Potentials for a Hertzian Oscillating Dipole

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(April 6, 2015; updated March 4, 2017)

1 Problem

Give expressions for the potentials of a Hertzian (point) oscillating dipole in various gauges.

2 Solution

The form of Maxwell's equations for the fields $\mathbf{E}$ and $\mathbf{B}$ permit these fields to be related to potentials V and $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B}\ \boldsymbol{\nabla} \times \mathbf{A}, \tag{1}$$

in Gaussian units, where c is the speed of light in vacuum, and the potentials obeys the wave equations,

$$\boldsymbol{\nabla}^2 V + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A} = -4\pi\varrho, \qquad \boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right), \tag{2}$$

in terms of source charge and current densities ϱ and $\mathbf{J}$.

Using the Lorenz-gauge condition [1],

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})} = \frac{1}{c}\frac{\partial V^{(\mathrm{L})}}{\partial t}, \tag{3}$$

the potentials obey the wave equations

$$\nabla^2 V^{(\mathrm{L})} - \frac{1}{c^2}\frac{\partial^2 V^{(\mathrm{L})}}{\partial t^2} = -4\pi\varrho, \qquad \nabla^2\mathbf{A}^{(\mathrm{L})} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}^{(\mathrm{L})}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}. \tag{4}$$

The solutions to these wave equations are the famous retarded potentials of Lorenz [1] and Riemann [2].

2.1 From Potentials in the Lorenz Gauge to Those in Any Other Gauge

As deduced in eq. (16) of [3], a formal expression for the vector potential in the any other gauge is given in terms of the Lorenz-gauge potentials, and the scalar potential in the other gauge, as

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \mathbf{A}^{(\mathrm{L})} + \boldsymbol{\nabla}\chi = \mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) + c\boldsymbol{\nabla}\int_{-\infty}^{t}\{V^{(\mathrm{L})}(\mathbf{r},t') - V(\mathbf{r},t')\}\,dt' \\
&= \mathbf{A}^{(\mathrm{L})}(\mathbf{r},-\infty) - c\int_{-\infty}^{t}\{\mathbf{E}(\mathbf{r},t') + \boldsymbol{\nabla}V(\mathbf{r},t')\}\,dt',
\end{aligned} \tag{5}$$

where c is the speed of light in vacuum.

We first review the potentials in the Lorenz gauge [1] (see, for example, Chap. 9 of [4]), and then transform these into other gauges following the prescription (5).

2.2 Potentials of a Hertzian Dipole in the Lorenz Gauge

This section follows [5].

We consider a time-dependent point electric dipole $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$, centered at the origin, as defined by

$$\mathbf{p}(t) = \lim_{q\to\infty,\ d\to 0,\ qd=p} q(t)\,\mathbf{d}, \tag{6}$$

for which the associated electric charge density ρ can be written

$$\varrho(\mathbf{r},t) = \lim_{q\to\infty,\ d\to 0,\ qd=p} q(t)\{\delta^3(\mathbf{r}-\mathbf{d}/2) - \delta^3(\mathbf{r}-\mathbf{d}/2)\} = \mathbf{p}(t)\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r}). \tag{7}$$

The current density $\mathbf{J}$ is related by the equation of continuity,

$$\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},t) = -\frac{\partial\varrho(\mathbf{r},t)}{\partial t} = \dot{\mathbf{p}}(t)\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r}) = \boldsymbol{\nabla}\cdot\{\dot{\mathbf{p}}(t)\delta^3(\mathbf{r})\}, \tag{8}$$

so that

$$\mathbf{J}(\mathbf{r},t) = \dot{\mathbf{p}}(t)\delta^3(\mathbf{r}). \tag{9}$$

The retarded (Lorenz-gauge) scalar potential $V^{(\mathrm{L})}$ is given by

$$\begin{aligned} V^{(\mathrm{L})}(\mathbf{r},t) &= \int \frac{\varrho(\mathbf{r}',t'=t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' = \int \frac{\mathbf{p}(t')\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' \\ &= -\int \delta^3(\mathbf{r}')\boldsymbol{\nabla}\cdot\frac{\mathbf{p}(t')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' = -\boldsymbol{\nabla}\cdot\frac{\mathbf{p}(t'=t-r/c)}{r} \\ &= \frac{[\mathbf{p}]\cdot\mathbf{r}}{r^3} + \frac{[\dot{\mathbf{p}}]\cdot\mathbf{r}}{c\,r^2}, \end{aligned} \tag{10}$$

where we write a retarded quantity $f(t-r/c)$ as $[f]$, and note that $\boldsymbol{\nabla} r = \mathbf{r}/r$ and

$$\boldsymbol{\nabla}\cdot\mathbf{p}(t-r/c) = -\frac{[\dot{\mathbf{p}}]}{c}\cdot\boldsymbol{\nabla} r = -\frac{[\dot{\mathbf{p}}]\cdot\mathbf{r}}{c\,r}. \tag{11}$$

Similarly, the retarded vector potential $\mathbf{A}$ is given by

$$\mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) = \int \frac{\mathbf{J}(\mathbf{r}',t'=t-|\mathbf{r}-\mathbf{r}'|/c)}{c\,|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' = \int \frac{\dot{\mathbf{p}}(t')\delta^3(\mathbf{r}')}{c\,|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' = \frac{[\dot{\mathbf{p}}]}{c\,r}. \tag{12}$$

For an oscillating dipole, $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$, $[\mathbf{p}] = \mathbf{p}_0\, e^{i(kr-\omega t)} = \mathbf{p}\, e^{ikr}$, and the Lorenz-gauge potentials are spherical waves with speed $c = \omega/k$,

$$V^{(\mathrm{L})}(\mathbf{r},t) = \mathbf{p}\cdot\mathbf{r}\, e^{ikr}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) = \mathbf{p}_0\cdot\mathbf{r}\, e^{i(kr-\omega t)}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right), \tag{13}$$

$$\mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) = -ik\,\mathbf{p}\,\frac{e^{ikr}}{r} = -ik\,\mathbf{p}_0\,\frac{e^{i(kr-\omega t)}}{r}. \tag{14}$$

If we are primarily interested in the fields and potentials far from the dipole, and keep only terms that vary as $1/r$, we have that

$$V_{\rm far}^{(\rm L)} = -ik\mathbf{p}_0 \cdot \hat{\mathbf{r}}\, \frac{e^{i(kr-\omega t)}}{r}\,, \qquad \mathbf{A}_{\rm far}^{(\rm L)} = \mathbf{A}^{(\rm L)} = -ik\,\mathbf{p}_0\, \frac{e^{i(kr-\omega t)}}{r}\,. \tag{15}$$

2.3 Electric and Magnetic Fields

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are obtained from the retarded potentials according to

$$\mathbf{E} = -\boldsymbol{\nabla} V^{(\rm L)} - \frac{1}{c}\frac{\partial \mathbf{A}^{(\rm L)}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(\rm L)}, \tag{16}$$

noting that $\boldsymbol{\nabla} \times \mathbf{r} = 0$,

$$\boldsymbol{\nabla} \times \mathbf{p}(t - r/c) = -\frac{\boldsymbol{\nabla} r}{c} \times [\dot{\mathbf{p}}] = -\frac{\mathbf{r}}{c\,r} \times [\dot{\mathbf{p}}],. \tag{17}$$

and

$$\begin{aligned} \boldsymbol{\nabla}([\mathbf{p}] \cdot \mathbf{r}) &= ([\mathbf{p}] \cdot \boldsymbol{\nabla})\mathbf{r} + (\mathbf{r} \cdot \boldsymbol{\nabla})[\mathbf{p}] + [\mathbf{p}] \times (\boldsymbol{\nabla} \times \mathbf{r}) + [\mathbf{r} \times (\boldsymbol{\nabla} \times \mathbf{p}]) \\ &= [\mathbf{p}] - [\dot{\mathbf{p}}]\,\frac{r}{c} + 0 + [\dot{\mathbf{p}}]\,\frac{r}{c} - \frac{([\dot{\mathbf{p}}] \cdot \mathbf{r})\,\mathbf{r}}{c\,r} = [\mathbf{p}] - \frac{([\dot{\mathbf{p}}] \cdot \mathbf{r})\,\mathbf{r}}{c\,r} \end{aligned} \tag{18}$$

Thus,

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\frac{[\mathbf{p}] \cdot \mathbf{r}}{r^3} - \boldsymbol{\nabla}\frac{[\dot{\mathbf{p}}] \cdot \mathbf{r}}{cr^2} - \frac{1}{c}\frac{\partial}{\partial t}\frac{[\dot{\mathbf{p}}]}{cr} \\ &= -\frac{1}{r^3}\boldsymbol{\nabla}([\mathbf{p}] \cdot \mathbf{r}) - ([\mathbf{p}] \cdot \mathbf{r})\boldsymbol{\nabla}\frac{1}{r^3} - \frac{1}{cr^2}\boldsymbol{\nabla}([\dot{\mathbf{p}}] \cdot \mathbf{r}) - ([\dot{\mathbf{p}}] \cdot \mathbf{r})\boldsymbol{\nabla}\frac{1}{cr^2} - \frac{[\ddot{\mathbf{p}}]}{c^2 r} \\ &= -\frac{[\mathbf{p}]}{r^3} + \frac{([\dot{\mathbf{p}}] \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{cr^2} + 3\frac{([\mathbf{p}] \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{r^3} - \frac{[\dot{\mathbf{p}}]}{cr^2} + \frac{([\ddot{\mathbf{p}}] \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{c^2 r} + 2\frac{([\dot{\mathbf{p}}] \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{cr^2} - \frac{[\ddot{\mathbf{p}}]}{c^2 r} \\ &= \frac{([\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}}{c^2 r} + \frac{3([\dot{\mathbf{p}}] \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - [\dot{\mathbf{p}}]}{cr^2} + \frac{3([\mathbf{p}] \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - [\mathbf{p}]}{r^3}\,, \end{aligned} \tag{19}$$

and

$$\mathbf{B} = \boldsymbol{\nabla} \times \frac{[\dot{\mathbf{p}}]}{cr} = \frac{1}{cr}\boldsymbol{\nabla} \times [\dot{\mathbf{p}}] + \left(\boldsymbol{\nabla}\frac{1}{cr}\right) \times [\dot{\mathbf{p}}] = -\frac{\hat{\mathbf{r}}}{c^2 r} \times [\ddot{\mathbf{p}}] - \frac{\hat{\mathbf{r}}}{cr^2} \times [\dot{\mathbf{p}}], \tag{20}$$

The fields for an oscillating dipole are

$$\begin{aligned} \mathbf{E} &= k^2(\mathbf{p} - (\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})\,\frac{e^{ikr}}{r} + (3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \\ &= k^2(\mathbf{p}_0 - (\mathbf{p}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})\,\frac{e^{i(kr-\omega t)}}{r} + \mathbf{E}_0(\mathbf{r})\,(1 - ikr)\,e^{i(kr-\omega t)}, \end{aligned} \tag{21}$$

where

$$\mathbf{E}_0(\mathbf{r}) = \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_0}{r^3} \tag{22}$$

is the static field of electric dipole $\mathbf{p}_0$, and

$$\mathbf{B} = k^2\,\hat{\mathbf{r}} \times \mathbf{p}\left(1 + \frac{i}{kr}\right)\frac{e^{ikr}}{r} = k^2\,\hat{\mathbf{r}} \times \mathbf{p}_0\left(1 + \frac{i}{kr}\right)\frac{e^{i(kr-\omega t)}}{r}\,, \tag{23}$$

which are also spherical waves with propagation speed c.

Far from the dipole,

$$\mathbf{B}_{\rm far} = k^2\,\hat{\mathbf{r}} \times \mathbf{p}_0\frac{e^{i(kr-\omega t)}}{r}\,, \qquad \mathbf{E}_{\rm far} = \mathbf{B}_{\rm far} \times \hat{\mathbf{r}}. \tag{24}$$

2.4 Gibbs Gauge

A gauge in which the prescription (5) readily applies is where the scalar potential is defined to be zero, $V^{(\rm G)} = 0$, such that $\mathbf{E} = -\partial\mathbf{A}^{(\rm G)}/\partial ct$, as first proposed by Gibbs [6, 7].[1]

Since the Gibbs-gauge vector potential is an integral of the electric field, $\mathbf{A}^{(\rm G)}(t) = -c\int_{t_0}^t \mathbf{E}(t')\,dt'$, this potential propagates at speed c. However, it differs from the Lorenz-gauge vector potential. Since $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho = -\partial\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm G)}/\partial ct$, the Gibbs-gauge vector potential obeys $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm G)} = 0$ away from charged particles (whereas the Coulomb-gauge vector potential obeys $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)} = 0$ everywhere).[2]

As the Gibbs-gauge scalar potential $V^{(\rm G)}$ is zero, the Gibbs-gauge vector potential can be computed via

$$\mathbf{E} = -\boldsymbol{\nabla}V^{(\rm G)} - \frac{1}{c}\frac{\partial\mathbf{A}^{(\rm G)}}{\partial t} = ik\mathbf{A}^{(\rm C)}. \tag{25}$$

Thus, using eqs. (21),

$$\begin{aligned}
\mathbf{A}^{(\rm G)} &= \frac{\mathbf{E}}{ik} = -ik\,(\mathbf{p} - (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\,\frac{e^{ikr}}{r} - (3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})\left(\frac{i\,e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2}\right) \\
&= ik(\mathbf{p}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r} - \left(r + \frac{i}{k}\right)\mathbf{E}_0(\mathbf{r})\,e^{i(kr-\omega t)},
\end{aligned} \tag{26}$$

Far from the dipole, the potentials in the Gibbs gauge are,

$$V_{\rm far}^{(\rm G)} = 0, \qquad \mathbf{A}_{\rm far}^{(\rm G)} = ik(\mathbf{p}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r}\,. \tag{27}$$

2.4.1 Gibbs-Gauge Potential from the Lorenz Gauge

The gauge-transformation function χ of eq. (5) is, using eqs. (13) and (32),

$$\begin{aligned}
\chi^{(\rm L\to G)} &= c\int_{-\infty}^t \{V^{(\rm L)}(\mathbf{r},t') - V^{(\rm G)}(\mathbf{r},t')\}\,dt' = c\int_{-\infty}^t \mathbf{p}(t')\cdot\mathbf{r}\,e^{ikr}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)dt' \\
&= i\,\mathbf{p}\cdot\mathbf{r}\,e^{ikr}\left(\frac{1}{kr^3} - \frac{i}{r^2}\right).
\end{aligned} \tag{28}$$

[1] Apparently the Gibbs gauge is also called the Hamiltonian or temporal gauge, as mentioned in sec. VIII of [8]. That is, the Gibbs gauge is handy in examples where the electric field is known, and the vector potential is needed for use in the Hamiltonian of the system, expressed in terms of canonical momenta of charges q as $\mathbf{p}_{\rm canonical} = \mathbf{p}_{\rm mech} + q\mathbf{A}/c$.

[2] The distinction between $\boldsymbol{\nabla}\cdot\mathbf{A}$ in the Coulomb and Gibbs gauges is slight, and may be why Gibbs thought [6] that his new gauge was the Coulomb gauge used by Maxwell.

We can now obtain the Gibbs-gauge vector potential from that in the Lorenz gauge via eq. (5),

$$\begin{aligned}
\mathbf{A}^{(\mathrm{G})} &= \mathbf{A}^{(\mathrm{L})} + \boldsymbol{\nabla}\chi^{(\mathrm{L}\to\mathrm{G})} = -ik\,\mathbf{p}\frac{e^{ikr}}{r} + i\boldsymbol{\nabla}\left(\mathbf{p}\cdot\mathbf{r}\,e^{ikr}\left(\frac{1}{kr^3} - \frac{i}{r^2}\right)\right) - i\boldsymbol{\nabla}\frac{\mathbf{p}\cdot\mathbf{r}}{kr^3} \\
&= -ik\mathbf{p}\frac{e^{ikr}}{r} - (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,e^{ikr}\left(\frac{1}{r^2} - \frac{ik}{r}\right) - i\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}}{kr^3}e^{ikr} + (\mathbf{p} - 2(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}})\frac{e^{ikr}}{r^2} \\
&= ik((\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})\,\frac{e^{ikr}}{r} - (3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})\left(\frac{i\,e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2}\right). \qquad (29)
\end{aligned}$$

This is the same as eq. (26), which further validates the transformation (5).

2.4.2 Potential in Any Gauge from That in the Gibbs Gauge

According to eq. (5), the vector potential in the Gibbs gauge is

$$\mathbf{A}^{(\mathrm{G})}(\mathbf{r},t) = \mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) + c\boldsymbol{\nabla}\int_{-\infty}^{t} V^{(\mathrm{L})}(\mathbf{r},t')\,dt', \qquad (30)$$

so that the vector potential in any other gauge, where the scalar potential is V, can be written as

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) + c\boldsymbol{\nabla}\int_{-\infty}^{t} \{V^{(\mathrm{L})}(\mathbf{r},t') - V(\mathbf{r},t')\}\,dt' \\
&= \mathbf{A}^{(\mathrm{G})} - c\boldsymbol{\nabla}\int_{-\infty}^{t} V(\mathbf{r},t')\,dt'. \qquad (31)
\end{aligned}$$

That is, if the vector potential in Gibbs gauge in known, this provides an even simpler prescription than eq. (5) for the vector potential in another gauge.

2.5 Coulomb Gauge

The Coulomb-gauge scalar potential is

$$V^{(\mathrm{C})}(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)}{r}\,d\mathrm{Vol}' = \int \frac{\mathbf{p}\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r})}{r}\,d\mathrm{Vol}' = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\,. \qquad (32)$$

The Coulomb-gauge vector potential can be computed via

$$\mathbf{E} = -\boldsymbol{\nabla}V^{(\mathrm{C})} - \frac{1}{c}\frac{\partial\mathbf{A}^{(\mathrm{C})}}{\partial t} = -\boldsymbol{\nabla}V^{(\mathrm{C})} + ik\mathbf{A}^{(\mathrm{C})}. \qquad (33)$$

Thus, using eqs. (21) and (32),

$$\begin{aligned}
\mathbf{A}^{(\mathrm{C})} &= \frac{\mathbf{E} + \boldsymbol{\nabla}V^{(\mathrm{C})}}{ik} = -ik(\mathbf{p} - (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\,\frac{e^{ikr}}{r} - (3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})\left(\frac{i\,e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2} - \frac{i}{kr^3}\right) \\
&= ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r} - \left(r + \frac{i}{k}\right)\mathbf{E}_0\,e^{i(kr-\omega t)} + \frac{i\,\mathbf{E}_0}{k}\,e^{-i\omega t}. \qquad (34)
\end{aligned}$$

While the first two terms of the second line of eq. (34) are waves with propagation speed c, the third term is instantaneous.

Since $\boldsymbol{\nabla} \times \mathbf{E}_0 = 0$, the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{C})}$ propagates at speed c. Likewise, the electric field $\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{C})} - \partial \mathbf{A}^{(\mathrm{C})}/\partial ct = -\boldsymbol{\nabla} V^{(\mathrm{C})} + ik\mathbf{A}^{(\mathrm{C})}$ propagates at speed c, as the term $-\boldsymbol{\nabla} V^{(\mathrm{C})}$ is canceled by the third term in $ik\mathbf{A}^{(\mathrm{C})}$ according to the form (46).

That is, the Coulomb-gauge vector potential $\mathbf{A}^{(\mathrm{C})}$ always contains an instantaneous term whose time derivative cancels the instantaneous term $-\boldsymbol{\nabla} V^{(\mathrm{C})}$, such that the electric field $\mathbf{E}$ propagates at speed c.[3]

Far from the dipole, the potentials are the same in the Coulomb gauge and the Gibbs gauge,

$$V_{\mathrm{far}}^{(\mathrm{C})} = V_{\mathrm{far}}^{(\mathrm{G})} = 0, \qquad \mathbf{A}_{\mathrm{far}}^{(\mathrm{C})} = \mathbf{A}_{\mathrm{far}}^{(\mathrm{G})} = ik(\mathbf{p}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}\, \frac{e^{i(kr-\omega t)}}{r}\, . \tag{35}$$

2.5.1 Coulomb-Gauge Vector Potential from the Lorenz-Gauge Potential

The gauge-transformation function χ of eq. (5) is, using eqs. (13) and (32),

$$\begin{aligned}
\chi^{(\mathrm{L}\to\mathrm{C})} &= c\int_{-\infty}^{t} [V^{(\mathrm{L})}(\mathbf{r},t') - V^{(\mathrm{C})}(\mathbf{r},t')]\, dt' = c\int_{-\infty}^{t} \left[\mathbf{p}(t')\cdot\mathbf{r}\, e^{ikr}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) - \frac{\mathbf{p}(t')\cdot\mathbf{r}}{r^3}\right] dt' \\
&= i\mathbf{p}\cdot\mathbf{r}\, e^{ikr}\left(\frac{1}{kr^3} - \frac{i}{r^2}\right) - i\frac{\mathbf{p}\cdot\mathbf{r}}{kr^3}\, .
\end{aligned} \tag{36}$$

We can now obtain the Coulomb-gauge vector potential from that in the Lorenz gauge via eq. (5),

$$\begin{aligned}
\mathbf{A}^{(\mathrm{C})} &= \mathbf{A}^{(\mathrm{L})} + \boldsymbol{\nabla}\chi^{(\mathrm{L}\to\mathrm{C})} = -ik\mathbf{p}\frac{e^{ikr}}{r} + i\boldsymbol{\nabla}\left(\mathbf{p}\cdot\mathbf{r}\, e^{ikr}\left(\frac{1}{kr^3} - \frac{i}{r^2}\right)\right) - i\boldsymbol{\nabla}\frac{\mathbf{p}\cdot\mathbf{r}}{kr^3} \\
&= -ik\mathbf{p}\frac{e^{ikr}}{r} - (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\, e^{ikr}\left(\frac{1}{r^2} - \frac{ik}{r}\right) - i\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{kr^3}e^{ikr} + (\mathbf{p} - 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\frac{e^{ikr}}{r^2} \\
&\quad + i\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{kr^3} \\
&= -ik(\mathbf{p} - (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\,\frac{e^{ikr}}{r} - (3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})\left(\frac{i\, e^{ikr}}{kr^3} + \frac{e^{ikr}}{r^2} - \frac{i}{kr^3}\right).
\end{aligned} \tag{37}$$

This is the same as eq. (34), which validates the transformation (5).

For another example of the use of eq. (5) to obtain the Coulomb-gauge vector potential, see [13].

2.5.2 Direct Computation of the Coulomb-Gauge Vector Potential

For comparison, we can also deduce the Coulomb-gauge vector potential using the classic prescription[4]

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) = \int \frac{[\mathbf{J}_t]}{c\,|\mathbf{r} - \mathbf{r}'|}\, d\mathrm{Vol}', \tag{38}$$

[3] This cancelation for the Coulomb-gauge potentials has been noted in [9, 10, 11, 12].

[4] See, for example, sec. 6.3 of [4].

where the transverse current density is defined by

$$\mathbf{J}_t(\mathbf{r},t) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int\frac{\mathbf{J}(\mathbf{r}',t)}{c\,|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}'. \tag{39}$$

The integral in eq. (39) is the nonretarded version of the Lorenz-gauge vector potential (13),

$$\int\frac{\mathbf{J}(\mathbf{r}',t)}{c\,|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = -\frac{ik\mathbf{p}(t)}{r}\,. \tag{40}$$

Hence,

$$\boldsymbol{\nabla}\times\int\frac{\mathbf{J}(\mathbf{r}',t)}{c\,|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = \boldsymbol{\nabla}\times\frac{-ik\mathbf{p}}{r} = \frac{\mathbf{r}}{r^3}\times ik\mathbf{p}, \tag{41}$$

and

$$\mathbf{J}_t(\mathbf{r},t) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int\frac{\mathbf{J}(\mathbf{r}',t)}{c\,|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = \frac{1}{4\pi}\boldsymbol{\nabla}\times\left(\frac{\mathbf{r}}{r^3}\times ik\mathbf{p}\right) = -\frac{ik}{4\pi}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3}\,. \tag{42}$$

Note that while the physical current associated with the point dipole is localized to the origin, the (nonphysical) transverse current (42) is nonzero everywhere in space. The Coulomb-gauge vector potentials is now given by eq. (38),

$$\begin{aligned}\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) &= \int\frac{[\mathbf{J}_t]}{c\,|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = -\frac{ik}{4\pi c}\int\frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}}')\,\hat{\mathbf{r}}'-\mathbf{p}]}{|\mathbf{r}-\mathbf{r}'|\,r'^3}\,d\mathrm{Vol}' \\ &= -\frac{ik}{4\pi c}\int\frac{(3(\mathbf{p}\cdot\hat{\mathbf{r}}')\,\hat{\mathbf{r}}'-\mathbf{p})\,e^{ikr'}}{|\mathbf{r}-\mathbf{r}'|\,r'^3}\,d\mathrm{Vol}'.\end{aligned} \tag{43}$$

However, it is not straightforward to go from eq. (43) to (34).

2.5.3 Coulomb-Gauge Vector Potential from the Gibbs-Gauge Potential

We can also compute the vector potential in the Coulomb gauge, from that in the Gibbs gauge according to the prescription (31),

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) = \mathbf{A}^{(\mathrm{G})}(\mathbf{r},t) - c\boldsymbol{\nabla}\int_{-\infty}^{t}V^{(\mathrm{C})}(\mathbf{r},t')\,dt'. \tag{44}$$

If the dipole moment oscillates according to $\mathbf{p} = \mathbf{p}_0\,e^{-i\omega t}$,

$$c\int_{-\infty}^{t}V^{(\mathrm{C})}(\mathbf{r},t')\,dt' = c\frac{(\mathbf{p_0}\cdot\hat{\mathbf{r}})}{r^2}\int_{-\infty}^{t}e^{-i\omega t'}\,dt' = \frac{i}{k}\frac{(\mathbf{p_0}\cdot\hat{\mathbf{r}})}{r^2}\,e^{-i\omega t} = \frac{i}{k}V^{(\mathrm{C})}(\mathbf{r},t), \tag{45}$$

taking the scalar potential at $t = -\infty$ to be its average value of zero at large negative times, such that the Coulomb-gauge vector potential is, recalling eq. (26),

$$\begin{aligned}\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) &= ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r} - \left(r+\frac{i}{k}\right)\mathbf{E}_0\,e^{i(kr-\omega t)} - \frac{i}{k}\boldsymbol{\nabla}V^{(\mathrm{C})}(\mathbf{r},t) \\ &= ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r} - \left(r+\frac{i}{k}\right)\mathbf{E}_0\,e^{i(kr-\omega t)} + \frac{i\mathbf{E}_0}{k}\,e^{-i\omega t}.\end{aligned} \tag{46}$$

The first two terms of eq. (46) propagate at speed $c = \omega/k$, while the third term is instantaneous.

2.6 Static-Voltage Gauge

A variant of the Gibbs gauge is that the scalar potential is not zero, but rather is the instantaneous Coulomb potential at some arbitrary time t_0,

$$V^{(\mathrm{SV})}(\mathbf{r}, t) = V^{(\mathrm{C})}(\mathbf{r}, t_0) = \int \frac{\rho(\mathbf{r}', t_0)}{|\mathbf{r} - \mathbf{r}'|}\, d\mathrm{Vol}'. \tag{47}$$

This is the static-voltage gauge [14], called the Coulomb-static gauge in [15].

From eq. (31), we see that the vector potential in the static-voltage gauge differs only slightly from that in the Gibbs gauge,

$$\mathbf{A}^{(\mathrm{SV})}(\mathbf{r}, t) = \mathbf{A}^{(\mathrm{G})}(\mathbf{r}, t) - ct\boldsymbol{\nabla} V^{(\mathrm{C})}(\mathbf{r}, t_0). \tag{48}$$

We can take the scalar potential in the static-voltage gauge to be that at time $t_0 = 0$, so for the present example,

$$V^{(\mathrm{SV})}(\mathbf{r}, t) = V^{(\mathrm{C})}(t = 0) = \frac{\mathbf{p}_0 \cdot \mathbf{r}}{r^2}\,. \tag{49}$$

The vector potential in the static-voltage gauge is then given by eq. (48) as

$$\begin{aligned} \mathbf{A}^{(\mathrm{SV})}(\mathbf{r}, t) &= \mathbf{A}^{(\mathrm{G})}(\mathbf{r}, t) - ct\boldsymbol{\nabla} V^{(\mathrm{C})}(t = 0) = \mathbf{A}^{(\mathrm{G})}(\mathbf{r}, t) + ct\mathbf{E}_0(\mathbf{r}) \\ &= ik(\mathbf{p}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}\, \frac{e^{i(kr - \omega t)}}{r} - \left(r + \frac{i}{k}\right) \mathbf{E}_0(\mathbf{r})\, e^{i(kr-\omega t)} + ct\mathbf{E}_0(\mathbf{r}). \end{aligned} \tag{50}$$

Far from the dipole, the potentials are the same in the Coulomb gauge, the Gibbs gauge and the static-voltage gauge,

$$V^{(\mathrm{C})}_{\mathrm{far}} = V^{(\mathrm{G})}_{\mathrm{far}} = V^{(\mathrm{SV})}_{\mathrm{far}} = 0, \qquad \mathbf{A}^{(\mathrm{C})}_{\mathrm{far}} = \mathbf{A}^{(\mathrm{G})}_{\mathrm{far}} = \mathbf{A}^{(\mathrm{SV})}_{\mathrm{far}} = ik(\mathbf{p}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}\, \frac{e^{i(kr-\omega t)}}{r}\,. \tag{51}$$

2.7 Kirchhoff Gauge

Kirchhoff [16] was apparently the first to define a gauge condition, writing

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{K})} = -\frac{1}{c}\frac{\partial V^{(\mathrm{K})}}{\partial t}\,, \tag{52}$$

which differs by a sign from the Lorenz-condition (3). Then, the wave equation (2) for the scalar potential becomes, in the Kirchhoff-gauge,

$$\nabla^2 V^{(\mathrm{K})} + \frac{1}{c^2}\frac{\partial^2 V^{(\mathrm{K})}}{\partial t^2} \boldsymbol{\nabla} \cdot \mathbf{A} = -4\pi\varrho, \tag{53}$$

which is the same as that in the Lorenz gauge, eq. (4), but with the substitution $c \to ic$. Hence, a formal solution for the Kirchhoff-gauge scalar potential is

$$\begin{aligned} V^{(\mathrm{K})}(\mathbf{r}, t) &= \int \frac{\varrho(\mathbf{r}', t' = t - |\mathbf{r} - \mathbf{r}'|\,/ic)}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}' = -\boldsymbol{\nabla} \cdot \frac{\mathbf{p}(t' = t - r/ic)}{r} \\ &= \mathbf{p} \cdot \mathbf{r}\, e^{-kr} \left(\frac{1}{r^3} - \frac{k}{r^2}\right) = \mathbf{p}_0 \cdot \mathbf{r}\, e^{-kr - i\omega t} \left(\frac{1}{r^3} - \frac{k}{r^2}\right). \end{aligned} \tag{54}$$

We use eq. (31) to relate the vector potential in the Kirchhoff gauge to that in the Gibbs gauge,

$$\begin{aligned}
\mathbf{A}^{(\mathrm{K})}(\mathbf{r},t) &= \mathbf{A}^{(\mathrm{G})}(\mathbf{r},t) - c\boldsymbol{\nabla}\int_{-\infty}^{t} V^{(\mathrm{K})}(\mathbf{r},t_0) = \mathbf{A}^{(\mathrm{G})}(\mathbf{r},t) - \frac{i}{k}\boldsymbol{\nabla}\left(\mathbf{p}\cdot\mathbf{r}\,e^{-kr}\left(\frac{1}{r^3}-\frac{k}{r^2}\right)\right) \\
&= ik((\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p})\,\frac{e^{ikr}}{r} - (3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p})\left(\frac{i\,e^{ikr}}{kr^3}+\frac{e^{ikr}}{r^2}\right) \\
&\quad -ik(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,\frac{e^{-kr}}{r} - i((\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p})\,\frac{e^{-kr}}{r^2} + i(3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p})\,\frac{e^{-kr}}{kr^3}\,. \qquad (55)
\end{aligned}$$

Both the scalar and vector potential in the Kirchhoff gauge have terms that die out as e^{-kr} away from the source, but these terms do not contribute to such behavior in the **E** and **B** fields.

Far from the dipole, the potentials are the same in the Coulomb gauge, the Gibbs gauge, the Kirchhoff gauge and the static-voltage gauge,

$$V_{\mathrm{far}}^{(\mathrm{C})} = V_{\mathrm{far}}^{(\mathrm{G})} = V_{\mathrm{far}}^{(\mathrm{K})} = V_{\mathrm{far}}^{(\mathrm{SV})} = 0, \qquad (56)$$

$$\mathbf{A}_{\mathrm{far}}^{(\mathrm{C})} = \mathbf{A}_{\mathrm{far}}^{(\mathrm{G})} = \mathbf{A}_{\mathrm{far}}^{(\mathrm{K})} = \mathbf{A}_{\mathrm{far}}^{(\mathrm{SV})} = ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r}\,. \qquad (57)$$

2.8 Poincaré Gauge

In cases where the fields **E** and **B** are known, we can compute the potentials in the so-called Poincaré gauge (see sec. 9A of [8] and [17, 18, 19]),[5]

$$V^{(\mathrm{P})}(\mathbf{r},t) = -\mathbf{r}\cdot\int_0^1 du\,\mathbf{E}(u\mathbf{r},t), \qquad \mathbf{A}^{(\mathrm{P})}(\mathbf{r},t) = -\mathbf{r}\times\int_0^1 u\,du\,\mathbf{B}(u\mathbf{r},t) \qquad (\text{Poincaré}). \quad (58)$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.[6]

The scalar potential in the Poincaré gauge can be computed from eqs. (58) and (21),

$$\begin{aligned}
V^{(\mathrm{P})}(\mathbf{r},t) &= -\mathbf{r}\cdot\int_0^{u_0=1} du\,\mathbf{E}(u\mathbf{r},t) = -r\int_0^1 2(\mathbf{p}\cdot\hat{\mathbf{r}})\left(\frac{1}{u^3r^3}-\frac{ik}{u^2r^2}\right)e^{ikur}\,du \\
&= -2(\mathbf{p}\cdot\hat{\mathbf{r}})\int_0^r\left(\frac{1}{s^3}-\frac{ik}{s^2}\right)e^{iks}\,ds. \qquad (59)
\end{aligned}$$

The integral is ill behaved at the lower limit, and the Poincaré potentials are not useful for this example. However, if the oscillating dipole were at not at the origin, the Poincaré potentials could be evaluated (with considerable effort) at any point not on the ray from the origin to the dipole.

[5]The Poincaré gauge is also called the multipolar gauge [20].

[6]The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the "other side" from the origin.

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