1 Problem

Give expressions for the potentials of a Hertzian (point) oscillating dipole in various gauges.

2 Solution

2.1 From Potentials in the Lorenz Gauge to Those in Any Other Gauge

As deduced in eq. (16) of [3], a formal expression for the vector potential in the any other gauge is given in terms of the Lorenz-gauge potentials, and the scalar potential in the other gauge, as

\[
A(r, t) = A^{(L)} + \nabla \chi = A^{(L)}(r, t) + c \nabla \int_{-\infty}^{t} [V^{(L)}(r, t') - V(r, t')] dt'
\]

\[
= A^{(L)}(r, -\infty) - c \int_{-\infty}^{t} [E(r, t') + \nabla V(r, t')] dt'.
\]

(1)

We first review the potentials in the Lorenz gauge [1] (see, for example, Chap. 9 of [2]), and then transform these into other gauges following the prescription (1).

2.2 Potentials of a Hertzian Dipole in the Lorenz Gauge

This section follows [4].

We consider a time-dependent point electric dipole \( p_0 e^{-i\omega t} \), centered at the origin, as defined by

\[
p(t) = \lim_{q \to \infty, d \to 0, qd = p} q(t) d,
\]

(2)

for which the associated electric charge density \( \rho \) can be written

\[
\rho(r, t) = \lim_{q \to \infty, d \to 0, qd = p} q(t) [\delta^3(r - d/2) - \delta^3(r - d/2)] = p(t) \cdot \nabla \delta^3(r).
\]

(3)

The current density \( J \) is related by the equation of continuity,

\[
\nabla \cdot J(r, t) = -\frac{\partial \rho(r, t)}{\partial t} = \dot{p}(t) \cdot \nabla \delta^3(r) = \nabla \cdot [\dot{p}(t) \delta^3(r)],
\]

(4)

so that

\[
J(r, t) = \dot{p}(t) \delta^3(r).
\]

(5)
The retarded (Lorentz-gauge) scalar potential $V^{(L)}$ is given (in Gaussian units) by

$$V^{(L)}(r, t) = \int \frac{\delta(r', t' = t - |r - r'|/c)}{|r - r'|} d^3r' = \int \frac{\mathbf{p}(t') \cdot \nabla \delta^3(r')}{r} d^3r'$$

$$= -\int \frac{\delta^3(r') \nabla \cdot \mathbf{p}(t')}{r} d^3r' = -\nabla \cdot \frac{\mathbf{p}(t' = t - r/c)}{r}$$

$$= \frac{[\mathbf{p} \cdot \mathbf{r}]}{r^3} + \frac{[\dot{\mathbf{p}} \cdot \mathbf{r}]}{cr^2}, \quad \text{(6)}$$

where we write a retarded quantity $f(t - r/c)$ as $[f]$, and note that $\nabla r = r/r$ and

$$\nabla \cdot \mathbf{p}(t - r/c) = -\frac{[\dot{\mathbf{p}}]}{cr} \cdot \nabla r = -\frac{[\dot{\mathbf{p}} \cdot \mathbf{r}]}{cr}. \quad \text{(7)}$$

Similarly, the retarded vector potential $\mathbf{A}$ is given by

$$\mathbf{A}^{(L)}(r, t) = \int \frac{\mathbf{J}(r', t' = t - |r - r'|/c)}{c |r - r'|} d^3r' = \int \frac{\dot{\mathbf{p}}(t') \delta^3(r')}{c |r - r'|} d^3r' = \frac{[\dot{\mathbf{p}}]}{cr}. \quad \text{(8)}$$

For an oscillating dipole, $\mathbf{p} = \mathbf{p}_0 e^{-i\omega t}$, $[\mathbf{p}] = \mathbf{p}_0 e^{i(kr - \omega t)} = \mathbf{p} e^{ikr}$, and the Lorenz-gauge potentials are

$$V^{(L)}(r, t) = \mathbf{p} \cdot \mathbf{r} e^{ikr} \left(\frac{1}{r^3} - \frac{ik}{r^2}\right), \quad \mathbf{A}^{(L)}(r, t) = -ik\mathbf{p} e^{ikr} r/c \cdot \mathbf{r}. \quad \text{(9)}$$

### 2.3 Electric and Magnetic Fields

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are obtained from the retarded potentials according to

$$\mathbf{E} = -\nabla V^{(L)} - \frac{1}{c} \frac{\partial \mathbf{A}^{(L)}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}^{(L)}, \quad \text{(10)}$$

noting that $\nabla \times \mathbf{r} = 0$,

$$\nabla \times \mathbf{p}(t - r/c) = -\frac{\nabla r}{c} \times [\dot{\mathbf{p}}] = -\frac{\mathbf{r}}{cr} \times [\dot{\mathbf{p}}]. \quad \text{(11)}$$

and

$$\nabla ([\mathbf{p} \cdot \mathbf{r}] = ([\mathbf{p} \cdot \nabla] \mathbf{r} + (\mathbf{r} \cdot \nabla) [\mathbf{p}] + [\mathbf{p}] \times (\nabla \times \mathbf{r}) + [\mathbf{r} \times (\nabla \times \mathbf{p})])$$

$$= [\mathbf{p}] - [\mathbf{p}] \frac{r}{c} \mathbf{r} + 0 + [\mathbf{p}] \frac{r}{c} \mathbf{r} - \frac{([\dot{\mathbf{p}} \cdot \mathbf{r}]) \mathbf{r}}{cr} = [\mathbf{p}] - \frac{([\dot{\mathbf{p}} \cdot \mathbf{r}]) \mathbf{r}}{cr} \quad \text{(12)}$$

Thus,

$$\mathbf{E} = -\nabla \frac{[\mathbf{p} \cdot \mathbf{r}]}{r^3} - \nabla \frac{[\dot{\mathbf{p}} \cdot \mathbf{r}]}{cr} - \frac{1}{c} \frac{\partial [\mathbf{p}]}{\partial t}$$

$$= -\frac{1}{r^3} \nabla ([\mathbf{p} \cdot \mathbf{r}] - ([\mathbf{p} \cdot \mathbf{r}] \nabla \frac{1}{r^3} - \frac{1}{cr^2} \nabla ([\dot{\mathbf{p}} \cdot \mathbf{r}] - ([\dot{\mathbf{p}} \cdot \mathbf{r}] \nabla \frac{1}{cr} - \frac{[\dot{\mathbf{p}}]}{c^2 r}$$

$$= -\frac{[\mathbf{p}]}{r^3} + \frac{[\dot{\mathbf{p}} \cdot \mathbf{r}]}{cr^2} + 3 \frac{([\mathbf{p}] \cdot \mathbf{r}) \mathbf{r} - [\mathbf{p}] + ([\dot{\mathbf{p}} \cdot \mathbf{r}]) \mathbf{r} + 2 ([\dot{\mathbf{p}} \cdot \mathbf{r}]) \mathbf{r} - \frac{[\dot{\mathbf{p}}]}{c^2 r}$$

$$= \frac{([\dot{\mathbf{p}} \times \mathbf{r}] \times \mathbf{r}) \times \mathbf{r} + 3([\dot{\mathbf{p}} \cdot \mathbf{r}]) \mathbf{r} - [\mathbf{p}] + 3([\dot{\mathbf{p}} \cdot \mathbf{r}]) \mathbf{r} - [\mathbf{p}]}{r^3}, \quad \text{(13)}$$
The fields for an oscillating dipole are
\[ \mathbf{E} = k^2 (\mathbf{p} - (\mathbf{p} \cdot \hat{r}) \hat{r}) \frac{e^{ikr}}{r} + (3(\mathbf{p} \cdot \hat{r}) \hat{r} - \mathbf{p}) \left( \frac{1}{r^3} - \frac{i}{r^2} \right) e^{ikr}, \]
and
\[ \mathbf{B} = k^2 \hat{r} \times \mathbf{p} \left( 1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r}. \]

2.4 Coulomb Gauge

The Coulomb-gauge scalar potential is
\[ V^{(C)}(r, t) = \int \frac{\rho(r', t)}{r} \, dV' = \int \frac{\mathbf{p} \cdot \nabla \delta^{(3)}(r)}{r} \, dV' = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}. \]
The Coulomb-gauge vector potential can be computed via
\[ \mathbf{E} = -\nabla V^{(C)} - \frac{1}{c} \frac{\partial \mathbf{A}^{(C)}}{\partial t} = -\nabla V^{(C)} + ik\mathbf{A}^{(C)}. \]

Thus, using eqs. (15) and (17),
\[ \mathbf{A}^{(C)} = \frac{\mathbf{E} + \nabla V^{(C)}}{ik} = -ik(\mathbf{p} - (\mathbf{p} \cdot \hat{r}) \hat{r}) \frac{e^{ikr}}{r} - (3(\mathbf{p} \cdot \hat{r}) \hat{r} - \mathbf{p}) \left( \frac{i e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2} - \frac{i}{kr^3} \right). \]

The gauge-transformation function \( \chi \) of eq. (1) is, using eqs. (9) and (17),
\[ \chi^{(L-C)} = \frac{c}{i} \int_{-\infty}^{t} [V^{(L)}(r, t') - V^{(C)}(r, t')] dt' = c \int_{-\infty}^{t} \left[ \mathbf{p}(t') \cdot \mathbf{r} e^{ikr} \left( \frac{1}{r^3} - \frac{i}{r^2} \right) - \frac{\mathbf{p}(t') \cdot \mathbf{r}}{r^3} \right] dt'. \]
We can now obtain the Coulomb-gauge vector potential from that in the Lorenz gauge via eq. (1),
\[ \mathbf{A}^{(C)} = \mathbf{A}^{(L)} + \nabla \chi^{(L-C)} = -ik \mathbf{p} e^{ikr} \frac{1}{r} + i \nabla \left( \mathbf{p} \cdot \mathbf{r} e^{ikr} \left( \frac{1}{kr^3} - \frac{i}{r^2} \right) \right) - i \nabla \frac{\mathbf{p} \cdot \mathbf{r}}{kr^3} \]
\[ = -ik \mathbf{p} e^{ikr} \frac{1}{r} - (\mathbf{p} \cdot \hat{r}) \hat{r} e^{ikr} \frac{1}{r^2} - \frac{3(\mathbf{p} \cdot \hat{r}) \hat{r} - \mathbf{p}}{kr^3} e^{ikr} + (\mathbf{p} - 2(\mathbf{p} \cdot \hat{r}) \hat{r}) e^{ikr} \frac{1}{r^2} \]
\[ + i \frac{3(\mathbf{p} \cdot \hat{r}) \hat{r} - \mathbf{p}}{kr^3} \]
\[ = -ik(\mathbf{p} - (\mathbf{p} \cdot \hat{r}) \hat{r}) e^{ikr} \frac{1}{r} - (3(\mathbf{p} \cdot \hat{r}) \hat{r} - \mathbf{p}) \left( \frac{i e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2} - \frac{i}{kr^3} \right). \]
This is the same as eq. (19), which validates the transformation (1).
For comparison, we can also deduce the Coulomb-gauge vector potential using the classic prescription

\[
A^{(C)}(\mathbf{r}, t) = \int \frac{[J_i]}{c |\mathbf{r} - \mathbf{r}'|} d\text{Vol}',
\]  

(22)

where the transverse current density is defined by

\[
J_i(\mathbf{r}, t) = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{J(\mathbf{r}', t)}{c |\mathbf{r} - \mathbf{r}'|} d\text{Vol}'.
\]  

(23)

The integral in eq. (23) is the nonretarded version of the Lorenz-gauge vector potential (9),

\[
\int \frac{J(\mathbf{r}', t)}{c |\mathbf{r} - \mathbf{r}'|} d\text{Vol}' = -\frac{ikp(t)}{r}. 
\]  

(24)

Hence,

\[
\nabla \times \int \frac{J(\mathbf{r}', t)}{c |\mathbf{r} - \mathbf{r}'|} d\text{Vol}' = \nabla \times \left( \frac{-ikp}{r} \right) = \frac{r}{r^3} \times ikp,
\]  

(25)

and

\[
J_i(\mathbf{r}, t) = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{J(\mathbf{r}', t)}{c |\mathbf{r} - \mathbf{r}'|} d\text{Vol}' = \frac{1}{4\pi} \nabla \times \left( \frac{r}{r^3} \times ikp \right) = -\frac{ik}{4\pi} \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}}')\hat{\mathbf{r}} - \mathbf{p}}{r^3}. 
\]  

(26)

Note that while the physical current associated with the point dipole is localized to the origin, the (nonphysical) transverse current (26) is nonzero everywhere in space. The Coulomb-gauge vector potentials is now given by eq. (22),

\[
A^{(C)}(\mathbf{r}, t) = \int \frac{[J_i]}{c |\mathbf{r} - \mathbf{r}'|} d\text{Vol}' = -\frac{ik}{4\pi c} \int \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}}')\hat{\mathbf{r}} - \mathbf{p}}{|\mathbf{r} - \mathbf{r}'| r'^3} d\text{Vol}'
\]

\[
= -\frac{ik}{4\pi c} \int \frac{(3(\mathbf{p} \cdot \hat{\mathbf{r}}')\hat{\mathbf{r}} - \mathbf{p}) e^{ikr'}}{|\mathbf{r} - \mathbf{r}'| r'^3} d\text{Vol}'.
\]  

(27)

However, it is not straightforward to go from eq. (27) to (19).

For another example of the use of eq. (1) to obtain the Coulomb-gauge vector potential, see [5].

2.5 Gibbs Gauge

Another case where the prescription (1) readily applies is the gauge where the scalar potential is defined to be zero, \(V^{(G)} = 0\), such that \(\mathbf{E} = -(1/c)\partial A^{(G)}/\partial t\), as first proposed by Gibbs [6, 7].

\[1\] See, for example, sec. 6.3 of [2].

\[2\] Apparently the Gibbs gauge is also called the Hamiltonian or temporal gauge, as mentioned in sec. VIII of [8]. That is, the Gibbs gauge is handy in examples where the electric field is known, and the vector potential is needed for use in the Hamiltonian of the system, expressed in terms of canonical momenta of charges \(q\) as \(\mathbf{p}_{\text{canonical}} = \mathbf{p}_{\text{mech}} + qA/c\).
Since the Gibbs-gauge vector potential is an integral of the electric field, \( \mathbf{A}^{(G)}(t) = -c \int_{t_0}^{t} \mathbf{E}(t') \, dt' \), this potential propagates at speed \( c \). However, it differs from the Lorenz-gauge vector potential. Since \( \nabla \cdot \mathbf{E} = 4\pi \rho = -(1/c)\partial \nabla \cdot \mathbf{A}^{(G)} / \partial t \), the Gibbs-gauge vector potential obeys \( \nabla \cdot \mathbf{A}^{(G)} = 0 \) away from charged particles (whereas the Coulomb-gauge vector potential obeys \( \nabla \cdot \mathbf{A}^{(C)} = 0 \) everywhere).\(^3\)

As the Gibbs-gauge scalar potential \( V^{(G)} \) is zero, the Gibbs-gauge vector potential can be computed via

\[
\mathbf{E} = -\nabla V^{(G)} - \frac{1}{c} \frac{\partial \mathbf{A}^{(G)}}{\partial t} = ik \mathbf{A}^{(C)}. \tag{28}
\]

Thus, using eqs. (15),

\[
\mathbf{A}^{(G)} = \frac{\mathbf{E}}{ik} = -ik(\mathbf{p} - (\mathbf{p} \cdot \hat{r})\hat{r}) \frac{e^{ikr}}{r} - (3(\mathbf{p} \cdot \hat{r})\hat{r} - \mathbf{p}) \left( \frac{i e^{ikr}}{kr^3} + \frac{e^{ikr}}{r} \right). \tag{29}
\]

The gauge-transformation function \( \chi \) of eq. (1) is, using eqs. (9) and (17),

\[
\chi^{(L\rightarrow G)} = c \int_{-\infty}^{t} [V^{(L)}(\mathbf{r}, t') - V^{(G)}(\mathbf{r}, t')] \, dt' = c \int_{-\infty}^{t} \mathbf{p}(t') \cdot \mathbf{r} e^{ikr} \left( \frac{1}{r^3} - \frac{i}{r^2} \right) \, dt'
\]

\[
= ip \cdot r e^{ikr} \left( \frac{1}{kr^3} - \frac{i}{r^2} \right). \tag{30}
\]

We can now obtain the Gibbs-gauge vector potential from that in the Lorenz gauge via eq. (1),

\[
\mathbf{A}^{(G)} = \mathbf{A}^{(L)} + \nabla \chi^{(L\rightarrow G)} = -ik \mathbf{p} \frac{e^{ikr}}{r} + i \nabla \left( \mathbf{p} \cdot \mathbf{r} e^{ikr} \left( \frac{1}{kr^3} - \frac{i}{r^2} \right) \right) - i \nabla \frac{\mathbf{p} \cdot \mathbf{r}}{kr^3}
\]

\[
= -ik \mathbf{p} \frac{e^{ikr}}{r} - (\mathbf{p} \cdot \hat{r}) e^{ikr} \left( \frac{1}{r^2} - \frac{ik}{r} \right) - i \frac{3(\mathbf{p} \cdot \hat{r})\hat{r} - \mathbf{p}}{kr^3} e^{ikr} + (\mathbf{p} - 2(\mathbf{p} \cdot \hat{r})\hat{r}) \frac{e^{ikr}}{r^2}
\]

\[
= ik((\mathbf{p} \cdot \hat{r})\hat{r} - \mathbf{p}) \frac{e^{ikr}}{r} - (3(\mathbf{p} \cdot \hat{r})\hat{r} - \mathbf{p}) \left( \frac{i e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2} \right). \tag{31}
\]

This is the same as eq. (29), which further validates the transformation (1).

According to eq. (1), the vector potential in the Gibbs gauge is

\[
\mathbf{A}^{(G)}(\mathbf{r}, t) = \mathbf{A}^{(L)}(\mathbf{r}, t) + c \nabla \int_{-\infty}^{t} V^{(L)}(\mathbf{r}, t') \, dt', \tag{32}
\]

so that the vector potential in any other gauge, where the scalar potential is \( V \), can be written as

\[
\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^{(G)} - c \nabla \int_{-\infty}^{t} V(\mathbf{r}, t') \, dt'. \tag{33}
\]

That is, if the vector potential in Gibbs gauge in known, this provides an even simpler prescription than eq. (1) for the vector potential in another gauge.

\(^3\)The distinction between \( \nabla \cdot \mathbf{A} \) in the Coulomb and Gibbs gauges is slight, and may be why Gibbs thought that his new gauge was the Coulomb gauge used by Maxwell.
2.6 Static-Voltage Gauge

A variant of the Gibbs gauge is that the scalar potential is not zero, but rather is the instantaneous Coulomb potential at some arbitrary time \( t_0 \),

\[
V^{(SV)}(r, t) = V^{(C)}(r, t_0) = \int \frac{\rho(r', t_0)}{|r - r'|} d\text{Vol}' .
\]  
\[ (34) \]

This is the static-voltage gauge [9], called the Coulomb-static gauge in [10].

From eq. (33), we see that the vector potential in the static-voltage gauge differs only slightly from that in the Gibbs gauge,

\[
A^{(SV)}(r, t) = A^{(G)}(r, t) - ct \nabla V^{(C)}(r, t_0)
\]

We can take the scalar potential in the static-voltage gauge to be that at time \( t_0 = 0 \), so for the present example,

\[
V^{(SV)}(r, t) = V^{(C)}(t = 0) = \frac{p_0 \cdot r}{r^2}.
\]  
\[ (36) \]

The vector potential in the static-voltage gauge is then given by eq. (35) as

\[
A^{(SV)}(r, t) = A^{(G)}(r, t) - ct \nabla V^{(C)}(r, t_0) = A^{(G)}(r, t) - ct \frac{3(p_0 \cdot \hat{r})\hat{r} - p_0}{r^3}
\]

\[ (37) \]

2.7 Kirchhoff Gauge

The scalar potential in the Kirchhoff gauge [11] is the same as that in the Lorenz gauge, but with the substitution \( c \rightarrow ic \),

\[
V^{(K)}(r, t) = \int \frac{\rho(r', t' = t - |r - r'| / ic)}{|r - r'|} d\text{Vol}' = -\nabla \cdot p(t' = t - r / ic)
\]

\[ (38) \]

We use eq. (33) to relate the vector potential in the Kirchhoff gauge to that in the Gibbs gauge,

\[
A^{(K)}(r, t) = A^{(G)}(r, t) - c \nabla \int_{-\infty}^{t} V^{(K)}(r, t_0) = A^{(G)}(r, t) - \frac{i}{k} \nabla \left( p \cdot r e^{-kr} \left( \frac{1}{r^3} - \frac{k}{r^2} \right) \right)
\]

\[ (39) \]

Both the scalar and vector potential in the Kirchhoff gauge have terms that die out as \( e^{-kr} \) away from the source, but these terms do not contribute to such behavior in the \( \mathbf{E} \) and \( \mathbf{B} \) fields.
2.8 Poincaré Gauge

In cases where the fields $E$ and $B$ are known, we can compute the potentials in the so-called Poincaré gauge (see sec. 9A of [8] and [12, 13, 14]),\(^4\)

$$V^{(P)}(r, t) = -r \cdot \int_0^1 du E(u r, t), \quad A^{(P)}(r, t) = -r \times \int_0^1 u du B(u r, t) \quad \text{(Poincaré).} \quad (40)$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.\(^5\)

The scalar potential in the Poincaré gauge can be computed from eqs. (40) and (15),

$$V^{(P)}(r, t) = -r \cdot \int_0^{\frac{r}{u_0}} du E(u r, t) = -r \int_0^1 2(\mathbf{p} \cdot \hat{r}) \left( \frac{1}{u^3 r^3} - \frac{ik}{u^2 r^2} \right) e^{ikur} du$$

$$= -2(\mathbf{p} \cdot \hat{r}) \int_0^r \left( \frac{1}{s^3} - \frac{ik}{s^2} \right) e^{iks} ds. \quad (41)$$

The integral is ill behaved at the lower limit, and the Poincaré potentials are not useful for this example. However, if the oscillating dipole were at not at the origin, the Poincaré potentials could be evaluated (with considerable effort) at any point not on the ray from the origin to the dipole.

References


\(^4\)The Poincaré gauge is also called the multipolar gauge [15].

\(^5\)The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the “other side” from the origin.


