### 1 Problem

Although phase volume is invariant under canonical transformations of a Hamiltonian system (see, for example, [1]), approximations to phase volume such as rms emittance are not. Deduce approximate expressions for the growth of the various 2-D rms emittances of a Gaussian bunch of particles of mass \(m\) and charge \(q\), initially centered on the origin and with \(\langle p_x \rangle = 0 = \langle p_y \rangle\) but with nonzero energy spread about a central energy \(E_0\), as this bunch propagates in a region of zero electromagnetic field. For the longitudinal emittance, consider both coordinates \((z, p_z)\) and \((t, p_t = -E_{\text{total}})\) [1].

### 2 Solution

This solution is an extension of [2, 3, 4]. Numerical examples of rms emittance growth during propagation of a “beam” in a field-free region are given on slide 8 of [5].

#### 2.1 Emittances when \(t\) Is the Independent Variable

When using time \(t\) as the independent variable the canonical coordinates are \(x, y, z, p_x, p_y, p_z\) and the initial conditions are at time \(t = 0 \equiv 0_t\). We suppose the initial bunch is Gaussian, with nonzero first and second moments,

\[
\begin{align*}
\langle p_z(t = 0) \rangle &\equiv \langle p_z(0_t) \rangle = p_{0z}, \quad E_{0t} \equiv c\sqrt{m^2c^2 + p_{0t,z}^2} < \langle E(0_t) \rangle, \\
\langle x^2(0_t) \rangle = \langle y^2(0_t) \rangle = \sigma_{x1}^2, \quad \langle z^2(0_t) \rangle &= \sigma_{z1}^2, \quad \langle p_x^2(0_t) \rangle = \langle p_y^2(0_t) \rangle = \sigma_{p\perp 1}^2, \quad \langle p_z^2(0_t) \rangle = \langle p_z(0_t) - p_{0z,z} \rangle^2 = \sigma_{p_{zt}}^2, \quad \langle p_{0z}(0_t) \rangle = p_{0z} + \sigma_{p_{zt}}^2. \quad (1)
\end{align*}
\]

That is, there are no cross correlations initially. We will also need to know some higher moments, such as

\[
\begin{align*}
\langle p_{z1}^4(0_t) \rangle &= \langle p_{y1}^4(0_t) \rangle = 3\sigma_{p\perp 1}^4, \quad \langle p_{x1}^6(0_t) \rangle = \langle p_{y1}^6(0_t) \rangle = 15\sigma_{p\perp 1}^6, \\
\langle (p_z(0_t) - p_{0z,z})^3 \rangle &= 0, \quad \langle p_{z1}^3(0_t) \rangle = p_{0z,z}^3 + 3\sigma_{p_{zt}}^2 p_{0z,z}, \\
\langle p_z(0_t) p_{z1}^2(0_t) - p_{0z,z}^2 \rangle &= 3\sigma_{p_{zt}}^4, \quad \langle p_{z1}^4(0_t) \rangle = p_{0z,z}^4 + 6\sigma_{p_{zt}}^2 p_{0z,z}^2 + 3\sigma_{p_{zt}}^4, \\
\langle p_z(0_t) p_{z1}^2(0_t) - p_{0z,z}^2 \rangle &= 5\sigma_{p_{zt}}^2 p_{0z,z}^2 + 3\sigma_{p_{zt}}^4, \quad \langle (p_z(0_t) - p_{0z,z})^2 \rangle = 4\sigma_{p_{zt}}^2 p_{0z,z}^2 + 3\sigma_{p_{zt}}^4, \\
\langle (p_z(0_t) - p_{0z,z})^5 \rangle &= 0, \quad \langle p_{z1}^5(0_t) \rangle = p_{0z,z}^5 + 10\sigma_{p_{zt}}^2 p_{0z,z}^3 + 15\sigma_{p_{zt}}^4 p_{0z,z}.
\end{align*}
\]
\[
\begin{aligned}
\langle p_z(0) \rangle &= \langle p_z^2(0) \rangle = \sigma^2_{p_z}, \\
\langle x^4(t) \rangle &= \langle y^4(t) \rangle \approx \sigma^4_{p_{z}} + \frac{c^4 \sigma^2_{p_{z}} t^2}{E_{0t}^2} \left( 1 - \frac{c^2}{E_{0t}^2} (4 \sigma^2_{p_{z}} + \sigma^2_{p_{z}}) \right) \\
&\quad+ \frac{c^4}{E_{0t}^2} (24 \sigma^4_{p_{z}} + 8 \sigma^2_{p_{z}} \sigma^2_{p_{z}} + 8 \sigma^2_{p_{z}} p_{0t,z}^2 + 4 \sigma^2_{p_{z}} p_{0t,z}^2 + 3 \sigma^2_{p_{z}}),
\end{aligned}
\]
\[ \langle x(t)p_x(t) \rangle = \langle y(t)p_y(t) \rangle \approx \frac{c^2 \sigma_{px}^2 t}{E_{0t}} \left( 1 - \frac{c^2}{2E_{0t}^2} (4 \sigma_{p_{xz}}^2 + \sigma_{p_{xt}}^2) \right. \\
+ \left. \frac{3c^4}{8E_{0t}^4} (24 \sigma_{p_{xt}}^4 + 8 \sigma_{p_{xt}}^2 \sigma_{p_{zxt}}^2 + 8 \sigma_{p_{zxt}}^2 p_{0t,z}^2 + 4 \sigma_{p_{zt}}^2 p_{0t,z}^2 + 3 \sigma_{p_{xt}}^4) \right), \]

\[ \langle x(t)p_x(t) \rangle^2 = \langle y(t)p_y(t) \rangle^2 \approx \frac{c^4 \sigma_{px}^2 t^2}{E_{0t}^2} \left( 1 - \frac{c^2}{2E_{0t}^2} (4 \sigma_{p_{xt}}^2 + \sigma_{p_{zxt}}^2) \right. \\
+ \left. \frac{c^4}{E_{0t}^4} (20 \sigma_{p_{xt}}^4 + 8 \sigma_{p_{zxt}}^2 \sigma_{p_{zxt}}^2 + 6 \sigma_{p_{zxt}}^2 p_{0t,z}^2 + 3 \sigma_{p_{zt}}^2 p_{0t,z}^2 + 5 \sigma_{p_{zt}}^4/2) \right). \] (5)

The rms x and y emittances are, keeping terms only up to order \(1/E_{0t}^6\),

\[
\epsilon_x(t) = \epsilon_y(t) = \sqrt{\langle x^2(t) \rangle \langle p_x^2(t) \rangle - \langle x(t)p_x(t) \rangle^2} \\
\approx \sqrt{\sigma_{x,t}^2 \sigma_{p_{xt}}^2 + \frac{c^4 \sigma_{p_{xt}}^2 t^2}{E_{0t}^6} (4 \sigma_{p_{xt}}^4 + 2 \sigma_{p_{xt}}^2 \sigma_{p_{zxt}}^2 + \sigma_{p_{zxt}}^2 p_{0t,z}^2 + \sigma_{p_{zt}}^4/2)} \\
\approx \sigma_{x,t} \sigma_{p_{xt}} \left[ 1 + \frac{c^4 \sigma_{p_{xt}}^2 t^2}{2 \sigma_{p_{xt}}^2 E_{0t}^6} (4 \sigma_{p_{xt}}^4 + 2 \sigma_{p_{zxt}}^2 p_{0t,z}^2 + \sigma_{p_{zxt}}^2 p_{0t,z}^2 + \sigma_{p_{zt}}^4/2) \right]. \] (6)

The emittance grows quadratically with time with a coefficient that is of fourth order of smallness. There is no growth of the x or y emittance at second order of smallness, which order corresponds to the first-order term in the expansion of the square root in \(1/E \propto 1/\sqrt{1 + \Delta} \approx 1 - \Delta/2\). This has led to the statement that there is no emittance growth in “linear” beam transport although I find this use of the adjective “linear” to be obscure.

We now turn to longitudinal quantities.

\[ \langle p_z(t) \rangle = p_{0t,z} + \sigma_{p_{zt}}^2, \]

\[ \langle \Delta p_z(t) \rangle = \langle (p_z(t) - \langle p_z(t) \rangle)^2 \rangle = \langle p_z^2(t) \rangle - \langle p_z(t) \rangle^2 = \sigma_{p_{zt}}^2, \]

\[ \langle z(t) \rangle \approx \frac{c^2 t p_{0t,z}}{E_{0t}} \left( 1 - \frac{c^2}{2E_{0t}^2} (2 \sigma_{p_{zt}}^2 + 3 \sigma_{p_{zt}}^2) \right. \\
+ \left. \frac{3c^4}{8E_{0t}^4} (8 \sigma_{p_{zt}}^4 + 12 \sigma_{p_{zt}}^2 \sigma_{p_{zxt}}^2 + 4 \sigma_{p_{zxt}}^2 p_{0t,z}^2 + 15 \sigma_{p_{zt}}^4) \right), \]

\[ \langle z^2(t) \rangle \approx \frac{c^4 t^2 p_{0t,z}^2}{E_{0t}^2} \left( 1 - \frac{c^2}{E_{0t}^2} (2 \sigma_{p_{zt}}^2 + 3 \sigma_{p_{zt}}^2) \right. \\
+ \left. \frac{c^4}{E_{0t}^4} (7 \sigma_{p_{zt}}^4 + 12 \sigma_{p_{zt}}^2 \sigma_{p_{zxt}}^2 + 3 \sigma_{p_{zxt}}^2 p_{0t,z}^2 + 27 \sigma_{p_{zt}}^4/2) \right), \]

\[ \langle z^2(t) \rangle \approx \sigma_{zt}^2 + \frac{c^4 t^2 p_{0t,z}^2}{E_{0t}^2} \left( 1 - \frac{c^2}{E_{0t}^2} (2 \sigma_{p_{zt}}^2 + 3 \sigma_{p_{zt}}^2) \right. \\
+ \left. \frac{c^4}{E_{0t}^4} (8 \sigma_{p_{zt}}^4 + 12 \sigma_{p_{zt}}^2 \sigma_{p_{zxt}}^2 + 3 \sigma_{p_{zxt}}^2 p_{0t,z}^2 + 27 \sigma_{p_{zt}}^4/2) \right) + \frac{c^4 \sigma_{p_{zt}}^2 t^2}{E_{0t}^2} \left( 1 - \frac{c^2}{E_{0t}^2} (2 \sigma_{p_{zt}}^2 + 3 \sigma_{p_{zt}}^2 + 2p_{0t,z}^2) \right) \]
\[ \langle \Delta z^2(t) \rangle \equiv \langle (z(t) - \langle z(t) \rangle)^2 \rangle = \langle z^2(t) \rangle - \langle z(t) \rangle^2 \]

\approx \sigma_z^2 + \frac{c^8 t^2 E_0^4}{E_0^6} \left( 1 - \frac{c^2}{E_0^2} (2 \sigma_{p_z}^2 + 3 \sigma_{p_{zt}}^2 + 2 p_{0t,z}^2) \right) \]

\[ \langle \Delta z^2(t) \rangle \langle \Delta p_z^2(t) \rangle \approx \sigma_z^2 \sigma_{p_z}^2 + \frac{c^4 t^2 p_{0t,z}^2}{E_0^6} \left( 1 - \frac{c^2}{E_0^2} (2 \sigma_{p_z}^2 + 3 \sigma_{p_{zt}}^2 + 2 p_{0t,z}^2) \right) \]

\[ \langle z(t) p_z(t) \rangle \approx \frac{c^2 t p_{0t,z}^2}{E_0} \left( 1 - \frac{c^2}{E_0^2} (2 \sigma_{p_z}^2 + 3 \sigma_{p_{zt}}^2) \right) \]

The rms z emittance is, keeping terms only up to order \(1/E_0^6\),

\[ \epsilon_z(t) = \sqrt{\langle \Delta z^2(t) \rangle \langle \Delta p_z^2(t) \rangle - \langle \Delta z(t) \Delta p_z(t) \rangle^2} \]

\approx \sigma_z^2 \sigma_{p_z}^2 + \frac{c^4 t^2}{E_0^6} \left( \sigma_{p_z}^4 + \sigma_{p_{zt}}^2 (p_{0t,z}^2 + \sigma_{p_{zt}}^2) + \frac{15 p_{0t,z}^2}{2} + 3 \sigma_{p_{zt}}^2 \right)
\[
\sigma_{zt} \approx \sigma_{zt} \sigma_{pzt} \left[ 1 + \frac{c^2 t^2}{2E_{0t}^2 \sigma_{zt}^2 \sigma_{pzt}^2} \left( \sigma_{pzt}^4 \sigma_{pzt}^2 (p_{0t,z}^2 + \sigma_{pzt}^2) + \sigma_{pzt}^6 \frac{15p_{0t,z}^2 + 3\sigma_{pzt}^2}{2} \right) \right].
\]

If \( \sigma_{pzt}^2 \ll p_{0t,z}^2 \) the forms of the emittances (6) and (8) can be simplified accordingly.

2.1.1 Eigenemittances

It has been suggested that it will somehow be advantageous to compute the so-called eigenemittances of the beam transport. See, for example, sec. 26.3 of [6].

We consider the \( 6 \times 6 \) second-moment matrix \( \Sigma \) defined by

\[
\Sigma_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = \langle \Delta x_i \Delta x_j \rangle,
\]

(9)

where \( x_1 = x, \ x_2 = p_x, \ x_3 = y, \ x_4 = p_y, \ x_5 = z, \ x_6 = p_z \) when using \( t \) as the independent variable. For the present case of a field-free drift with no initial cross correlations and \( x-y \) symmetry, the \( 6 \times 6 \) matrix \( \Sigma \) is block diagonal with three \( 2 \times 2 \) submatrices

\[
\Sigma(t) = \begin{pmatrix}
\Sigma_x(t) & 0 & 0 \\
0 & \Sigma_y(t) & 0 \\
0 & 0 & \Sigma_z(t)
\end{pmatrix},
\]

(10)

\[
\Sigma_x(t) = \Sigma_y(t) = \begin{pmatrix}
\langle x^2(t) \rangle & \langle x(t)p_x(t) \rangle \\
\langle x(t)p_x(t) \rangle & \sigma_{p_{\perp}}^2
\end{pmatrix},
\]

(11)

\[
\Sigma_z(t) = \begin{pmatrix}
\langle \Delta z^2(t) \rangle & \langle \Delta z(t) \Delta p_z(t) \rangle \\
\langle \Delta z(t) \Delta p_z(t) \rangle & \sigma_{p_z}^2 - p_{0t,z}^2
\end{pmatrix}.
\]

(12)

The eigenemittances at time \( t \) are \( |\lambda_i| \) where \( \lambda_i \) are the eigenvalues of the matrix \( J \Sigma(t) \), and

\[
J = \begin{pmatrix}
J_2 & 0 & 0 \\
0 & J_2 & 0 \\
0 & 0 & J_2
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

(13)

In the present case the matrix \( J \Sigma(t) \) is block diagonal with the three \( 2 \times 2 \) submatrices

\[
J_2 \Sigma_x(t) = J_2 \Sigma_y(t) = \begin{pmatrix}
\langle x(t)p_x(t) \rangle & \sigma_{p_{\perp}}^2 \\
-\langle x^2(t) \rangle & -\langle x(t)p_x(t) \rangle
\end{pmatrix},
\]

(14)

\[
J_2 \Sigma_z(t) = \begin{pmatrix}
\langle \Delta z(t) \Delta p_z(t) \rangle & \sigma_{p_z}^2 - p_{0t,z}^2 \\
-\langle \Delta z^2(t) \rangle & -\langle \Delta z(t) \Delta p_z(t) \rangle
\end{pmatrix},
\]

(15)
whose eigenvalues are

\[ \lambda_1 = -\lambda_2 = \lambda_3 = -\lambda_4 = i \sqrt{\langle x^2(t) \rangle} \sigma_{p \perp}^2 - \langle x(t)p_x(t) \rangle^2 = i \epsilon_x = i \epsilon_y, \]  
(16)

\[ \lambda_5 = -\lambda_6 = i \sqrt{\langle \Delta z^2(t) \rangle} (\sigma_{p_z}^2 - p_{0z,t}^2) - \langle \Delta z(t) \Delta p_z(t) \rangle^2 = i \epsilon_z. \]  
(17)

It appears to me that eigenemittances are the same as rms emittances in the present example.

References


http://www.physics.umd.edu/dsat/