

Index of Refraction of a Moving Medium

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1 Problem

A linear, isotropic medium at rest can be characterized by a (frequency-dependent) index of refraction $n(\omega)$. What is the index of refraction in a frame in which this medium has velocity \mathbf{v} ?

2 Solution

We write a 4-vector as $a_\mu = (a_0, \mathbf{a})$ and the square of its invariant length as $a_\mu a^\mu = a_0^2 - \mathbf{a}^2$. In particular, the position 4-vector is $x_\mu = (ct, \mathbf{x})$, where c is the speed of light in vacuum.

The Lorentz transformation of a 4-vector a_μ from one inertial frame to another with velocity \mathbf{v} with respect to the first can be written

$$a'_0 = \gamma(a_0 - a_{\parallel}v/c), \quad (1)$$

$$\mathbf{a}'_{\parallel} = \gamma(\mathbf{a}_{\parallel} - a_0\mathbf{v}/c), \quad (2)$$

$$\mathbf{a}'_{\perp} = \mathbf{a}_{\perp}, \quad (3)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}, \quad \mathbf{a}_{\parallel} = (\mathbf{a} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \quad (4)$$

and the 4-vector has components (a_0, \mathbf{a}) and (a'_0, \mathbf{a}') in the first and second frames.

We recall that the 3-velocity \mathbf{u} of a particle obeys $u < c$ and can be embedded in the 4-velocity

$$u_\mu = \gamma_u(c, \mathbf{u}), \quad \text{where} \quad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}. \quad (5)$$

The Lorentz transformation (1)-(3) of the particle 4-velocity is

$$\gamma_{u'} = \gamma\gamma_u(1 - \mathbf{u} \cdot \mathbf{v}/c^2), \quad (6)$$

$$\mathbf{u}'_{\parallel} = \frac{\gamma\gamma_u(\mathbf{u}_{\parallel} - \mathbf{v})}{\gamma_{u'}} = \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u} \cdot \mathbf{v}/c^2} = \frac{(\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \mathbf{v}}{1 - \mathbf{u} \cdot \mathbf{v}/c^2}, \quad (7)$$

$$\mathbf{u}'_{\perp} = \mathbf{u}_{\perp} = \mathbf{u} - (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}. \quad (8)$$

Equations (7) and (8) can be combined to give

$$\mathbf{u}' = \frac{\mathbf{u} - \mathbf{v} - \mathbf{u}_{\perp}(\mathbf{u} \cdot \mathbf{v})/c^2}{1 - \mathbf{u} \cdot \mathbf{v}/c^2}, \quad (9)$$

which reverts to the Galilean transformation, $\mathbf{u}' = \mathbf{u} - \mathbf{v}$, of particle velocity in the low-velocity limit.

2.1 Lorentz Transformation of Phase Velocity

For earlier discussions of this topic see [2, 3].

As will be confirmed below, phase velocity is not the spatial part of a 4-vector, and does not transform according to eq. (9). However, we can still display an explicit form for the Lorentz transformation of phase velocity via considerations of the (Lorentz invariant) phase of a wave.

In particular, we can associate the plane wave, $\cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$, with a wave 4-vector $k_\mu = (\omega/c, \mathbf{k})$ such that the phase ϕ of the wave is a Lorentz scalar,

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t = -k_\mu x^\mu. \quad (10)$$

Surfaces of constant phase propagate with phase velocity

$$\mathbf{u}_p(\omega, \mathbf{k}) = \frac{\omega}{k} \hat{\mathbf{u}}_p. \quad (11)$$

If the phase velocity depends on the angular frequency ω (or the wave number k), the medium is said to be **dispersive**, while if it depends on the direction $\hat{\mathbf{k}} = \hat{\mathbf{u}}_p$ of the wave vector it is said to be **anisotropic** (with respect to wave propagation).

In a frame that moves with velocity \mathbf{v} with respect to the original frame the phase velocity is

$$\mathbf{u}'_p = \frac{\omega'}{k'} \hat{\mathbf{u}}'_p. \quad (12)$$

The Lorentz transformation (1)-(3) of the wave 4-vector k_μ is

$$\omega' = \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) = \gamma k(u_p - \hat{\mathbf{u}}_p \cdot \mathbf{v}), \quad (13)$$

$$\mathbf{k}'_{\parallel} = \gamma(\mathbf{k}_{\parallel} - \omega \mathbf{v}/c^2) = \gamma \hat{\mathbf{v}} [\mathbf{k} \cdot \hat{\mathbf{v}} - \omega v/c^2] = \gamma k \hat{\mathbf{v}} (\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}} - u_p v/c^2), \quad (14)$$

$$\mathbf{k}'_{\perp} = \mathbf{k}_{\perp} = \mathbf{k} - (\mathbf{k} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} = k[\hat{\mathbf{u}}_p - (\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}}], \quad (15)$$

noting that $\omega = k u_p$ and $\hat{\mathbf{k}} = \hat{\mathbf{u}}_p$. Combining eqs. (14) and (15) we find

$$k'^2 = k^2 \{1 - (\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}})^2 + \gamma^2 [\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}} - u_p v/c^2]^2\}, \quad (16)$$

and hence the Lorentz transformation of phase velocity is¹

$$u'_p = \frac{\gamma(u_p - \hat{\mathbf{u}}_p \cdot \mathbf{v})}{\sqrt{1 - (\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}})^2 + \gamma^2 [\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}} - u_p v/c^2]^2}}, \quad (17)$$

where

$$\hat{\mathbf{u}}'_p = \hat{\mathbf{k}}' = \frac{\mathbf{k}'_{\parallel} + \mathbf{k}'_{\perp}}{k} = \frac{\hat{\mathbf{u}}_p + \hat{\mathbf{v}} [(\gamma - 1) \hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}} - \gamma u_p v/c^2]}{\sqrt{1 - (\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}})^2 + \gamma^2 [\hat{\mathbf{u}}_p \cdot \hat{\mathbf{v}} - u_p v/c^2]^2}}. \quad (18)$$

For the special case that $u_p = c$, eq. (13) becomes $\omega' = \gamma k c (1 - \beta \hat{\mathbf{u}}_p \cdot \mathbf{v})$ where $\beta = v/c$, and eq. (16) becomes $k'^2 = \gamma^2 k^2 (1 - \beta \hat{\mathbf{u}}_p \cdot \mathbf{v})^2 = \omega'^2/c^2$. That is, if the phase velocity is c in one inertial frame it is also c in any other inertial frame.

¹The cumbersome expression (17) reduces to the (counterintuitive) Galilean result, eq. (8) of [1], in the low-velocity limit.

The comments in the rest of this section suppose that the phase velocity differs from c in the rest frame of the medium.

From eq. (17) we see that the phase velocity in the moving medium depends on the direction of the wave, so that the moving medium appears to be anisotropic even if the medium at rest is isotropic.

We defer discussion of dispersion in the moving medium to sec. 2.2.

If the boost velocity \mathbf{v} is parallel to the phase velocity \mathbf{u}_p , then from eqs. (13) and (14), or from eq. (17), we find that

$$\mathbf{u}'_p = \frac{\mathbf{u}_p - \mathbf{v}}{1 - \mathbf{u}_p \cdot \mathbf{v}/c^2} \quad (\mathbf{u}_p \parallel \mathbf{v}), \quad (19)$$

so that the Lorentz transformation of phase velocity is the same as that for particle velocity, eq. (9), in this special case.

However, since the phase speed ω/k can have any value less than, equal to, or even greater than c ,² the wave 4-vector k_μ can be spacelike, lightlike or timelike, respectively. This permits possibly surprising scenarios such as $\mathbf{u}_p = 2c\hat{\mathbf{x}}$, $\mathbf{v} = c\hat{\mathbf{x}}/2$, for which the transformed wave vector \mathbf{k}' vanishes, the transformed waveform is the standing wave $\cos(\sqrt{3/4}\omega t')$, and the transformed phase velocity is formally infinite.

2.2 Propagation of Light in a Moving Medium

In the case of propagation of light in a medium with an index of refraction different from 1, the rest frame of that medium is a preferred frame. In this frame, the phase velocity (11) can be written as

$$\mathbf{u}_p = \frac{\omega}{k} \hat{\mathbf{u}}_p = \frac{c}{n} \hat{\mathbf{u}}_p, \quad (20)$$

such that the phase velocity (17) in a frame where the medium has velocity \mathbf{v} at angle α to the wave vector \mathbf{k} (in the rest frame of the medium) can be written as

$$\mathbf{u}'_p = \frac{1 - n\beta \cos \alpha}{\sqrt{n^2(1 - \beta^2) + (n \cos \alpha - \beta)^2}} c \hat{\mathbf{u}}'_p \equiv \frac{c}{n'} \hat{\mathbf{u}}'_p. \quad (21)$$

such that the index of refraction n' in the moving medium is given by³

$$n' = \frac{\sqrt{n^2(1 - \beta^2) + (n \cos \alpha - \beta)^2}}{1 - n\beta \cos \alpha} = \frac{\sqrt{n^2(1 + \cos^2 \alpha) - 2n\beta \cos \alpha}}{1 - n\beta \cos \alpha} = \frac{k'c}{\omega'}, \quad (22)$$

which is a function of the angle α and index of refraction n in the rest frame of the medium.⁴ If the medium at rest is nondispersive, the index n is independent of the angular frequency

²A well-known case with phase speed greater than c is a waveguide. See, for example, sec. 8.3 of [4].

³Equation (22) is implicit in eq. (311c) of Pauli's 1921 review [5], so no doubt was first deduced earlier than this.

⁴The naïve assumption that the index of refraction of a moving medium is isotropic if it is so in its rest frame can lead to spurious deductions, as in [6].

ω , and likewise the index n' is independent of ω' , so the moving medium also is nondispersive (but anisotropic).⁵

Only for the special case that the phase velocity is c are the waves both nondispersive and isotropic in all inertial frames.

Appendix: Lorentz Transformation of Group Velocity

This Appendix follows [8].

A suitable 4-vector generalization of the 3-dimensional group velocity, $\mathbf{u}_g = \partial\omega/\partial\mathbf{k}$, is based on rewriting the dispersion relation $\omega = \omega(\mathbf{k})$ as

$$F(k_\mu) = F(\omega/c, \mathbf{k}) = 0, \quad (23)$$

and then taking the 4-vector gradient,

$$\left. \frac{\partial F}{\partial k_\mu} \right|_{F=0} = \left(c \frac{\partial F}{\partial \omega}, \frac{\partial F}{\partial \mathbf{k}} \right)_{F=0} = \left. \frac{\partial F}{\partial \omega} \right|_{F=0} \left(c, \frac{\partial \omega}{\partial \mathbf{k}} \right) = \left. \frac{\partial F}{\partial \omega} \right|_{F=0} (c, \mathbf{u}_g). \quad (24)$$

The invariant length of this gradient is

$$\sqrt{\left. \frac{\partial F}{\partial k_\mu} \frac{\partial F}{\partial k^\mu} \right|_{F=0}} = c \left. \frac{\partial F}{\partial \omega} \right|_{F=0} \sqrt{1 - u_g^2/c^2} = \frac{c}{\gamma_{u_g}} \left. \frac{\partial F}{\partial \omega} \right|_{F=0}. \quad (25)$$

Dividing the 4-vector (24) by the Lorentz scalar (25) and multiplying by c we obtain the group-velocity 4-vector,

$$u_{g,\mu} = \gamma_{u_g} (c, \mathbf{u}_g). \quad (26)$$

This 4-vector is formally identical to that of the 4-velocity (5) of a particle, so the Lorentz transformation of the group velocity \mathbf{u}_g has the form of eq. (9),

$$\mathbf{u}'_g = \frac{\mathbf{u}_g - \mathbf{v} - \mathbf{u}_{g\perp}(\mathbf{u}_g \cdot \mathbf{v})/c^2}{1 - \mathbf{u}_g \cdot \mathbf{v}/c^2}. \quad (27)$$

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⁵This conclusion is reached in [7], where a much less compact expression for the index (22) in the moving medium is given.

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