

# The Helical Wiggler

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## 1 Problem

A variant on the electro- or magnetostatic boundary value problem arises in accelerator physics, where a specified field, say  $\mathbf{B}(0, 0, z)$ , is desired along the  $z$  axis. In general there exist static fields  $\mathbf{B}(x, y, z)$  that reduce to the desired field on the axis, but the “boundary condition”  $\mathbf{B}(0, 0, z)$  is not sufficient to insure a unique solution.<sup>1</sup>

For example, find a field  $\mathbf{B}(x, y, z)$  that reduces to

$$\mathbf{B}(0, 0, z) = B_0 \cos kz \hat{\mathbf{x}} + B_0 \sin kz \hat{\mathbf{y}} \quad (1)$$

on the  $z$  axis. In this, the magnetic field rotates around the  $z$  axis as  $z$  advances.

Show how the use of rectangular or cylindrical coordinates leads “naturally” to different forms for  $\mathbf{B}$ .

One 3-dimensional field extension of (1) is the so-called helical wiggler [2, 3], which obeys the auxiliary requirement that the field at  $z + \delta$  be the same as the field at  $z$ , but rotated by angle  $k\delta$ . Show that this field pattern can be realized by a current-carrying wire that is wound in a helix of period  $\lambda = 2\pi/k$  [4].

## 2 Solution

### 2.1 Solution in Rectangular Coordinates

We first seek a solution in rectangular coordinates, and expect that separation of variables will apply. Thus, we consider the form

$$B_x = f(x)g(y) \cos kz, \quad (2)$$

$$B_x = F(x)G(y) \sin kz, \quad (3)$$

$$B_z = A(x)B(y)C(z). \quad (4)$$

Then

$$\nabla \cdot \mathbf{B} = 0 = f'g \cos kz + FG' \sin kz + ABC', \quad (5)$$

where the  $'$  indicates differentiation of a function with respect to its argument. Equation (5) can be integrated with respect to  $z$  to give

$$ABC = -\frac{f'g}{k} \sin kz + \frac{FG'}{k} \cos kz. \quad (6)$$

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<sup>1</sup>If the axial field has only an axial component a unique solution obtains [1].

The  $z$  component of  $\nabla \times \mathbf{B} = 0$  tells us that

$$\frac{\partial B_x}{\partial y} = fg' \cos kz = \frac{\partial B_y}{\partial x} = F'G \sin kz. \quad (7)$$

For this to hold at all  $x$  and  $y$  we must have  $g' = 0 = F'$ , which implies that  $g$  and  $F$  are constant, say 1. Likewise,

$$\frac{\partial B_x}{\partial z} = -fk \sin kz = \frac{\partial B_z}{\partial x} = A'BC = -\frac{f''}{k} \sin kz, \quad (8)$$

using eqs. (6)-(7). Thus,  $f'' - k^2 f = 0$ , so

$$f = f_1 e^{kx} + f_2 e^{-kx}. \quad (9)$$

Finally,

$$\frac{\partial B_y}{\partial z} = Gk \cos kz = \frac{\partial B_z}{\partial y} = AB'C = \frac{G''}{k} \sin kz, \quad (10)$$

so

$$G = G_1 e^{ky} + G_2 e^{-ky}. \quad (11)$$

The “boundary conditions”  $f(0) = B_0 = G(0)$  are satisfied by

$$f = B_0 \cosh kx, \quad G = B_0 \cosh ky, \quad (12)$$

which together with eq. (6) leads to the solution

$$B_x = B_0 \cosh kx \cos kz, \quad (13)$$

$$B_y = B_0 \cosh ky \sin kz, \quad (14)$$

$$B_z = -B_0 \sinh kx \sin kz + B_0 \sinh ky \cos kz, \quad (15)$$

This satisfies the last “boundary condition” that  $B_z(0, 0, z) = 0$ .

However, this solution does not have helical symmetry.

## 2.2 Solution in Cylindrical Coordinates

Suppose instead, we look for a solution in cylindrical coordinates  $(r, \theta, z)$ . We again expect separation of variables, but we seek to enforce the helical symmetry that the field at  $z + \delta$  be the same as the field at  $z$ , but rotated by angle  $k\delta$ . This symmetry implies that the argument  $kz$  should be replaced by  $kz - \theta$ , and that the field has no other  $\theta$  dependence.

We begin constructing our solution with the hypothesis that

$$B_r = F(r) \cos(kz - \theta), \quad (16)$$

$$B_\theta = G(r) \sin(kz - \theta). \quad (17)$$

To satisfy the condition (1) on the  $z$  axis, we first transform this to rectangular components,

$$B_z = F(r) \cos(kz - \theta) \cos \theta + G(r) \sin(kz - \theta) \sin \theta, \quad (18)$$

$$B_y = -F(r) \cos(kz - \theta) \sin \theta + G(r) \sin(kz - \theta) \cos \theta, \quad (19)$$

from which we learn that the “boundary conditions” on  $F$  and  $G$  are

$$F(0) = G(0) = B_0. \quad (20)$$

A suitable form for  $B_z$  can be obtained from  $(\nabla \times \mathbf{B})_r = 0$ :

$$\frac{1}{r} \frac{\partial B_z}{\partial \theta} = \frac{\partial B_\theta}{\partial z} = kG \cos(kz - \theta), \quad (21)$$

so

$$B_z = -krG \sin(kz - \theta), \quad (22)$$

which vanishes on the  $z$  axis as desired.

From either  $(\nabla \times \mathbf{B})_\theta = 0$  or  $(\nabla \times \mathbf{B})_z = 0$  we find that

$$F = \frac{d(rG)}{dr} = \frac{d(krG)}{dkr}. \quad (23)$$

Then,  $\nabla \cdot \mathbf{B} = 0$  leads to

$$(kr)^2 \frac{d^2(krG)}{d(kr)^2} + kr \frac{d(krG)}{d(kr)} - [1 + (kr)^2](krG) = 0. \quad (24)$$

This is the differential equation for the modified Bessel function of order 1 [5]. Hence,

$$G = C \frac{I_1(kr)}{kr} = \frac{C}{2} \left[ 1 + \frac{(kr)^2}{8} + \dots \right], \quad (25)$$

$$F = C \frac{dI_1}{d(kr)} = C \left( I_0 - \frac{I_1}{kr} \right) = \frac{C}{2} \left[ 1 + \frac{3(kr)^2}{8} + \dots \right]. \quad (26)$$

The “boundary conditions” (20) require that  $C = 2B_0$ , so our second solution is

$$B_r = 2B_0 \left( I_0(kr) - \frac{I_1(kr)}{kr} \right) \cos(kz - \theta), \quad (27)$$

$$B_\theta = 2B_0 \frac{I_1}{kr} \sin(kz - \theta), \quad (28)$$

$$B_z = -2B_0 I_1 \sin(kz - \theta), \quad (29)$$

which is the form discussed in [3].

### 2.3 Magnetic Field Due to a Double Helix

This section follows [6].

We consider a wire that carries current  $I$  and is wound in the form of a helix of radius  $a$  and period  $\lambda = 2\pi/k$ . A suitable equation of this helix is

$$x_1 = a \sin kz, \quad y_1 = -a \cos kz. \quad (30)$$

The magnetic field due to this winding has a nonzero  $z$  component along the axis, which is not desired. Therefore, we also consider a second helical winding,

$$x_2 = -a \sin kz, \quad y_2 = a \cos kz, \quad (31)$$

which is offset from the first by half a period and which carries current  $-I$ . The combined magnetic field from the two helices has no component along their common axis.

The unit vector  $\hat{\mathbf{i}}_{1,2}$  that is tangent to helix 1(2) at a point

$$\mathbf{r}'_{1,2} = (x'_{1,2}, y'_{1,2}, z') = (\pm a \sin kz', \mp a \cos kz', z') \quad (32)$$

has components

$$\hat{\mathbf{i}}_{1,2} = \frac{(\pm 2\pi a \cos kz', \pm 2\pi a \sin kz', \lambda)}{\sqrt{\lambda^2 + (2\pi a)^2}}, \quad (33)$$

and the element  $d\mathbf{l}'_{1,2}$  of arc length along the helix is related by

$$d\mathbf{l}'_{1,2} = \hat{\mathbf{i}}_{1,2} dz' \frac{\sqrt{\lambda^2 + (2\pi a)^2}}{\lambda} = dz' (\pm ka \cos kz', \pm ka \sin kz', 1). \quad (34)$$

The magnetic field  $\mathbf{B}$  at a point  $\mathbf{r} = (0, 0, z)$  on the axis is given by

$$\begin{aligned} \mathbf{B}(0, 0, z) &= \frac{I}{c} \int_1 \frac{d\mathbf{l}'_1 \times (\mathbf{r}'_1 - \mathbf{r})}{|\mathbf{r}'_1 - \mathbf{r}|^3} - \frac{I}{c} \int_2 \frac{d\mathbf{l}'_2 \times (\mathbf{r}'_2 - \mathbf{r})}{|\mathbf{r}'_2 - \mathbf{r}|^3} \\ &= \frac{2Ia}{c} \int_{-\infty}^{\infty} \frac{dz'}{[a^2 + (z' - z)^2]^{3/2}} [\hat{\mathbf{x}}(k(z' - z) \sin kz' + \cos kz') \\ &\quad + \hat{\mathbf{y}}(-k(z' - z) \cos kz' + \sin kz')] \\ &= \frac{2I}{ca} \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^{3/2}} [\hat{\mathbf{x}}(kat \sin(kat + kz) + \cos(kat + kz)) \\ &\quad + \hat{\mathbf{y}}(-kat \cos(kat + kz) + \sin(kat + kz))] \\ &= \frac{4Ik}{c} (\hat{\mathbf{x}} \cos kz + \hat{\mathbf{y}} \sin kz) \left[ \frac{1}{ka} \int_0^{\infty} \frac{\cos kat}{(1 + t^2)^{3/2}} dt + \int_0^{\infty} \frac{t \sin kat}{(1 + t^2)^{3/2}} dt \right], \quad (35) \end{aligned}$$

where we made the substitution  $z' - z = at$  in going from the second line to the third. Equation 9.6.25 of [5] tells us that

$$\int_0^{\infty} \frac{\cos kat}{(1 + t^2)^{3/2}} dt = kaK_1(ka), \quad (36)$$

where  $K_1$  also satisfies eq. (24). We integrate the last integral by parts, using

$$u = \sin kat, \quad dv = \frac{t dt}{(1 + t^2)^{3/2}}, \quad \text{so} \quad du = ka \cos kat dt, \quad v = -\frac{1}{\sqrt{1 + t^2}}. \quad (37)$$

Thus,

$$\int_0^{\infty} \frac{t \sin kat}{(1 + t^2)^{3/2}} dt = ka \int_0^{\infty} \frac{\cos kat}{\sqrt{1 + t^2}} dt = kaK_0(ka), \quad (38)$$

using 9.6.21 of [5]. Hence

$$\mathbf{B}(0, 0, z) = \frac{4Ik}{c} [kaK_0(ka) + K_1(ka)] (\hat{\mathbf{x}} \cos kz + \hat{\mathbf{y}} \sin kz). \quad (39)$$

Both  $K_0(ka)$  and  $K_1(ka)$  have magnitudes  $\approx 0.5e^{-ka}$  for  $ka \approx 1$ . That is, the field on the axis of the double helix is exponentially damped in the radius  $a$  for a fixed current  $I$ .

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