

Heat Flow from a Point Source at the End of a Bar

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(November 18, 2014; updated November 22, 2014)

1 Problem

As a simple example of 3-dimensional heat flow, deduce the steady-state temperature distribution inside a semi-infinite square bar with a point source of heat somewhere on its square face, assuming no heat flow across other surfaces (except the square face at infinity).¹

2 Solution

The heat flux vector \mathbf{J} obeys

$$\mathbf{J} = -\kappa \nabla T, \quad (1)$$

where κ is the thermal conductivity and T is the temperature distribution. Energy is conserved in the interior of the bar, so $\nabla \cdot \mathbf{J} = 0$ there in the steady state, and hence $\nabla^2 T = 0$.

We consider a separation-of-variable solution in a rectangular coordinate system, taking the heat source Q to be at $(x_0, y_0, 0)$, with the bar extending over the $z \geq 0$ with square cross section $|x|, |y| \leq a/2$. The normal derivative of the temperature is zero at the surfaces across which no heat flows, so the boundary conditions are²

$$\frac{\partial T(x, y, 0)}{\partial z} = -\frac{Q}{\kappa} \delta(x - x_0, y - y_0), \quad (2)$$

$$\frac{\partial T(0, y, z)}{\partial x} = \frac{\partial T(a, y, z)}{\partial x} = 0 = \frac{\partial T(x, 0, z)}{\partial y} = \frac{\partial T(x, a, z)}{\partial y}. \quad (3)$$

A separated form that obeys $\nabla^2 T = 0$ and satisfies condition (3) is

$$T = \sum_{m,n=0}^{\infty} C_{mn} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} - Az. \quad (4)$$

Condition (2) is then

$$A + \frac{2\pi}{a} \sum_{m,n=0}^{\infty} \sqrt{m^2 + n^2} C_{mn} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} = \frac{Q}{\kappa} \delta(x - x_0, y - y_0). \quad (5)$$

¹Equivalently, consider a square bar of infinite length with a point source somewhere inside.

²It seems not possible to obtain an analytic solution for a bar of finite length with a point source on one end and the other end at fixed temperature, if no heat flows across its other surfaces.

On multiplying by $\cos \frac{2k\pi x}{a} \cos \frac{2l\pi y}{a}$ and integrating over the area of the square cross section of the bar, we find that

$$A = \frac{Q}{a^2\kappa}, \quad C_{kl} = \frac{2Q}{\pi a\kappa} \begin{cases} \text{undefined} & (k = l = 0), \\ \frac{1}{l} \cos \frac{l\pi y_0}{a} & (k = 0, l \geq 1), \\ \frac{1}{k} \cos \frac{k\pi x_0}{a} & (l = 0, k \geq 1), \\ \frac{2}{\sqrt{k^2+l^2}} \cos \frac{k\pi x_0}{a} \cos \frac{l\pi y_0}{a} & (k, l \geq 1). \end{cases} \quad (6)$$

Hence, on redefinit the undetermined constant C_{00} as T_0 ,

$$T = T_0 - \frac{Qz}{a^2\kappa} + \frac{2Q}{\pi a\kappa} \left(\sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \frac{e^{-2m\pi z/a}}{m} + \sum_{n=1}^{\infty} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} \frac{e^{-2n\pi z/a}}{n} \right. \\ \left. + 2 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right). \quad (7)$$

and the heat-flow vector (1) has components

$$J_x = \frac{2Q}{a^2} \left(\sum_{m=1}^{\infty} \sin \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} e^{-2m\pi z/a} \right. \\ \left. + 2 \sum_{m,n=1}^{\infty} m \sin \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right), \quad (8)$$

$$J_y = \frac{2Q}{a^2} \left(\sum_{n=1}^{\infty} \sin \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2n\pi z/a} \right. \\ \left. + 2 \sum_{m,n=1}^{\infty} n \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \sin \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right), \quad (9)$$

$$J_z = \frac{Q}{a^2} \left(1 + 2 \sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} e^{-2m\pi z/a} + 2 \sum_{n=1}^{\infty} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2n\pi z/a} \right. \\ \left. + 4 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right). \quad (10)$$

For the particular case that the point source is at the center of the end face, $x_0 = y_0 = 0$,

$$T = T_0 - \frac{Qz}{a^2\kappa} + \frac{Q}{\pi a\kappa} \left(\sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} \frac{e^{-2m\pi z/a}}{m} + \sum_{n=1}^{\infty} \cos \frac{2n\pi y}{a} \frac{e^{-2n\pi z/a}}{n} \right. \\ \left. + 2 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right). \quad (11)$$

and the heat-flow vector (1) has components

$$J_x = \frac{2Q}{a^2} \left(\sum_{m=1}^{\infty} \sin \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{m,n=1}^{\infty} m \sin \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right), \quad (12)$$

$$J_y = \frac{2Q}{a^2} \left(\sum_{n=1}^{\infty} \sin \frac{2n\pi y}{a} e^{-2n\pi z/a} + 2 \sum_{m,n=1}^{\infty} n \cos \frac{2m\pi x}{a} \sin \frac{2n\pi y}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right), \quad (13)$$

$$J_z = \frac{Q}{a^2} \left(1 + 2 \sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{n=1}^{\infty} \cos \frac{2n\pi y}{a} e^{-2n\pi z/a} + 4 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right). \quad (14)$$

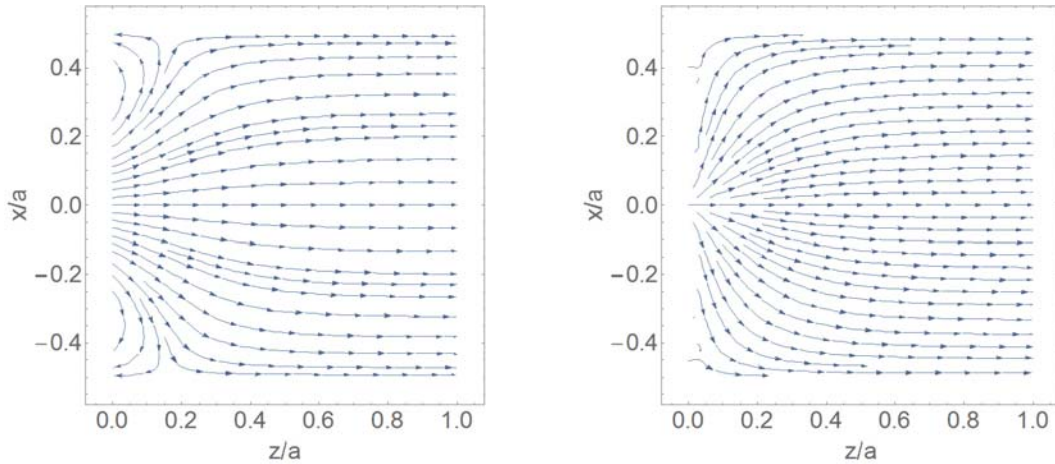
The figures below³ shows the lines of the heat-flow vector \mathbf{J} in the midplane $y = 0$,

$$J_x(x, 0, z) = \frac{2Q}{a^2} \left(\sum_{m=1}^{\infty} \sin \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{m,n=1}^{\infty} m \sin \frac{2m\pi x}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right), \quad (15)$$

$$J_y(x, 0, z) = 0, \quad (16)$$

$$J_z(x, 0, z) = \frac{Q}{a^2} \left(1 + 2 \sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{n=1}^{\infty} e^{-2n\pi z/a} + 4 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right). \quad (17)$$

Indices m and n were evaluated up to 1 in the left figure and to 20 in the right; indices higher than 1 mainly affect the region close to $z = 0$ where the delta-function boundary condition (5) is being approximated.



³The figures were generated via the Mathematica notebook <http://physics.princeton.edu/~mcdonald/examples/heatflow.nb>.

The figure indicates that the heat flow is essentially parallel to the z -axis for $z \gtrsim a/2$ from the point source, which is agreeable with naïve expectations.⁴

References

- [1] H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, 2nd ed. (Clarendon Press, 1959).

⁴This example does not appear in the great compendium [1] of lore on heat conduction, although the ingredients of the solution are, of course, well represented there. For example, sec. 14.3-III gives a 2-dimensional version of the present problem.