Electromagnetic Helicopter?

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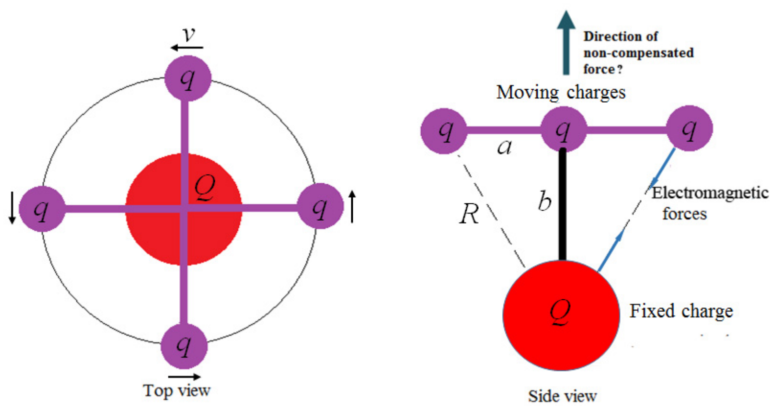
1 Problem

Because of the finite speed c of propagation of electromagnetic effects, the fields of a moving charge are associated with its retarded position rather than its present position, as noted by Liénard [1] and Wiechert [2]. In the case of a moving charge and a fixed charge, it would seem that the forces on them are not equal and opposite, which might permit a net propulsion of the system.

In particular, consider the example sketched below (due to Vladimir Onoochin) in which four electric charges q rotate in a circle of radius a with azimuthal velocity $\mathbf{v}$, while charge Q is fixed on the axis of rotation at distance b from the center of the circle. Does this system experience a net axial force, creating a kind of electromagnetic helicopter?



2 Solution

We consider the system when its axis is held fixed.

The forces on the moving charges q due to the fixed charge Q are $\mathbf{F}_q = q\mathbf{E}_Q$, and the forces on the fixed charge are $\mathbf{F}_Q = Q\mathbf{E}_q$.

The (static) Coulomb field of charge Q at charge q is

$$\mathbf{E}_Q = \frac{Q}{R^3}\,\mathbf{R}, \tag{1}$$

where $\mathbf{R} = \mathbf{R}_q - \mathbf{R}_Q$ points from Q to q. Hence, the axial force on a charge q is, in a cylindrical coordinate system (r, θ, z) with origin at the center of the circle and z-axis perpendicular to it,

$$F_{q,z} = \frac{qQb}{R^3}\,. \tag{2}$$

2.1 Point Charges with Q on Axis

In case of point charges (with Q on the z-axis), the distance between charges q and Q is $R = \sqrt{a^2 + b^2}$ at all times.

The electric field of a moving charge q can be deduced from a scalar potential ϕ and a vector potential $\mathbf{A}$ according to

$$\mathbf{E}_q = -\boldsymbol{\nabla}\phi_q - \frac{1}{c}\frac{\partial \mathbf{A}_q}{\partial t}\,, \tag{3}$$

in Gaussian units, where the retarded potentials (in the Lorenz gauge) are

$$\phi_q = \frac{q}{[R]}\,, \qquad \mathbf{A}_q = \frac{q\,[\mathbf{v}]}{[R]}\,, \qquad \text{with} \qquad [f(t)] = f(t' = t - R(t')/c). \tag{4}$$

For point charges, the retarded distance $[R]$ is equal to the present distance R (although $[\mathbf{R}]$ is different from $\mathbf{R}$), so $\phi_q = q/R$, *i.e.*, the retarded scalar potential is simply the instantaneous Coulomb potential. Also, since the velocity $\mathbf{v}$ has no z-component, the axial force on charge Q is due only to the scalar potential ϕ_q,

$$F_{Q,z} = -Q\frac{\partial \phi_q}{\partial z_Q} = -qQ\frac{\partial}{\partial z_Q}\frac{1}{R} = -\frac{qQb}{R^3} = -F_{q,z}. \tag{5}$$

Hence, for point charges (with fixed charge Q on the z-axis) there is no net axial force on the system and the "electromagnetic helicopter" doesn't fly.

2.2 Extended Charges with Q off Axis

If the fixed charge Q extends away from the z-axis, the retarded distance $[R]$ between a small element δQ of Q and an element δq of the moving charge q is no longer equal to their present separation R.

It is convenient to express the retarded field $\mathbf{E}_{\delta q}$ in terms of present quantities via an expansion in powers of v/c, and to content ourselves with this expansion to order v^2/c^2 (which is a very small quantity for velocities less than, say, the speed of sound in air). Such an expansion was first given by Heaviside in art. 48 of [3], but is better known as the Darwin approximation [4].[1]

Darwin [4] worked in the Coulomb gauge, and kept terms only to order v^2/c^2. Then, the scalar and vector potentials due to a charge e that has velocity $\mathbf{v}$ are,[2]

$$\phi \approx \frac{e}{R}\,, \qquad \mathbf{A} \approx \frac{e[\mathbf{v} + (\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR}\,, \tag{6}$$

where $\hat{\mathbf{n}}$ is directed from the charge to the observer, whose (present) distance is R.

[1] For some comments on the Darwin approximation, see [5].

[2] See, for example, sec. 65 of [6] or sec. 12.6 of [7]. A derivation of eq. (6) based on an approximation to Maxwell's equations rather than the Darwin Lagrangian is given in [8].

The electric and magnetic fields of a charge e at distance R from an observer follow in the Darwin approximation from the potentials (6) as,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial ct} \approx \frac{e}{R^2}\,\hat{\mathbf{n}} - \frac{e}{2c^2 R}\left(\dot{\mathbf{v}} + (\dot{\mathbf{v}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \frac{3(\mathbf{v}\cdot\hat{\mathbf{n}})^2 - v^2}{R}\hat{\mathbf{n}}\right), \tag{7}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \approx \frac{e\mathbf{v}\times\hat{\mathbf{n}}}{cR^2}, \tag{8}$$

where $\dot{\mathbf{v}} = d\mathbf{v}/dt$ is the (present) acceleration of the charge.[3]

In the present example, we desire the retarded electric field $\mathbf{E}_{\delta q}$ at the position of element δQ of the fixed charge. We write the Cartesian location of charge δq as $\mathbf{R}_q = (x_q, y_q, z_q) = \mathbf{r}_q + z_q\,\hat{\mathbf{z}}$ with $\mathbf{r}_q = (x_q, y_q, 0)$, and that of charge δQ as $\mathbf{R}_Q = (x_Q, y_Q, z_Q) \equiv \mathbf{r}_Q + z_q\,\hat{\mathbf{z}}$. Then, $\hat{\mathbf{n}} = -\mathbf{R}/R$ where $\mathbf{R} = \mathbf{R}_q - \mathbf{R}_Q = \mathbf{r}_q - \mathbf{r}_Q + (z_q - z_Q)\,\hat{\mathbf{z}}$, the velocity of element δq is $\mathbf{v}_q = \boldsymbol{\omega}\times\mathbf{r}_\mathbf{q} = \omega(y_q, -x_q, 0)$ with $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$, $\omega = v/a$, such that $v_q = \omega r_q$, and the present acceleration of element δq is $\dot{\mathbf{v}}_q = \boldsymbol{\omega}\times\mathbf{v}_q = -\omega^2\mathbf{r}_q$. Then, $\dot{\mathbf{v}}_q\cdot\hat{\mathbf{n}} = \omega^2\mathbf{r}_q\cdot(\mathbf{r}_q - \mathbf{r}_Q)/R$, $\mathbf{v}_q\cdot\hat{\mathbf{n}} = \boldsymbol{\omega}\times\mathbf{r}_q\cdot(\mathbf{r}_Q - \mathbf{r}_q)/R = \boldsymbol{\omega}\cdot\mathbf{r}_q\times\mathbf{r}_Q/R = \omega(x_q y_Q - x_Q y_q)/R$, and

$$\begin{aligned}\mathbf{E}_{\delta q} &= -\frac{\delta q}{R^3}\mathbf{R} - \frac{\delta q}{2c^2 R}\left(-\omega^2\mathbf{r}_q - \frac{\omega^2(r_q^2 - \mathbf{r}_q\cdot\mathbf{r}_Q)}{R^2}\mathbf{R} - \frac{3\omega^2(x_q y_Q - x_Q y_q)^2}{R^4}\mathbf{R} + \frac{\omega^2 r_q^2}{R^2}\mathbf{R}\right)\\ &= -\frac{\delta q}{R^3}\mathbf{R} + \frac{\omega^2\delta q}{2c^2 R}\left(\mathbf{r}_q - \frac{(\mathbf{r}_q\cdot\mathbf{r}_Q)R^2 - 3(x_q y_Q - x_Q y_q)^2}{R^4}\mathbf{R}\right).\end{aligned} \tag{9}$$

We are interested in the z-component of the force on element δQ due to element δq,

$$\begin{aligned}F_{\delta Q,z} &= \delta Q E_{\delta q,z}\\ &= -\frac{\delta q\,\delta Q}{R^3}(z_q - z_Q)\left(1 + \frac{\omega^2}{2c^2R^2}\left((x_q x_Q + y_q y_Q)R^2 - 3(x_q y_Q - x_Q y_q)^2\right)\right).\end{aligned} \tag{10}$$

Meanwhile, the force on element δq due to the fixed charge element δQ is

$$F_{\delta q,z} = \delta q E_{\delta Q,z} = \frac{\delta q\,\delta Q}{R^3}(z_q - z_Q). \tag{11}$$

Thus, it appears that the z-component, $F_z = F_{\delta Q,z} + F_{\delta q,z}$, of the self force on the system is nonzero, if the fixed charge element δQ is not located on the z-axis, such that both x_Q and y_Q are nonzero.

The location of the moving charge element δ_q can be written as $(r_q\cos\omega t, r_q\sin\omega t, z_q)$, such the the combined axial force is oscillatory,

$$\begin{aligned}F_z &= F_{\delta q,z} + F_{\delta Q,z}\\ &= -\frac{\omega^2 r_q^2\delta q\,\delta Q}{2c^2R^5}(z_q - z_Q)\left((x_Q\cos\omega t + y_Q\sin\omega t)R^2 - 3r_q(y_Q\cos\omega t - x_Q\sin\omega t)^2\right)\end{aligned} \tag{12}$$

The "electromagnetic helicopter" does not fly if unconstrained, but it does vibrate (its center of mass oscillates) vertically under the influence of its self force.

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[3] Sec. 65 of [6] shows that in the Darwin approximation the Liénard-Wiechert potentials (Lorenz gauge) reduce to $\phi = e/R + (e/2c^2)\partial^2 R/\partial t^2$ and $\mathbf{A} = e\mathbf{v}/cR$, from which eqs. (7)-(8) also follow.

See [9] for applications of these relations to considerations of electromagnetic momentum.

For example, if $r_q \ll R$ then $|z_q - z_Q| \approx R$, so for $y_Q = 0$,

$$F_z \approx -\frac{\omega^2 r_q^2 x_Q \delta q\, \delta Q}{2c^2 R^3} \cos\omega t = M \frac{d^2 z_{\rm cm}}{dt^2}, \tag{13}$$

where M is the mass of the system. Then, the center of mass oscillates in z according to

$$z_{\rm cm} \approx z_0 + \frac{r_q^2 x_Q}{2R^2} \frac{\delta q\, \delta Q/R}{Mc^2} \cos\omega t = z_0 + \frac{x_Q}{2} \frac{r_q^2}{R^2} \frac{U_E}{Mc^2} \cos\omega t, \tag{14}$$

where $U_E = \delta q\, \delta Q/R$ is the electrostatic energy of the charges. The amplitude of this oscillation is extremely small, which justifies the neglect of its possible effect on the oscillatory force (12).

It remains counterintuitive that the system can have a nonzero self force, which appears to violate Newton's third law (of action and reaction).

This is a famous historical issue, and led Ampère to devise a magnetic force law that obeys Newton's third law, although this is not the force law used here (which can be called the Lorentz force law). See [10] for discussion of an experiment of Ampère which seems to reinforce his view (and that of some others to this day) that the Biot-Savart-Grassmann-Lorentz law is not always valid.[4]

2.3 Electromagnetic Momentum

2.3.1 Maxwell

In Maxwell's earliest publication (at age 24) on electromagnetism [12], Part II is titled *On Faraday's "Electro-tonic State"*. On p. 52 Maxwell says:

> Considerations of this kind led Professor Faraday to connect with his discovery of the induction of electric currents the conception of a state in which all bodies are thrown by the presence of magnets and currents. This state does not manifest itself by any known phenomena as long as it is undisturbed, but any change in this state is indicated by a current or tendency towards a current. To this state he gave the name of the "Electro-tonic State".

Then on p. 65 Maxwell gives his theory of the electro-tonic state:

> The entire electro-tonic intensity round the boundary of an element of surface measures the quantity of magnetic induction which passes through that surface, or, in other words, the number of lines of magnetic force which pass through the surface.

In vector notation, with **A** as the electro-tonic intensity and **B** as the magnetic induction,

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area}. \tag{15}$$

[4]For another electromechanical example with similar issues to the present case, see [11],

Thus, we recognize Maxwell's electro-tonic intensity as the vector potential,[5] and it is often said that Faraday's electro-tonic state is the vector potential.

Maxwell continued discussion of his vector $\mathbf{A}$ on p. 290 of [13], now calling it the "electrotonic state" and seeking a mechanical interpretation in terms of "molecular vortices". On p. 389 he associates the vector $\mathbf{A}$ with a kind of "reduced momentum", arguing that if $\mathbf{A}$ is changing there is a force on a unit electric charge given by $d\mathbf{A}/dt$.[6,7]

In his great paper of 1865 [14], Maxwell reinforced this interpretation of his electro-tonic intensity/electrotonic state, now calling it the "electromagnetic momentum". His discussion in sec. 57, p. 481 concerns the mechanical momentum $\mathbf{P}_{\rm mech}$ of a unit charge q subject to the electric field induced by a changing $\mathbf{A}$,

$$\frac{d\mathbf{P}_{\rm mech}}{dt} = \mathbf{F} = q\mathbf{E} = -\frac{q}{c}\frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{P}_{\rm mech} = \mathbf{P}_0 - \frac{q\mathbf{A}}{c}\,. \tag{16}$$

A charge q that somehow arrives at a point where the vector potential is $\mathbf{A}$, it will have extracted momentum $-q\mathbf{A}/c$ from the electromagnetic field. Supposing that all charges (and possible other masses) started from rest, with zero total initial momentum, for momentum to be conserved it must be that the field now stores momentum $q\mathbf{A}/c$, leading to the interpretation that $\mathbf{A}$ is a kind of electromagnetic momentum.

The total electromagnetic momentum stored in the field must be

$$\mathbf{P}_{\rm EM}^{\rm (M)} = \sum \frac{q_i \mathbf{A}_i}{c} \to \int \frac{\rho \mathbf{A}}{c}\, d{\rm Vol}, \tag{17}$$

where the superscript M indicates that this form is due to Maxwell.[8]

2.3.2 Electromagnetic Momentum of Charge Elements δq and δQ

From eq. (17), using the Darwin approximation (6) for the vector potential, the electromagnetic momentum of the combined system of charges q_1 and q_2 is,

$$\mathbf{P}_{\rm EM} = \frac{q_1 q_2}{2c^2 R}[\mathbf{v}_1 + \mathbf{v}_2 + (\mathbf{v}_1\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} + (\mathbf{v}_2\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}], \tag{18}$$

For the case of charge elements δq and δQ, where the latter is at rest, the electromagnetic momentum is

$$\mathbf{P}_{\rm EM} = \frac{\delta q\,\delta Q}{2c^2 R}\,(\mathbf{v}_q + (\mathbf{v}_q\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}) = \frac{\delta q\,\delta Q}{2c^2 R}\left(\mathbf{v}_q + \frac{\mathbf{v}_q\cdot\mathbf{R}}{R^2}\,\mathbf{R}\right) \tag{19}$$

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[5] Maxwell noted on p. 59 of [12] that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, and that $\mathbf{A}$ can be subject to what is now called a gauge transformation while leaving $\mathbf{B}$ unchanged, so he is free to set $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, this being the first appearance of the Coulomb gauge. On p. 62, Maxwell attributed an energy $\int \mathbf{J}\cdot\mathbf{A}\, d{\rm Vol}$ to the interaction of a current density with the electro-tonic state, and on p. 64 he remarked that a changing $\mathbf{A}$ leads to an electric field $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$ (in Gaussian units, which we employ in this note), where c is the speed of light in vacuum. On p. 73 he gives a sample computation of $\mathbf{A}$ for a sphere of radius a with a $\sin\theta$ winding, finding that $A_\phi = (rB_0/2)\sin\theta$ for $r < a$ and $(a^3 B_0/2r^2)\sin\theta$ for $r > a$, where B_0 is the uniform magnetic field inside the sphere.

[6] Maxwell appears to have reversed the sign of $\mathbf{A}$ in [13] compared to the now-usual convention.

[7] Maxwell discusses the force on a unit charge on p. 342 of [13], giving in eq. (77) what is now called the Lorentz force law.

[8] We note that Maxwell worked in the Coulomb gauge, where $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$.

For other expressions for the electromagnetic momentum, see [15].

Noting that $\mathbf{R} = \mathbf{R}_q - \mathbf{R}_Q$, we find,

$$\frac{d\mathbf{R}}{dt} = \mathbf{v}_q, \qquad \frac{d}{dt}\frac{1}{R} = -\frac{\mathbf{v}_q \cdot \mathbf{R}}{R^3}, \qquad \frac{d}{dt}\frac{1}{R^2} = \frac{2}{R}\frac{d}{dt}\frac{1}{R} = -\frac{2\mathbf{v}_q \cdot \mathbf{R}}{R^4}, \tag{20}$$

and the time rate of change of the electromagnetic momentum is

$$\begin{aligned} \frac{d\mathbf{P}_{\mathrm{EM}}}{dt} &= -\frac{\delta q\, \delta Q}{2c^2 R^3}\left(\mathbf{v}_q(\mathbf{v}_q \cdot \mathbf{R}) + \frac{(\mathbf{v}_q \cdot \mathbf{R})^2}{R^2}\mathbf{R}\right) \\ &\quad + \frac{\delta q\, \delta Q}{2c^2 R}\left(\dot{\mathbf{v}}_q + \frac{\dot{\mathbf{v}}_q \cdot \mathbf{R}}{R^2}\mathbf{R} + \frac{v_q^2}{R^2}\mathbf{R} + \frac{\mathbf{v}_q(\mathbf{v}_q \cdot \mathbf{R})}{R^2} - \frac{2(\mathbf{v}_q \cdot \mathbf{R})^2}{R^4}\mathbf{R}\right) \\ &= \frac{\delta q\, \delta Q}{2c^2 R}\left(\dot{\mathbf{v}}_q + \frac{\dot{\mathbf{v}}_q \cdot \mathbf{R}}{R^2}\mathbf{R} + \frac{v_q^2}{R^2}\mathbf{R} - \frac{3(\mathbf{v}_q \cdot \mathbf{R})^2}{R^4}\mathbf{R}\right). \end{aligned} \tag{21}$$

We compare this to the self force on these charge elements, recalling the Darwin approximation (7),

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_{\delta q} + \mathbf{F}_{\delta Q} = \delta_q \mathbf{E}_{\delta Q} + \delta_Q \mathbf{E}_{\delta q} \\ &= -\frac{\delta q\, \delta Q}{2c^2 R}\left(\dot{\mathbf{v}}_q + \frac{\dot{\mathbf{v}}_q \cdot \mathbf{R}}{R^2}\mathbf{R} - \frac{3(\mathbf{v}_q \cdot \mathbf{R})^2 - v_q^2}{R^4}\mathbf{R}\right) = -\frac{d\mathbf{P}_{\mathrm{EM}}}{dt}. \end{aligned} \tag{22}$$

Since $\mathbf{F} = d\mathbf{P}_{\mathrm{mech}}/dt$, we have that the total momentum of the system is constant in time,

$$\frac{d\mathbf{P}_{\mathrm{total}}}{dt} = \frac{d\mathbf{P}_{\mathrm{mech}}}{dt} + \frac{d\mathbf{P}_{\mathrm{EM}}}{dt} = 0. \tag{23}$$

Hence, it is not a violation of momentum conservation that there exists a nonzero axial force on the system, as found in sec. 2.2.

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