps the simplest. If a torque analysis is desired, it is simplest to use the center of mass as the reference point so that no "coordinate" forces appear in the calculation. The use of reference points not on the ladder, such as points A and E is complicated by the fact that the ladder is not in simple rigid-body rotation about these points, so the angular momentum must be calculated as the sum of that of the center of mass plus that relative to the center of mass.

Thanks to Amin Rezaeezadeh for discussions of this problem.

3 References

- [1] See, for example, sec. 39 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed., (Pergamon Press, Oxford, 1960).