Gravitational Acceleration of a Moving Object at the Earth's Surface

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 4, 2016; updated August 29, 2018)

1 Problem

In Newton's theory of gravity the acceleration of a (small) mass at the surface of the Earth is g, independent of the velocity of the mass, where in the approximation of a spherical Earth of mass M_E and radius R_E, $g = GM_E/R_E^2$, with G being Newton's constant of gravitation. Deduce the velocity dependence of the acceleration in Einstein's theory of general relativity.

2 Solution

Einstein deduced (very briefly on p. 834 of [1], and in slightly more detail on pp. 820-822 of [2]) that the gravitational deflection of light according to general relativity is twice the "Newtonian" value.[1] Einstein did not deduce the acceleration of light (or that of other fast-moving particles), which has perhaps left the impression that the acceleration at the surface of the Earth is no different from that in Newton's theory (because curvature of space occurs only over large distances in "outer space").

Indeed, the topic of the acceleration due to gravity in Einstein's theory seems to be rarely discussed. It appears in somewhat disguised form in sec. VC of [7] (titled "Laboratory experiments to test relativistic gravity"), where I interpret eq. (6.18) as implying

$$a = g(1 + v^2/c^2), \tag{1}$$

for horizontal velocity v of a test mass at the Earth's surface, with c as the speed of light in vacuum (at the Earth's surface). The only other discussions that I have found is at the end

[1] In [1, 2], Einstein did not refer to Newton, but to his earlier computations of the gravitational deflection of light [3, 4] based on the effect of gravity on the speed of light. In his book of 1920 [5], Einstein stated (p. 153 of the English translation):

> It may be added that, according to the theory, half of this deflection is produced by the Newtonian field of attraction of the sun, and the other half by the geometrical modification ("curvature") of space caused by the sun.

Newton may not have explicitly stated that light is subject to the acceleration due to gravity, but he came very close to this in Questions 29 and 31, pp. 345 and 350, of his *Opticks* [6]:

> Quest. 29. Are not the Rays of Light very small Bodies emitted from shining Substances?

> Quest. 31. Have not the small Particles of Bodies certain Powers, Virtues, or Forces, by which they act at a distance, not only upon the Rays of Light for reflecting, refracting, and inflecting them, but also upon one another for producing a great Part of the Phænomena of Nature? For it's well known, that Bodies act one upon another by the Attractions of Gravity, Magnetism, and Electricity; and these Instances shew the Tenor and Course of Nature, and make it not improbable but that there may be more attractive Powers than these.

of [8] (Hilbert, 1916), which results are transcribed in sec. 6.7 of [9], where it is claimed that the acceleration for vertical motion at the surface of the Earth is,[2,3]

$$a = g(1 - 3v^2/c^2) \to \begin{cases} g & (v = 0), \\ -2g & (v = c). \end{cases} \tag{2}$$

However, as remarked on p. 191 of [17]:

> Whenever we obtain a prediction from general relativity the question always arises (or should arise) whether the result obtained really refers to an objective physical measurement or whether it has folded into it arbitrary subjective elements dependent on our choice of coordinate systems.

And, as somewhat sourly observed in the second paragraph of [18]:

> ... the fact that (Einstein's equations) retain their form under general coordinate transformation is an embarrassment rather than an advantage.

As will be discussed below, corrections to the flat-space metric tensor at the surface of the Earth are of order $R_M/R_E \approx 10^{-9}$, where $R_M = 2GM/c^2$ is the Schwarzschild radius corresponding to mass M, which is about 1 cm for the Earth, and yet general-relativistic corrections to the Newtonian gravitational acceleration of a fast-moving mass at the surface of a spherical mass are of order unity. This is rather counterintuitive, but is little discussed in the literature.

Einstein deftly avoided this issue in his discussion [1, 2] of the gravitational deflection of light by only considering the deflection according to observers far from the surface of the deflecting mass.

Einstein's original discussion [1] was based on an approximation to the metric of a static, spherical mass, but shortly thereafter an "exact" metric was found by Schwarzschild [19]. In principle, the rotation of the Earth could add an effect of "magnetic gravity", but this is very small as first noted by de Sitter [20], and we will only consider the case of spherical symmetry here.[4]

2.1 Spherically Symmetric Metrics

This section is something of a historical digression.

Schwarzschild [19] looked for a static, spherically symmetric solution to Einstein's equations for a point mass M at the origin in spherical coordinates (t, r, θ, ϕ) such that the square of the invariant interval (line element) for small motion of a test mass would have the form[5]

$$ds^2 = A\, c^2\, dt^2 - B\, dr^2 - C^2\, d\Omega^2, \qquad d\Omega^2 = (d\theta^2 + \sin^2\theta\, d\phi^2), \tag{3}$$

---

[2]The gravitational deflection of high-speed particles was considered in [10] (1920) using what appears to be a spurious version of isotropic coordinates, with a claim of upwards acceleration for particles with $v > c/\sqrt{3}$, as in eq. (2). This was objected to in [11], followed by a more usual discussion of the gravitational deflection of light without mention of acceleration.

[3]This paper emphasizes the acceleration of "point" masses. For extended objects, internal motion can affect the acceleration of their center of mass [12, 13, 14, 13, 16].

[4]It has been shown in [22, 21, 23, 24] that the only spherically symmetric solution to Einstein's equation is for a static metric. Kerr [25] has found a static metric for a rotating spherical mass.

[5]Strictly, Schwarzschild wrote F for A, $G + Hr^2$ for B, and Gr^2 for C^2.

where $A,\ B \overset{r\to\infty}{\longrightarrow} 1$, and $C \overset{r\to\infty}{\longrightarrow} r$, such that spacetime is asymptotically flat at large distances from the mass. The circumference of a circle of radius r is $2\pi C$ (not $2\pi r$),[6] which is a reminder that the physical interpretation of the coordinate r (and t) is not obvious.

A general result[7] is that A and B can be written in terms of $C(r)$ and the Schwarzschild radius R_M, such that the line element (3) takes the form,

$$ds^2 = \left(1 - \frac{R_M}{C}\right) c^2\, dt^2 - \frac{C'^2}{1 - R_M/C}\, dr^2 - C^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{4}$$

Over the years, many variants of the Schwarzschild metric have been considered, with various forms of the function $C(r)$,

1. Schwarzschild [19] (Jan. 16, 1916): $C = (r^3 + R_M^3)^{1/3}$.

   Note that Schwarzschild's form is not that commonly associated with the "Schwarzschild" metric, which latter is actually the form of item 2.

2. Droste [28] (May 27, 1916); Flamm [29] (Sept. 3, 1916); Hilbert [8] (Dec. 23, 1916); Weyl [30] (Aug. 8, 1917): $C = r$.

3. "Isotropic" Coordinates:[8] $C = r(1 + R_M/4r)^2$.

4. Brillouin [31] (Jan. 1923): $C = r + R_M$.

Although it is not our main concern here, we note that for all of the above forms the metric is "peculiar" (if not "singular") at $r = R_M$, and the physical meaning of the solutions is unclear for smaller r. An eventual understanding emerged [33, 34] that a test mass cannot have constant r at values less than R_M, such that the notion of a static mass of radius $0 < r < R_M$ is not valid, and physically reasonable metrics in this regime must be time-dependent.

The first hint of this resolution to the (non)issue of a "singularity" at $r = R_M$ was given by Lemaître [35] (1933), when he found the (diagonal), time-dependent metric:

$$ds^2 = c^2\, dt^2 - \left(\frac{2R_M}{3(r-t)}\right)^{2/3} dr^2 - \left(\frac{3\sqrt{R_M}(r-t)}{2}\right)^{4/3} d\Omega^2. \tag{5}$$

The metric tensor is also diagonal for all of the Schwarzschild metrics. Time-dependent metrics with off-diagonal elements have also been considered:

---

[6] Apparently, C is sometimes called the "curvature radius".

[7] This result is deduced in Appendix A of the paper [26], whose polemic against the notion of a "black hole" was carried further in [27].

[8] For isotropic coordinates, $B = C'^2/(1 - R_m/C) = C^2/r^2$, such that the spatial part of the line element (4) is $(1 + R_M/4r)^2(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2)$, which is isotropic. Hence, these coordinates may be the "closest" to those for flat space, if one emphasizes spatial geometry. These coordinates appear without attribution in eq. (421b) of [32] (Pauli, 1920). For completeness, $A = (1 - R_M/4r)^2/(1 + R_M/4r)^2$.

In [10], $A = 1 - R_M/r$ and $B - Cr^2 = 1 + R_M/r$, which are a kind of isotropic coordinates, but they do not correspond to a static mass M.

1. *Gullstrand [36] (May 25, 1921); Painlevé [37] (Oct. 1921):*

$$ds^2 = \left(1 - \frac{R_M}{r}\right) c^2\, dt^2 - 2\sqrt{\frac{r}{R_M}}\, c\, dt\, dr - dr^2 - r^2 d\Omega^2. \tag{6}$$

2. *Szekeres [38] (May 26, 1959); Kruskal [39] (Dec. 21, 1959); (extending earlier work of Synge [18]):*

$$ds^2 = \frac{4R_M}{r}\, e^{-r/R_M}(c^2\, \mathsf{d}\mathsf{t}^2 - d\mathsf{r}^2) - r^2 d\Omega^2, \tag{7}$$

$$c^2\, \mathsf{t}^2 - \mathsf{r}^2 = \left(1 - \frac{r}{R_M}\right) e^{r/R_M}, \qquad \frac{2c\,\mathsf{r}\,\mathsf{t}}{\mathsf{r}^2 + c^2\,\mathsf{t}^2} = \tanh\frac{ct}{R_M}\,. \tag{8}$$

3. *Penrose [40] (1964) built on earlier work by Finkelstein [41] (1958) and a much earlier hint by Eddington [42] (1924) to provide yet another time-dependent, nondiagonal metric.*

2.2 Equations of Motion

We now consider the equations of motion for a test particle (of small mass/energy compared to M) located on the x axis at $(r, \theta, \phi) = (r, \pi/2, 0)$ at time $t = 0$, and which moves in the "equatorial" plane with $\theta = \pi/2$ always.[9] We only consider metrics of the form (4), and compute the geodesic equations (eqs. (2) and (7) of [1]),

$$\frac{d^2x^\lambda}{dp^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\,, \qquad \text{where} \qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right), \tag{9}$$

in which the parameter p can be taken as the proper time s for a particle with nonzero mass.

The generalized accelerations d^2x^λ/dp^2 for geodesic motion in "curved" spacetime are the sums of terms that are products of generalized velocities, dx^μ/dp and dx^ν/dp with factors, $\Gamma^\lambda_{\mu\nu}$ = Christoffel symbols, that characterize the distortions of spacetime due to the presence of mass/energy.

In this view, it is less surprising that the acceleration due to "gravity" is velocity dependent, than that there is any effect independent of velocity. The latter occurs in that the "velocity" dt/dp for the time coordinate is essentially a constant (except in regions of extreme curvature). With this understanding, we should then expect that the accelerations have "corrections" of order unity for spatial velocities $d\mathbf{x}/dp$ that are of order unity (which in "relativistic" language means velocities of order the speed of light).

Thus, it is surprising (to this author) that 100 years of discussion of general relativity place such little emphasis on these "obvious" aspects of the geodesic equations (9).[10]

[9] This section follows sec. 3.8.4 of [17], which follows p. 68 ff of [8].

[10] While applications of the special relativity tend to emphasize physics associated with high velocities, applications of general relativity tend to emphasize rather low-velocity situations. An exception is the gravitational deflection of light, which is usually treated in a manner that makes no use of the notion of the gravitational acceleration of a fast-moving object.

For the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ we can refer to the compilation in [43], identifying $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = t$. Then, in the notation of [43], the line element (4) tells us that,

$$A = \frac{C'^2}{1 - R_M/C(r)}, \qquad B = C^2(r), \qquad C = C^2(r)\sin^2\theta, \qquad D = c^2\left(1 - \frac{R_M}{C(r)}\right). \quad (10)$$

The nonzero Christoffel symbols are then, from p. 560 of [43],

$$\Gamma^r_{rr} = \frac{A'}{2A} = \frac{C''}{C'} - \frac{R_M C'}{2C(C - R_M)}, \qquad \Gamma^r_{\theta\theta} = -\frac{B'}{2A} = \frac{R_M - C}{C'}, \quad (11)$$

$$\Gamma^r_{\phi\phi} = -\frac{CC'\sin^2\theta}{A} = \frac{R_M - C}{C'}\sin^2\theta, \qquad \Gamma^r_{tt} = \frac{D'}{2A} = \frac{c^2 R_M(C - R_M)}{2C'C^3}, \quad (12)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{C'}{C}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad (13)$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta, \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{D'}{2D} = \frac{R_M C'}{2C(C - R_M)}, \quad (14)$$

where $A' = \partial A/\partial r$, *etc.* Using these, the four equations of motion (9) become,

$$\frac{d^2 r}{dp^2} = -\frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 + \frac{B'}{2A}\left(\frac{d\theta}{dp}\right)^2 + \frac{CC'\sin^2\theta}{A}\left(\frac{d\phi}{dp}\right)^2 - \frac{D'}{2A}\left(\frac{dt}{dp}\right)^2, \quad (15)$$

$$\frac{d^2\theta}{dp^2} = -\frac{2C'}{C}\frac{dr}{dp}\frac{d\theta}{dp} + \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2, \quad (16)$$

$$\frac{d^2\phi}{dp^2} = -\frac{2C'}{C}\frac{dr}{dp}\frac{d\phi}{dp} - 2\cot\theta\frac{d\phi}{dp}\frac{d\theta}{dp} = -\frac{2}{C}\frac{dC}{dp}\frac{d\phi}{dp} - 2\cot\theta\frac{d\phi}{dp}\frac{d\theta}{dp}, \quad (17)$$

$$\frac{d^2 t}{dp^2} = -\frac{R_M C'}{C(C - R_M)}\frac{dr}{dp}\frac{dt}{dp} = -\frac{R_M}{C(C - R_M)}\frac{dC}{dp}\frac{dt}{dp}. \quad (18)$$

2.2.1 Constants of the Motion

As we consider only the case of motion in the plane $\theta = \pi/2$, the θ-equation (16) is automatically satisfied, and $d\theta/dp = 0$ in eqs. (15) and (17). Then, eq. (17) leads to

$$\frac{d(d\phi/dp)/dp}{d\phi/dp} + 2\frac{dC/dp}{C} = 0, \qquad \Rightarrow \qquad \ln\frac{d\phi}{dp} + \ln C^2 = \text{const}, \qquad \Rightarrow \qquad C^2\frac{d\phi}{dp} = J, \quad (19)$$

where the constant J is related to the angular momentum of the test particle.

Similarly, eq. (18) leads to, using Dwight 101.1,

$$\frac{d(dt/dp)/dp}{dt/dp} + R_M\frac{dC/dp}{C(C - R_M)} = 0, \qquad \Rightarrow \qquad \ln\frac{dt}{dp} + \ln\left(1 - \frac{R_M}{C}\right) = \text{const}. \quad (20)$$

We take the constant to be zero, as an aspect of the definition of parameter p, such that the "velocity" dt/dp becomes,

$$\frac{dt}{dp} = \frac{1}{1 - R_M/C} = \frac{C}{C - R_M} = \frac{c^2}{D}. \quad (21)$$

For large r, $dt/dp \to 1$, and $p \to t$.

We can now use eqs. (19) and (21) in eq. (15) for $\theta = \pi/2$, and multiply it by $2A\ dr/dp$ to find

$$2A\frac{dr}{dp}\frac{d^2r}{dp^2} + \frac{dA}{dp}\left(\frac{dr}{dp}\right)^2 - \frac{2J^2}{C^3}\frac{dC}{dp} + \frac{c^4}{D^2}\frac{dD}{dp} = 0, \tag{22}$$

$$A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{C^2} - \frac{c^4}{D} = -E = \text{const.} \tag{23}$$

$$\frac{C'^2}{1 - R_M/C}\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{C^2} - \frac{c^2}{1 - R_M/C} = -E. \tag{24}$$

The line element (4) can now be written (for $\theta = \pi/2$) in terms of parameter p as

$$\begin{aligned} ds^2 &= dp^2\left[\left(1 - \frac{R_M}{C}\right)\left(\frac{c\,dt}{dp}\right)^2 - \frac{C'^2}{1 - R_M/C}\left(\frac{dr}{dp}\right)^2 - C^2\left(\frac{d\phi}{dp}\right)^2\right] \\ &= dp^2\left[-\frac{C'^2}{1 - R_M/C}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{C^2} + \frac{c^2}{1 - R_M/C}\right] = E\,dp^2. \end{aligned} \tag{25}$$

For a massless particle, the constant E is zero, while for a particle with nonzero rest mass, $E > 0$.

For a particle with nonzero mass m at large r, where $p \to t$ and $C \to r$, eq. (24) becomes

$$-E \approx \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2 - c^2 - c^2R_M/r = v^2 - c^2 - \frac{2GM}{r} \tag{26}$$

$$mc^2 + \frac{mv^2}{2} - \frac{GMm}{r} \approx \frac{m(3c^2 - E)}{2}, \tag{27}$$

such that for small velocities, $v \ll c$, the constant E is related to the total energy U of the particle by $U = m(3c^2 - E)/2$. In general, $c^2 - v^2$ is much larger than $2GM/r$, such that $E \approx c^2 - v^2$, but for v extremely close to c, $mE \to 2GMm/r = -2$ PE.

Applying eq. (26) to a massless particle, we have that $v \approx c$.

However, in our quick approximation of eq. (24) by (26), we have missed some subtle features.

2.2.2 "Antigravity" in the Radial Equation of Motion

We can now re-express derivatives with respect to parameter p as derivatives with respect to coordinate t,

$$\frac{dr}{dp} = \frac{dt}{dp}\frac{dr}{dt} = \frac{C}{C - R_M}\frac{dr}{dt}, \qquad \frac{d\phi}{dp} = \frac{dt}{dp}\frac{d\phi}{dt} = \frac{C}{C - R_M}\frac{d\phi}{dt}, \tag{28}$$

$$\frac{d^2r}{dp^2} = \frac{d}{dp}\frac{dr}{dp} = \frac{dt}{dp}\frac{d}{dt}\left[\frac{C}{C - R_M}\frac{dr}{dt}\right] = \left(\frac{C}{C - R_M}\right)^2\left[\frac{d^2r}{dt^2} - \frac{R_M C'}{C(C - R_M)}\left(\frac{dr}{dt}\right)^2\right], \tag{29}$$

Then, eq. (15) can be written as,

$$\frac{d^2r}{dt^2} = -\left(\frac{C''}{C'} - \frac{3R_M C'}{2C(C - R_M)}\right)\left(\frac{dr}{dt}\right)^2 - \frac{R_M - C}{C'}\left(\frac{d\phi}{dt}\right)^2 - c^2 \frac{R_M(C - R_M)}{2C'C^3}\,. \tag{30}$$

If gravity is negligible, then $R_M = 0$, $C = r$, and eq. (30) simplifies to

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 = 0. \tag{31}$$

That is, the radial component of the acceleration is zero when there is no gravity (or any other force) acting on the test particle. Further, eq. (19) becomes $r^2\, d\phi/dt$ = constant, such that $a_\phi(\theta = \pi/2) = r\, d^2\phi/dt^2 + 2(dr/dt)(d\phi/dt) = 0$. The total acceleration is zero, spacetime is flat, and the particle's trajectory is a straight line (for nonzero velocity).

In Einstein's theory, the existence of a mass M (other than the mass m of the test particle) "curves" the spacetime experienced by the test particle, which adds velocity-dependent terms of order R_M to the equations of motion, that represent the effect of gravity of mass M on the test particle.[11]

For the "standard" Schwarzschild line element (4) with $C = r$, eq. (30) leads to

$$\begin{aligned} a_r \equiv \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 &= -\frac{c^2 R_M}{2r^2}\left[1 - \frac{R_M}{r} - \frac{3v_r^2}{c^2(1 - R_M/r)} + 2\frac{v_\phi^2}{c^2}\right] \qquad (C = r) \\ &\approx -g\left(1 - \frac{3v_r^2}{c^2} + 2\frac{v_\phi^2}{c^2}\right), \end{aligned} \tag{32}$$

setting $v_r = dr/dt$, $v_\phi = r\, d\phi/dt$, $g = c^2 R_M/2r^2 = GM/r^2$, and the approximation holds for $r \gg R_M$.

This is suggestive,[12] but a bit odd in that the dependence of the radial acceleration a_r on velocity is not isotropic.

Note that in eq. (30), a contribution to C of order R_M will affect the strength of the term in $(d\phi/dt)^2$, so we must make more careful approximations. And, these approximations will depend on the choice of coordinates!

The result (32) holds for $C = r$, which is the "standard" parameterization of the Schwarzschild metric,[13] and also holds for Schwarzschild's original choice, $C = (r^3 + R_M^3)^{1/3} \approx r + R_M^3/3r^2$.

For Brillouin's coordinates, with $C = r + R_M$, the gravitational acceleration a_r is independent of v_ϕ (but not v_r)

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \approx -g\left(1 - \frac{3v_r^2}{c^2}\right) \qquad (C = r + R_M). \tag{33}$$

[11] As summarized by Wheeler, p. xi of [44]: "Spacetime grips mass, telling it how to move; mass grips spacetime telling it how to curve."

[12] We certainly should be impressed that the radial acceleration predicted for a test particle with $r \gg R_M$ and $v \ll c$ is the Newtonian value g. This should perhaps be called the zeroth experimental test of general relativity.

[13] Thus, we verify the claim of eq. (2), where "standard" Schwarzschild coordinates were used, with $v_\phi = 0$ and $v_r \equiv v$.

If we use isotropic coordinates, $C = r(1 + R_M/4r)^2 \approx r + R_M/2$, $C' \approx 1$, $C'' \approx 0$, and now the approximation to eq. (30) is

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \approx -g\left(1 - \frac{3v_r^2}{c^2} + \frac{v_\phi^2}{c^2}\right) \qquad (C = r[1 + R_M/4r]^2). \tag{34}$$

Thus, we reproduce the claim (1) of [7] that objects with large horizontal velocity fall with $2g$ if we use isotropic coordinates, but not for "standard" Schwarzschild coordinates. It remains slightly disconcerting that switching from isotropic to "standard" coordinates at the surface of the Earth, where $R_M/r \ll 1$ makes a 50% difference in the acceleration due to gravity for fast, horizontally moving particles.

It is somewhat comforting that all static Schwarzschild metrics lead to the same expression for vertical acceleration at the surface of the Earth, but it is perhaps surprising that this acceleration is upwards for $|v_r| > c/\sqrt{3}$, as in eq. (2).[14] We could say that the gravitational force is repulsive, rather than attractive, in case of a high-speed particle moving vertically (either up or down); more dramatically, we could say that "antigravity" occurs in this case.

The prediction of "antigravity" for vertical motion of high-speed particles is one of the most dramatic consequences of Einstein's theory of general relativity, yet it is almost unknown (despite being deduced by Hilbert in 1916 [8]). This prediction has been confirmed experimentally, but the results are never discussed as providing this confirmation.

In particular, if a fast-moving particle falls from the Earth towards the Sun, then (once the effect of Earth's gravity can be ignored) its velocity decreases, rather than increases, according to eqs. (2), (32) and (34). Hence, the fall time of the particle is longer than that expected in Newton's theory of gravity (or that in the case of zero gravity), If the particle is a photon, and is reflected back to the Earth, the time of the return trip is the same, according to time-reversal invariance.

This has been investigated in the radar-time-delay experiments of Shapiro *et al.* [45, 46], in which signals sent from the Earth to Venus, and then reflected back to Earth, were observed to take longer than the Newtonian prediction, by amounts in agreement with Einstein's theory. However, the acceleration of the fast particles (photons) was never mentioned.[15]

2.2.3 The Azimuthal Equation of Motion

Similarly to eq. (29), we have that

$$\frac{d^2\phi}{dp^2} = \frac{d}{dp}\frac{d\phi}{dp} = \frac{dt}{dp}\frac{d}{dt}\left[\frac{C}{C - R_M}\frac{d\phi}{dt}\right] = \left(\frac{C}{C - R_M}\right)^2 \left[\frac{d^2\phi}{dt^2} - \frac{R_M C'}{C(C - R_M)}\frac{dr}{dt}\frac{d\phi}{dt}\right]. \tag{35}$$

[14] If a particle is shot upwards with $v > c/\sqrt{3}$, which far exceeds the escape velocity of Earth, its velocity increases, rather than decreases, with time. However, the relative increase in velocity is small.

[15] The gravitational deflection of light and the time-delay for light during largely vertical travel could be considered as an examples of effects that combine general relativity and quantum physics, but apparently Einstein hesitated to make this connection (and instead emphasized an analysis via a Huygens' construction for wave fronts in [1, 2].

A quantum correction to the gravitational deflection of light has been considered in [47].

Then, for constant θ as considered here, eq (17) can be written as

$$a_\phi = C\frac{d^2\phi}{dt^2} + 2C'\frac{dr}{dt}\frac{d\phi}{dt} = -\frac{R_M C'}{C(C-R_M)}\frac{dr}{dt}\frac{d\phi}{dt}\,, \tag{36}$$

noting that $2\pi C$ rather than $2\pi r$ is the circumference of a circle of radius r, For $r \gg R_M$, where $C \approx r$, this becomes

$$a_\phi = r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} = 0, \tag{37}$$

which vanishes as might be expected.

However, for motion near a black hole eq. (36) indicates that their exists a nonzero azimuthal acceleration, opposite to the direction of the azimuthal velocity $C\,d\phi/dt$ if dr/dt is negative. This azimuthal acceleration diverges as $r \to R_M$ in "standard" Schwarzschild coordinates, but not in the other Schwarzschild coordinate systems considered in sec. 2.1.

3 Comments

The result (34), that in isotropic coordinates the vertical acceleration due to gravity at the surface of the Earth is $2g$ downwards for horizontal velocity $v \approx c$ but $2g$ upwards for vertical motion, is hardly compatible with Galileo's hypothesis of universal gravitational acceleration (at g downwards).[16]

Galileo's law of universal acceleration is the basis of the equivalence principle, so the result (34), that the acceleration due to gravity depends on velocity, is in conflict with that principle. That is, an accelerated observer in flat spacetime would regard the motion of "free" bodies as having the same acceleration independent of their velocity. Thus, in principle one could determine whether or not a gravitational field is present (*i.e.*, that spacetime is curved) by performing experiments inside a small box as to the acceleration of high-speed particles moving in different directions.[17]

A speculation is that the "antigravity" between masses with large relative radial velocity might be related to the "dark energy" problem.

[16]On p. 72 of the English translation of Galileo's *Discorsi* [48] Salviati says:

> ... in a medium totally devoid of resistance all would bodies fall with the same speed.

The meaning of "speed" is perhaps not clear here, so Galileo began a long discussion of "naturally accelerated motion" on p. 160, including on p. 164:

> Imagine a heavy stone held in the air at rest; the support is removed and the stone set free; then since it is heavier than the air it begins to fall, not with uniform motion but slowly at the beginning with a continuously accelerated motion.

[17]These experiments would be very difficult to perform in a small space, but in the classical view that infinite precision of measurement is possible, they could be done. The one attempt [49, 50] to perform such an experiment at the surface of the Earth yielded a null result.

A Appendix: Proper, Coordinate and Geometric Acceleration

This appendix was written on July 15, 2017, following a suggestion from Ricardo Vieira.

When considering a test particle with nonzero mass, one often speaks of its proper 4-velocity,

$$u^\alpha = \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds}\,, \tag{38}$$

for $x^\alpha = (x^0, \mathbf{x}) = (x^0, x^1, x^2, x^3)$. In coordinate systems where $x^0 = ct$ with t as the "time", one speaks of the ordinary (or coordinate) 3-velocity, $\mathbf{v} = d\mathbf{x}/dt$. Introducing $\gamma = 1/\sqrt{1 - v^2/c^2}$, one can identity the proper-time interval as $d\tau = dt/\gamma$, and the proper 4-velocity can be written as

$$u^\alpha = \gamma(c, \mathbf{v}). \tag{39}$$

While one might then suppose that the 4-vector $du^\alpha/d\tau$ would be called the proper 4-acceleration, this seems not to be the convention in general relativity, where it is (sometimes) called the coordinate 4-acceleration. Instead, the geodesic equation (9) is sometimes written as

$$A^\alpha = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0, \tag{40}$$

and the 4-vector $A^\lambda = (0, 0, 0, 0)$ is called the proper 4-acceleration. In this view, the proper 4-acceleration is zero for a particle with no other interaction than "gravity".

The geodesic equation

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu}u^\mu u^\nu, \tag{41}$$

can then be interpreted as saying that the coordinate 4-acceleration $du^\lambda/d\tau$ is equal and opposite to the geometric 4-acceleration $\Gamma^\alpha_{\mu\nu}u^\mu u^\nu$ associated with the curvature of spacetime.

We can also use eq. (39) to write

$$\frac{du^\alpha}{d\tau} = \gamma\frac{du^\alpha}{dt} = \left(\frac{\gamma^4}{c}\mathbf{v}\cdot\mathbf{a},\, \gamma^2\mathbf{a} + \frac{\gamma^4}{c^2}(\mathbf{v}\cdot\mathbf{a})\mathbf{v}\right), \qquad \text{where} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt}\,. \tag{42}$$

Comparing this with eq. (41), we see that

$$\frac{\gamma^4}{c}\mathbf{v}\cdot\mathbf{a} = -\Gamma^0_{\mu\nu}u^\mu u^\nu, \qquad \gamma^2 a^k = \Gamma^0_{\mu\nu}u^\mu u^\nu\frac{v^k}{c} - \Gamma^k_{\mu\nu}u^\mu u^\nu \qquad (k = 1, 2, 3). \tag{43}$$

This provides an alternative approach to the accelerations a_r and a_ϕ as found in secs. 2.2.2-3 above.

Acknowledgment

The author thanks Malcolm Perry for discussions of this problem — 30 years ago.

References

[1] A. Einstein, *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, S.K.P. Akad. Wiss. 831 (1915),
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_skpaw_831_15.pdf
Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity,
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_skpaw_831_15_english.pdf

[2] A. Einstein, *Die Grundlage der allgemeinem Relativitätstheorie*, Ann. d. Phys. **49**, 769 (1916), http://physics.princeton.edu/~mcdonald/examples/GR/einstein_ap_49_769_16.pdf
The Foundation of the General Theory of Relativity,
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_ap_49_769_16_english.pdf

[3] A. Einstein, *Über das Relativitätsprinzip und die aus demselben gezogene Folgerungen*, Jahrb. Radioakt. Elektron. **4**, 411 (1908),
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_jre_4_411_08.pdf
On the Relativity Principle and Conclusions Drawn from It,
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_jre_4_411_08_english.pdf
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_jre_4_411_08_english2.pdf

[4] A. Einstein, *Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes*, Ann. d. Phys. **35**, 898 (1911), http://physics.princeton.edu/~mcdonald/examples/GR/einstein_ap_35_898_11.pdf
On the Influence of Gravitation on the Propagation of Light,
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_ap_35_898_11_english2.pdf

[5] A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie*, (Vieweg & Sohn, 1920), http://physics.princeton.edu/~mcdonald/examples/GR/einstein_relativititatstheorie_20.pdf
Relativity: The Special and General Theory (Holt, 1920),
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_relativity_20_english.pdf

[6] I. Newton, *Optics*, 4th ed. (1730),
http://physics.princeton.edu/~mcdonald/examples/optics/newton_opticks.pdf

[7] V.B. Braginsky, C.M. Caves and K.S. Thorne, *Laboratory experiments to test relativistic gravity*, Phys. Rev. D **15**, 2047 (1977),
http://physics.princeton.edu/~mcdonald/examples/GR/braginsky_prd_15_2047_77.pdf

[8] D. Hilbert, *Die Grundlagen der Physik*, Nachr. Gesell. Wiss. Göttingen, 53 (1917),
http://physics.princeton.edu/~mcdonald/examples/GR/hilbert_ngwg_53_17.pdf

[9] K. Brown, *Reflections on Relativity* (2016), http://www.mathpages.com/rr/

[10] L. Page, *Gravitational Deflection of High-speed Particles*, Nature **104**, 692 (1920),
http://physics.princeton.edu/~mcdonald/examples/GR/page_nature_104_692_20.pdf
Nature **105**, 233 (1920),
http://physics.princeton.edu/~mcdonald/examples/GR/page_nature_105_233_20.pdf

[11] A.S. Eddington, *Gravitational Deflection of High-speed Particles*, Nature **105**, 37 (1920), http://physics.princeton.edu/~mcdonald/examples/GR/page_nature_105_37_20.pdf

[12] J. Wisdom, *Swimming in Spacetime: Motion by Cyclic Changes in Body Shape*, Science **299**, 1865 (2003), http://physics.princeton.edu/~mcdonald/examples/GR/wisdom_science_299_1865_03.pdf

[13] E. Gueŕon, C.A.S. Maia and G.E.A. Matsas, *Swimming versus swinging effects in spacetime*, Phys. Rev. D **73**, 024020 (2006),
http://physics.princeton.edu/~mcdonald/examples/GR/gueron_prd_73_024020_06.pdf

[14] J.E. Avron and O. Kenneth, *Swimming in curved space or the Baron and the cat*, New J. Phys. **8**, 69 (2006), http://physics.princeton.edu/~mcdonald/examples/GR/avron_njp_8_69_06.pdf

[15] E. Gueŕon and R.A. Mosna, *Relativistic glider*, Phys. Rev. D **75**, 081501 (2007),
http://physics.princeton.edu/~mcdonald/examples/GR/gueron_prd_75_081501_07.pdf

[16] L. Bergamin1, P. Delva and A. Hees, *Vibrating systems in Schwarzschild spacetime: toward new experiments in gravitation?* Class. Quant. Grav. **26**, 195006 (2009),
http://physics.princeton.edu/~mcdonald/examples/GR/bergamin_cqg_26_185006_09.pdf

[17] S. Weinberg, *Gravitation and Cosmology* (Wiley, 1972),
http://physics.princeton.edu/~mcdonald/examples/GR/weinberg_chap8.pdf

[18] J.L. Synge, *The Gravitational Field of a Particle*, Proc. Roy. Irish Acad. A **53**, 83 (1950), http://physics.princeton.edu/~mcdonald/examples/GR/synge_pria_a53_83_50.pdf

[19] K. Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINschen Theorie*, S.K.P. Akad. Wiss. 189 (1916),
http://physics.princeton.edu/~mcdonald/examples/GR/schwarzschild_skpaw_189_16.pdf
On the Gravitational Field of a Point-Mass, According to Einstein's Theory, http://physics.princeton.edu/~mcdonald/examples/GR/schwarzschild_skpaw_189_16_english.pdf

[20] W. de Sitter, *Einstein's Theory of Gravitation, and its Astronomical Consequences*, Mon. Not. Roy. Astron. Soc. **76**, 699 (1916),
http://physics.princeton.edu/~mcdonald/examples/GR/desitter_mnras_76_699_16.pdf
Mon. Not. Roy. Astron. Soc. **77**, 155 (1916),
http://physics.princeton.edu/~mcdonald/examples/GR/desitter_mnras_77_155_16.pdf
Mon. Not. Roy. Astron. Soc. **78**, 3 (1917),
http://physics.princeton.edu/~mcdonald/examples/GR/desitter_mnras_78_3_17.pdf

[21] J.T. Jebsen, *Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum*, Ark. Math. Astr. Fys. **15**(18), 1 (1921),
On the general spherically symmetric solutions of Einstein's gravitational equations in vacuo, http://physics.princeton.edu/~mcdonald/examples/GR/jebsen_amaf_15(18)_1_21_english.pdf

[22] J. Eiesland, *The Group of Motions of an Einstein Space*, Trans. Am. Math. Soc. **27**, 213 (1925), http://physics.princeton.edu/~mcdonald/examples/GR/eiesland_tams_27_213_25.pdf

[23] G.D. Birkhoff, *Relativity and Modern Physics* (Harvard U.P., 1923), p. 253.

[24] N.J. Johansen and F. Ravndal, *On the discovery of Birkhoff's theorem*, Gen. Rel. Grav. **38**, 213 (2006), http://physics.princeton.edu/~mcdonald/examples/GR/johansen_grg_38_537_06.pdf

[25] R.P. Kerr, *Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics*, Phys. Rev. Lett. **11**, 237 (1963),
http://physics.princeton.edu/~mcdonald/examples/GR/kerr_prl_11_237_63.pdf
Discovering the Kerr and Kerr-Schild metrics (Jan. 14, 2008), http://arxiv.org/abs/0706.1109

[26] L.S. Abrams, *Black holes: the legacy of Hilbert's error*, Can. J. Phys. **67**, 919 (1989),
http://physics.princeton.edu/~mcdonald/examples/GR/abrams_cjp_67_919_89.pdf

[27] S.J. Crothers, *On the General Solution to Einsteins Vacuum Field and Its Implications for Relativistic Degeneracy*, Prog. Phys. **1**(1), 68 (2005),
http://physics.princeton.edu/~mcdonald/examples/GR/crothers_pp_1(1)_68_05.pdf
http://www.sjcrothers.plasmaresources.com/

[28] J. Droste, *The field of a single centre in EINSTEIN'S theory of gravitation, and the motion of a particle in that field*, Proc. Roy. Neth. Acad. Arts. Sci. **19**, 197 (1917),
http://physics.princeton.edu/~mcdonald/examples/GR/droste_pknaw_19_197_17.pdf

[29] L. Flamm, *Beiträge zur Einsteinschen Gravitationstheorie* Phys. Z. **19**, 197 (1917),
http://physics.princeton.edu/~mcdonald/examples/GR/flamm_pz_17_448_16.pdf
Contributions to Einsteins theory of gravitation,
http://physics.princeton.edu/~mcdonald/examples/GR/flamm_pz_17_448_16_english.pdf

[30] H. Weyl, *Zur Gravitationstheorie*, Ann. d. Phys. **54**, 117 (1917),
http://physics.princeton.edu/~mcdonald/examples/GR/weyl_ap_54_117_17.pdf
On the theory of gravitation,
http://physics.princeton.edu/~mcdonald/examples/GR/weyl_ap_54_117_17_english.pdf

[31] M. Brillouin, *Les Points Singuliers de l'Univers d'Einstein*, J. Phys. Rad. **4**, 43 (1923),
http://physics.princeton.edu/~mcdonald/examples/GR/brillouin_jpr_4_43_23.pdf
The singular points of Einstein's universe,
http://physics.princeton.edu/~mcdonald/examples/GR/brillouin_jpr_4_43_23_english.pdf

[32] W. Pauli, *Relativitätstheorie*, Ency. Math. Phys. **5**(2), 539 (1921),
http://physics.princeton.edu/~mcdonald/examples/GR/pauli_emp_5_2_539_21.pdf
Theory of Relativity (Pergamon, 1958).

[33] E. Serini, *Euclideità dello spazio completamente vuoto nella relatività generale di Einstein*, Rend. Reale Acad. Lincei **27**, 235 (1918),
http://physics.princeton.edu/~mcdonald/examples/GR/serini_rral_27_235_13.pdf

[34] A. Einstein and W. Pauli, *On the Non-Existence of Regular Stationary Solutions of Relativistic Field Equations*, Ann. Math. **44**, 131 (1943),
http://physics.princeton.edu/~mcdonald/examples/GR/einstein_am_44_131_43.pdf

[35] G. Lemaître, *L'Univers en expansion*, Ann. Soc. Sci. Bruxelles A **53**, 51 (1933),
http://physics.princeton.edu/~mcdonald/examples/GR/lemaitre_assb_a53_51_33.pdf
The Expanding Universe,
http://physics.princeton.edu/~mcdonald/examples/GR/lemaitre_assb_a53_51_33_english.pdf

[36] A. Gullstrand, *Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie*, Ark. Math. Astr. Fys. **16**(8), 1 (1922),
http://physics.princeton.edu/~mcdonald/examples/GR/gullstrand_amaf_16(8)_1_22.pdf

[37] P. Painlevé, *La Mécanique classique et la théorie de la relativité*, Comp. Rend. Acad. Sci. **173**, 677 (1921), http://physics.princeton.edu/~mcdonald/examples/GR/painleve_cras_173_677_21.pdf
La gravitation dans la Mécanique de Newton et dans la Mécanique d'Einstein, Comp. Rend. Acad. Sci. **173**, 873 (1921),
http://physics.princeton.edu/~mcdonald/examples/GR/painleve_cras_173_873_21.pdf

[38] G. Szekeres, *On the singularities of a Riemannian manifold*, Pub. Math. Debrecen **7**, 285 (1960), http://physics.princeton.edu/~mcdonald/examples/GR/szekeres_pmd_7_285_60.pdf

[39] M.D. Kruskal, *Maximal Extension of Schwarzschild Metric*, Phys. Rev. **119**, 1743 (1960), http://physics.princeton.edu/~mcdonald/examples/GR/kruskal_pr_119_1743_60.pdf

[40] R. Penrose, *Gravitational Collapse and Space-Time Singularities*, Phys. Rev. Lett. **14**, 57 (1965), http://physics.princeton.edu/~mcdonald/examples/GR/penrose_prl_14_57_65.pdf

[41] D. Finkelstein, *Past-Future Asymmetry of the Gravitational Field of a Point Particle*, Phys. Rev. **110**, 965 (1958),
http://physics.princeton.edu/~mcdonald/examples/GR/finkelstein_pr_110_965_58.pdf

[42] A.S. Eddington, *A Comparison of Whitehead's and Einstein's Formulæ*, Nature **113**, 192 (1924), http://physics.princeton.edu/~mcdonald/examples/GR/eddington_nature_113_192_24.pdf

[43] H. Dingle, *Values of T_μ^ν, and the Christoffel Symbols for a Line Element of Considerable Generality*, Proc. Nat. Acad. Sci. **19**, 959 (1933),
http://physics.princeton.edu/~mcdonald/examples/GR/dingle_pnas_19_959_33.pdf

[44] J.A. Wheeler, *A Journey Into Gravity and Spacetime* (W.H. Freeman, 1990).

[45] I.I. Shapiro, *Fourth Test of General Relativity*, Phys. Rev. Lett. **13**, 789 (1964),
http://physics.princeton.edu/~mcdonald/examples/GR/shaprio_prl_13_789_64.pdf

[46] I.I. Shapiro *et al.*, *Fourth Test of General Relativity: Preliminary Results*, Phys. Rev. Lett. **20**, 265 (1968),
http://physics.princeton.edu/~mcdonald/examples/GR/shapiro_prl_20_1265_68.pdf

[47] N.E.J. Bjerrum-Bohr *et al.*, *Bending of Light in Quantum Gravity*, Phys. Rev. Lett. **114**, 061301 (2015),
http://physics.princeton.edu/~mcdonald/examples/GR/bjerrum-bohr_prl_114_061301_15.pdf

[48] G. Galilei, *Discorsi e Dimonstazioni Matematiche intorno a due nuove scienze attenenti alla mecanica & i movimenti locali* (1638),
http://physics.princeton.edu/~mcdonald/examples/mechanics/galileo_discorsi_38.pdf
Dialogues concerning Two New Sciences (Macmillan, 1914),
http://physics.princeton.edu/~mcdonald/examples/mechanics/galileo_two_new_sciences.pdf

[49] A.C. Melissinos, *Experimental Tests of Newtonian Gravity at Relativistic Velocities*, Nuovo Cim. B **62**, 190 (1981),
http://physics.princeton.edu/~mcdonald/examples/GR/meslissinos_nc_62b_190_81.pdf

[50] P. Reiner *et al.*, *Search for Long Range Interactions at Highly Relativistic Velocities*, Phys. Lett. B **176**, 233 (1986),
http://physics.princeton.edu/~mcdonald/examples/GR/reiner_pl_176b_233_86.pdf