Formal Expressions for the Electromagnetic Potentials in Any Gauge

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1 Problem

Deduce expressions for the electromagnetic vector potential \( A \) in an arbitrary gauge, where the scalar potential \( V(x, t) \) has a specified form, via a gauge transformation from the potentials in the Lorenz gauge [1].

For an illustration of such expressions for a Hertzian (point) oscillating dipole, see [2].

2 Solution

In the Hamiltonian dynamics of a charged particle, the Hamiltonian (and Lagrangian) \( H = U_{\text{mech}} + qV \), where \( U_{\text{mech}} \) is the “mechanical” energy of the particle, \( q \) is its electric charge, and \( V \) is the “external” electromagnetic scalar potential in some gauge. Yet, the equations of motion deduced from this Hamiltonian do not depend on the choice of gauge,\(^1\) so we are free to use whatever gauge is convenient.

2.1 Gauge Transformations

We first review the notion of gauge transformations in classical electrodynamics.\(^2\) We consider microscopic electrodynamics, and work in Gaussian units.

In electrostatics, Coulomb’s law can be written as

\[
E(x) = \int \frac{\rho(x') r}{r^3} d\text{Vol}' = -\nabla V, \quad \text{where} \quad V = \int \frac{\rho(x')}{r} d\text{Vol}'
\]  

(1)

\( \rho \) is the volume density of electric charge, and \( r = x - x' \).

Faraday discovered (as later interpreted by Maxwell) that\(^3\)

\[
\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t},
\]  

(2)

\(^1\)See, for example, sec. 2.1 of [3].
\(^2\)For a historical survey, see [4].
\(^3\)Faraday’s law, eq. (2), implies that the condition \( \nabla \times E = 0 \) of electrostatics (where \( E \) is independent of time) requires that the magnetic field \( B \) also be independent of time. That is, electrostatics and magnetostatics are equivalent (as reinforced by the 4\(^{th}\) Maxwell equation (9)).
where \( c \) is the speed of light in vacuum, which implies that time-dependent magnetic fields \( \mathbf{B} \) are associated with additional electric fields beyond those deducible from the scalar potential \( \mathbf{V} \). The nonexistence (so far as we know) of magnetic charges (Gilbertian monopoles) implies that
\[
\nabla \cdot \mathbf{B} = 0,
\]
and hence that the magnetic field can be related to a vector potential \( \mathbf{A} \) according to
\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]
Using eq. (4) in (2), we can write
\[
\nabla \times \left( \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0,
\]
which implies that \( \mathbf{E} + (1/c)\partial \mathbf{A}/\partial t \) can be related to a scalar potential \( V \) as
\[
\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.
\]
Then, using eq. (6) in the Maxwell equation
\[
\nabla \cdot \mathbf{E} = 4\pi \rho
\]
leads to
\[
\nabla^2 V + \frac{1}{c} \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -4\pi \rho.
\]
Similarly, using eqs. (4) and (6) in the Maxwell equation
\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t},
\]
where \( \mathbf{J} \) is the volume density of electrical current, leads to
\[
\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} + \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \mathbf{V}}{\partial t} \right).
\]

The differential equations (8) and (10) do not uniquely determine the potentials \( V \) and \( \mathbf{A} \). As perhaps first noted by Lorentz \([5, 6]\),\(^4\) if \( V_0, \mathbf{A}_0 \) are valid electromagnetic potentials, then so are
\[
V = V_0 - \frac{1}{c} \frac{\partial \chi}{\partial t}, \quad \mathbf{A} = \mathbf{A}_0 + \nabla \chi,
\]
where \( \chi \) is an arbitrary scalar function, now called the gauge-transformation function. That is, eqs. (4) and (6) give the same values for the electromagnetic fields \( \mathbf{B} \) and \( \mathbf{E} \) for either the potentials \( V, \mathbf{A} \) or \( V_0, \mathbf{A}_0 \).

\(^4\)The notion of a gauge transformation of the vector potential of magnetism was discussed by Lord Kelvin (1851) in sec. 82 of \([7]\), without consideration of the electric field/potential. In sec. 98 of \([8]\), Maxwell noted that if potentials \( V_0, \mathbf{A}_0 \) do not obey \( \nabla \cdot \mathbf{A}_0 = 0 \), then a function \( \chi \) can be found such that the potentials of eq. (11) obey \( \nabla \cdot \mathbf{A} = 0 \) (Coulomb gauge), which he thereafter considered to be the proper type of potentials.


2.2 From Lorenz-Gauge Potentials to Those in Any Other Gauge

As deduced by Lorenz in 1867 [1],

\[
V^{(L)}(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t') = t - |\mathbf{r} - \mathbf{r}'|/c}{|\mathbf{r} - \mathbf{r}'|} \, d\text{Vol}' \quad \text{(Lorenz),} \tag{12}
\]

\[
A^{(L)}(\mathbf{r}, t) = \int \frac{\mathbf{J}(\mathbf{r}', t') = t - |\mathbf{r} - \mathbf{r}'|/c}{c |\mathbf{r} - \mathbf{r}'|} \, d\text{Vol}' \quad \text{(Lorenz),} \tag{13}
\]

are formal expressions for the (retarded) potentials in what is now called the Lorenz gauge, for which the so-called gauge condition is

\[
\nabla \cdot A^{(L)} = -\frac{1}{c} \frac{\partial V^{(L)}}{\partial t} \quad \text{(Lorenz).} \tag{14}
\]

Another set of potentials can be defined by the gauge condition that the scalar potential \( V(\mathbf{x}, t) \) is a specified, but arbitrary scalar function. Then, we can formally integrate the first of eq. (11) for \( V_0 = V^{(L)} \) to write the gauge-transformation function as

\[
\chi(\mathbf{x}, t) = c \int_{-\infty}^{t} \{ V^{(L)}(\mathbf{x}, t') - V(\mathbf{x}, t') \} \, dt'. \tag{15}
\]

Hence, a formal expression for the vector potential in the new gauge is given by the second of eq. (11),

\[
A(\mathbf{r}, t) = A^{(L)} + \nabla \chi = A^{(L)}(\mathbf{r}, t) + c \nabla \int_{-\infty}^{t} \{ V^{(L)}(\mathbf{r}, t') - V(\mathbf{r}, t') \} \, dt'.
\]

\[
= A^{(L)}(\mathbf{r}, -\infty) - c \int_{-\infty}^{t} \{ \mathbf{E}(\mathbf{r}, t') + \nabla V(\mathbf{r}, t') \} \, dt'. \tag{16}
\]

2.2.1 Propagation of the Fields and Potentials

The Lorenz-gauge potentials (12)-(13) can be said to propagate with speed \( c \), with the consequence that the fields \( \mathbf{B} \) and \( \mathbf{E} \) derived from them via eqs. (4) and (6) can also be said to propagate with speed \( c \) (as expected from Maxwell’s electrodynamics [9]).

When using eq. (16) to compute the magnetic field from the vector potential in an arbitrary gauge, we have that \( \mathbf{B} = \nabla \times \mathbf{A} = \nabla \times A^{(L)} \). Similarly, the electric field, as computed from the potentials \( V \) and \( \mathbf{A} \) in an arbitrary gauge, is

\[
\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla V - \frac{1}{c} \frac{\partial A^{(L)}}{\partial t} - \nabla V^{(L)} + \nabla V = -\nabla V^{(L)} - \frac{1}{c} \frac{\partial A^{(L)}}{\partial t}. \tag{17}
\]

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5 Equation (15) is a slight generalization of the procedure given in sec. IIIA of [10]. The arguments there depend in part on eq. (2.10), which relations are more obvious if it is understood that eq. (2.9) is applied to Lorenz-gauge potentials, and that the function \( \Psi \) also serves as the gauge-transformation function from the Coulomb gauge to the Lorenz gauge.

6 General expressions for \( \mathbf{B} \) and \( \mathbf{E} \) deduced from the Lorenz-gauge potentials in terms of retarded quantities that propagate with speed \( c \) are given in eqs. (14-34) and (14-42) of [11]. See also [12], where it is shown in the Appendix that the general expressions for \( \mathbf{B} \) and \( \mathbf{E} \) can also be deduced from Maxwell’s equations without use of potentials.
which reaffirms that $\mathbf{B}$ and $\mathbf{E}$ propagate with speed $c$. While the general scalar potential $V$ can propagate arbitrarily, when considering the electric field $\mathbf{E}$ according to eq. (6), this arbitrary behavior is canceled by that of the time derivative of the third term in (the first line of) eq. (16) for the general vector potential $\mathbf{A}$.\(^7\)

### 2.3 Examples

While the expression (16) applies for “any” gauge, to use it we must first know the scalar potential $V$ in that gauge, which is not obvious in general. That is, the prescription (16) is a “solution looking for a problem.”

The present general result (16) can be contrasted with prescriptions for transformations from the Lorenz-gauge potentials to those in several other gauges as given in [10].

#### 2.3.1 Velocity Gauge

One application of eq. (16) is to the so-called velocity gauge in which the scalar potential is assumed to propagate with arbitrary speed $v$ rather than $c$.\(^8\)

$$V^{(v)}(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t' = t - |\mathbf{r} - \mathbf{r}'|/v)}{|\mathbf{r} - \mathbf{r}'|} d\text{Vol}' \quad \text{(velocity gauge)}.$$ \hspace{1cm} (18)

#### 2.3.2 Coulomb Gauge

A special case of a velocity gauge is the famous Coulomb gauge, corresponding to $v = \infty$.\(^9\)

$$V^{(C)}(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d\text{Vol}' \quad \text{(Coulomb)}.$$ \hspace{1cm} (19)

The use of eq. (16) to compute the vector potential in the Coulomb gauge can be simpler than the classic prescription\(^{10}\)

$$\mathbf{A}^{(C)}(\mathbf{r}, t) = \int \frac{[\mathbf{J}_t]}{c|\mathbf{r} - \mathbf{r}'|} d\text{Vol}',$$ \hspace{1cm} (20)

where the transverse current density is defined by

$$\mathbf{J}_t(\mathbf{r}, t) = \frac{1}{4\pi} \nabla \times \mathbf{V} \times \int \frac{\mathbf{J}(\mathbf{r}', t)}{c|\mathbf{r} - \mathbf{r}'|} d\text{Vol}'. $$ \hspace{1cm} (21)

For an example of the vector potential in the Coulomb gauge obtained by transforming the vector potential from the Lorenz gauge via eq. (16), see sec. 2.4 of [2]. For the case of a uniformly moving charge, see [21, 22]. For an interesting dynamic example where it is simpler to use the Coulomb gauge than the Lorenz gauge, see [23].

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\(^7\)Such a cancelation for the Coulomb-gauge potentials has been noted in [13, 14, 15, 16].

\(^8\)The velocity gauge was initially called the $\alpha$-Lorenz gauge in [17, 18]. See also [19].

\(^9\)The potentials used by Maxwell were always in the Coulomb gauge, as in sec. 617 of [9].

In eq. (68) of [8], Maxwell nearly discovered the Lorenz gauge, which reads $kJ + 4\pi \mu d\Psi/dt = 0$ in the notation there. Instead, he argued after eq. (79) that $J (= \nabla \cdot \mathbf{A})$ is either zero or constant for wave propagation. He was not bothered by the implication of eq. (79) that in this case the scalar potential $\varphi$ “propagates” instantaneously, perhaps because of the great success that his assumptions about the potentials lead to propagation of the electric and magnetic fields at lightspeed $\sqrt{k/4\pi \mu}$.

\(^{10}\)See, for example, sec. 6.3 of [20].
2.3.3 Gibbs Gauge

Another case where the prescription (16) readily applies is the gauge where the scalar potential is defined to be zero, \( V^{(G)} = 0 \), such that \( E = -(1/c)\partial A^{(G)}/\partial t \), as first proposed by Gibbs [24, 25].11

Since the Gibbs-gauge vector potential is an integral of the electric field, \( A^{(G)}(t) = -c \int_{t_0}^{t} E(t') \, dt' \), this potential propagates at speed \( c \). However, it differs from the Lorenz-gauge vector potential. Since \( \nabla \cdot E = 4\pi \rho = -(1/c)\partial \nabla \cdot A^{(G)}/\partial t \), the Gibbs-gauge vector potential obeys \( \nabla \cdot A^{(G)} = 0 \) away from charged particles (whereas the Coulomb-gauge vector potential obeys \( \nabla \cdot A^{(C)} = 0 \) everywhere).12

According to eq. (16), the vector potential in the Gibbs gauge is

\[
A^{(G)}(r, t) = A^{(L)}(r, t) + c\nabla \int_{-\infty}^{t} V^{(L)}(r, t') \, dt',
\]

(22)

so that the vector potential in any other gauge, where the scalar potential is \( V \), can be written as

\[
A(r, t) = A^{(G)} - c\nabla \int_{-\infty}^{t} V(r, t') \, dt'.
\]

(23)

That is, if the vector potential in Gibbs gauge in known, this provides an even simpler prescription than eq. (16) for the vector potential in another gauge.

For an example of the vector potential in the Gibbs gauge, which can also be obtained by transforming the vector potential from the Lorenz gauge via eq. (16), see sec. 2.5 of [2].

2.3.4 Static-Voltage Gauge

A variant of the Gibbs gauge is that the scalar potential is not zero, but rather is the instantaneous Coulomb potential at some arbitrary time \( t_0 \),

\[
V^{(SV)}(r, t) = V^{(C)}(r, t_0) = \int \frac{\rho(r', t_0)}{|r - r'|} \, dVol'.
\]

(24)

This is the static-voltage gauge [28], called the Coulomb-static gauge in [29].

From eq. (23), we see that the vector potential in the static-voltage gauge differs only slightly from that in the Gibbs gauge,

\[
A^{(SV)}(r, t) = A^{(G)}(r, t) - c t \nabla V^{(C)}(r, t_0)
\]

(25)

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11 Apparently the Gibbs gauge is also called the Hamiltonian or temporal gauge, as mentioned in sec. VIII of [10]. That is, the Gibbs gauge is handy in examples where the electric field is known, and the vector potential is needed for use in the Hamiltonian of the system, expressed in terms of canonical momenta of charges \( q \) as \( p_{\text{canonical}} = p_{\text{mech}} + qA/c \).

The Gibbs gauge, where \( V = 0 \), was considered, but not so named, in [26]. See also [27], where if one takes the “mechanical” potential energy \( U \) of electric charge \( q \) to be \( qV \), then the condition that \( V = -U/q \) requires that \( V = 0 \).

12 The distinction between \( \nabla \cdot A \) in the Coulomb and Gibbs gauges is slight, and may be why Gibbs thought that his new gauge was the Coulomb gauge used by Maxwell.
2.3.5 Kirchhoff Gauge

The earliest statement of a gauge condition appears to have been made by Kirchhoff in 1857 [30], when he specified that

$$\nabla \cdot A^{(K)} = \frac{1}{c} \frac{\partial V^{(K)}}{\partial t} \quad \text{(Kirchhoff).} \quad (26)$$

Using this gauge condition in the general wave equation (8) for the scalar potential, we have

$$\nabla^2 V^{(K)} + \frac{1}{c^2} \frac{\partial^2 V^{(K)}}{\partial t^2} = -4\pi \rho, \quad (27)$$

such that the Kirchhoff-gauge scalar potential can be said to propagate with imaginary speed, $v_K = ic$. In this sense, the Kirchhoff gauge is a special case of the velocity gauge of sec. 2.3.1.

The scalar potential in the Kirchhoff gauge can be written as a “retarded” potential which speed of propagation $ic$. Recalling eq. (12), we have

$$V^{(K)}(r, t) = \int \frac{\rho(r', t') = t - |r - r'|/ic}{|r - r'|} d\text{Vol}' \quad \text{(Kirchhoff),} \quad (28)$$

For further discussion of the Kirchhoff gauge, see [31].

2.3.6 Poincaré Gauge

In cases where the fields $E$ and $B$ are known, we can compute the potentials in the so-called Poincaré gauge (see sec. 9A of [10] and [32, 33, 34]),

$$V^{(P)}(r, t) = -r \cdot \int_0^1 du E(u r, t), \quad A^{(P)}(r, t) = -r \times \int_0^1 u du B(u r, t) \quad \text{(Poincaré).} \quad (29)$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.\textsuperscript{14,15}

The Poincaré-gauge condition can be stated as

$$r \cdot A^{(P)} = 0 \quad \text{(Poincaré).} \quad (32)$$

\textsuperscript{13}The Poincaré gauge is also called the multipolar gauge [35].

\textsuperscript{14}The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the “other side” from the origin.

\textsuperscript{15}We transcribe Appendices C and D of [32] to verify that $E$ and $B$ indeed follow from the Poincaré potentials (29).

$$-\nabla V^{(P)} - \frac{1}{c} \frac{\partial A^{(P)}}{\partial t} = \int_0^1 du \left\{ \nabla [r \cdot E(u r, t)] + r \times \frac{u}{c} \frac{\partial B(u r, t)}{\partial t} \right\} = \int_0^1 du \left\{ \nabla [r \cdot E(u r, t)] - r \times [\nabla \times E(u r, t)] \right\}$$

$$= \int_0^1 du \left\{ (r \cdot \nabla) E(u r, t) + |E(u r, t) \cdot \nabla| r + E(u r, t) \times (\nabla \times r) \right\} = \int_0^1 du \left\{ u \frac{d(u x_i)}{du} \frac{\partial E(u r, t)}{\partial (u x_i)} + E(u r, t) \right\}$$

$$= \int_0^1 du \frac{d}{du} u E(u r, t) = E(r, t). \quad (30)$$
If the scalar potential in the Poincaré gauge can be computed, it may then be simpler to deduce the vector potential in this gauge via eq. (16) than via eq. (29).

See [36] for an application of the spirit of the Poincaré potentials to a relation between a physical charge and current densities $\rho$ and $J$ and effective polarization and magnetization densities $P$ and $M$, such that $\rho = -\nabla \cdot P$ and $J = c\nabla \times M + \partial P/\partial t$.

References

On the Identity of the Vibration of Light with Electrical Currents, Phil. Mag. 34, 287 (1867),


\[ \nabla \times A^{(P)} = -\int_0^1 u \, du \, \nabla \times \left[ r \times B(ur, t) \right] \]

\[ = -\int_0^1 u \, du \, \left[ r \left[ \nabla \cdot B(ur, t) \right] - B(ur, t) \left[ \nabla \cdot r \right] + \left[ B(ur, t) \cdot \nabla \right] r - \left( r \cdot \nabla \right) B(ur, t) \right] \]

\[ = \int_0^1 u \, du \, \left\{ 2B(ur, t) + ur \frac{\partial B(ur, t)}{\partial (ux_1)} \right\} = \int_0^1 u \, du \, \left\{ \frac{1}{u} \frac{d}{du} u^2 B(ur, t) \right\} = B(r, t). \]


