The Grating Accelerator

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(Sept. 14, 1984)

1 Problem

In optics, a reflective grating is a conducting surface with a ripple. For example, consider the surface defined by

$$z = a \sin \frac{2\pi x}{d}. \tag{1}$$

The typical use of such a grating involves an incident electromagnetic wave with wave vector $\mathbf{k}$ in the x-z plane, and interference effects lead to a discrete set of reflected waves also with wave vectors in the x-z plane.

Consider, instead, an incident plane electromagnetic wave with wave vector in the y-z plane and polarization in the x direction:

$$\mathbf{E}_{\rm in} = E_0 \hat{\mathbf{x}} e^{i(k_y y - k_z z - \omega t)}, \tag{2}$$

where $k_y > 0$ and $k_z > 0$. Show that for small ripples ($a \ll d$), this leads to a reflected wave as if $a = 0$, plus two surface waves that are attenuated exponentially with z. What is the relation between the grating wavelength d and the optical wavelength λ such that the x component of the phase velocity of the surface waves is the speed of light, c?

In this case, a charged particle moving with $v_x \approx c$ could extract energy from the wave, which is the principle of the proposed "grating accelerator" [1, 2, 3].

2 Solution

The interaction between particle beams and diffraction gratings was first considered by Smith and Purcell [4], who emphasized energy transfer from the particle to free electromagnetic waves. The excitation of surface waves by particles near conducting structures was first discussed by Pierce [5], which led to the extensive topic of wakefields in particle accelerators. The presence of surface waves in the Smith-Purcell effect was noted by di Francia [6]. A detailed treatment of surface waves near a diffraction grating was given by van den Berg [7]. Here, we construct a solution containing surface waves by starting with only free waves, then adding surface waves to satisfy the boundary condition at the grating surface.

If the (perfectly) conducting surface were flat, the reflected wave would be

$$\mathbf{E}_{\rm r} = -E_0 \hat{\mathbf{x}} e^{i(k_y y + k_z z - \omega t)}. \tag{3}$$

However, the sum $\mathbf{E}_{\rm in} + \mathbf{E}_{\rm r}$ does not satisfy the boundary condition that $\mathbf{E}_{\rm total}$ must be perpendicular to the wavy surface (1). Indeed,

$$[\mathbf{E}_{\rm in} + \mathbf{E}_{\rm r}]_{\rm surface} = 2iE_0 \hat{\mathbf{x}} e^{i(k_y y - \omega t)} \sin k_z z \approx 2iak_z E_0 \hat{\mathbf{x}} e^{i(k_y y - \omega t)} \sin k_x x, \tag{4}$$

where the approximation holds for $a \ll d$, and we have defined $k_x = 2\pi/d$.

Hence, we require additional fields near the surface to cancel that given by (4). For $z \approx 0$, these fields therefore have the form

$$\mathbf{E} = -ak_z E_0 \hat{\mathbf{x}} e^{i(k_y y - \omega t)} \left(e^{ik_x x} - e^{-ik_x x} \right). \tag{5}$$

This can be decomposed into two waves $\mathbf{E}_\pm$ given by

$$\mathbf{E}_\pm = \mp ak_z E_0 \hat{\mathbf{x}} e^{i(\pm k_x x + k_y y - \omega t)}. \tag{6}$$

Away from the surface, we suppose that the z dependence of the additional waves can be described by including a factor $e^{ik'_z z}$. Then, the full form of the additional waves is

$$\mathbf{E}_\pm = \mp ak_z E_0 \hat{\mathbf{x}} e^{i(\pm k_x x + k_y y + k'_z z - \omega t)}. \tag{7}$$

The constant k'_z is determined on requiring that each of the additional waves satisfy the wave equation,

$$\nabla^2 \mathbf{E}_\pm = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_\pm}{\partial t^2}. \tag{8}$$

This leads to the dispersion relation

$$k_x^2 + k_y^2 + k_z'^2 = \frac{\omega^2}{c^2}. \tag{9}$$

The component k_y of the incident wave vector can be written in terms of the angle of incidence $\theta_{\rm in}$ and the wavelength λ as

$$k_y = \frac{2\pi}{\lambda} \sin\theta_{\rm in}. \tag{10}$$

Combining (9) and (10), we have

$$k'_z = 2\pi i \sqrt{\frac{1}{d^2} - \left(\frac{\cos\theta_{\rm in}}{\lambda} \right)^2}. \tag{11}$$

For short wavelengths, k'_z is real and positive, so the reflected wave (3) is accompanied by two additional plane waves with direction cosines (k_x, k_y, k'_z). But for long enough wavelengths, k'_z is imaginary, and the additional waves are exponentially attenuated in z.

When surface waves are present, consider the fields along the line $y = 0$, $z = \pi/2k_z$. Here, the incident plus reflected fields vanish (see the first form of (4)), and the surface waves are

$$\mathbf{E}_\pm = \mp ak_z e^{-\pi |k'_z|/2k_z} E_0 \hat{\mathbf{x}} e^{i(\pm k_x x - \omega t)}. \tag{12}$$

The phase velocity of these waves is

$$v_p = \frac{\omega}{k_x} = \frac{d}{\lambda} c. \tag{13}$$

When $d = \lambda$, the phase velocity is c, and $k'_z = ik_y$ according to (11). The surface waves are then,

$$\mathbf{E}_{\pm} = \mp \frac{2\pi a \cos\theta_{\rm in}}{d} e^{-(\pi/2)\tan\theta_{\rm in}} E_0 \hat{\mathbf{x}} e^{i(\pm k_x x - \omega t)}. \qquad (14)$$

A relativistic charged particle that moves in, say, the $+x$ direction remains in phase with the wave $\mathbf{E}_+$, and can extract energy from that wave for phases near π. On average, the particle's energy is not affected by the counterpropagating wave $\mathbf{E}_-$. In principle, significant particle acceleration can be achieved via this technique. For a small angle of incidence, and with $a/d = 1/2\pi$, the accelerating field strength is equal to that of the incident wave.

3 References

[1] Y. Takeda and I. Matsui, *Laser Linac with Grating*, Nucl. Instr. and Meth. **62**, 306-310 (1968), http://puhep1.princeton.edu/~mcdonald/examples/accel/takeda_nim_62_306_68.pdf

[2] K. Mizuno, S. Ono and O. Shimoe, *Interaction between coherent light waves and free electrons with a reflection grating*, Nature **253**, 180-181 (1975), http://puhep1.princeton.edu/~mcdonald/examples/accel/mizuno_nature_253_184_75.pdf

[3] R.B. Palmer, *A Laser-Driven Grating Linac*, Part. Accel. **11**, 81-90 (1980), http://puhep1.princeton.edu/~mcdonald/examples/accel/palmer_pa_11_81_80.pdf

[4] S.J. Smith and E.M. Purcell, *Visible Light from Localized Surface Charges Moving across a Grating*, Phys. Rev. **62**, 1069 (1953), http://puhep1.princeton.edu/~mcdonald/examples/accel/smith_pr_92_1069_53.pdf

[5] J.R. Pierce, *Interaction of Moving Charges with Wave Circuits*, J. Appl. Phys. **26**, 627-638 (1955), http://puhep1.princeton.edu/~mcdonald/examples/accel/pierce_jap_26_627_55.pdf

[6] G. Toraldo di Francia, *On the Theory of some Čerenkovian Effects*, Nuovo Cim. **16**, 61-77 (1960), http://puhep1.princeton.edu/~mcdonald/examples/accel/toraldo_di_francia_nc_16_61_60.pdf

[7] P.M. van den Berg, *Diffraction Theory of a Reflective Grating*, Appl. Sci. Res. **24**, 261-293 (1971), http://puhep1.princeton.edu/~mcdonald/examples/EM/vandenberg_asr_24_261_71.pdf