Radiation Damping of a Refrigerator Magnet

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(January 20, 2018)

1 Problem

A refrigerator magnet in the form of a thin disc with magnetic moment $\mu$ parallel to its axis is initially arranged for that axis to precess at angle $\theta$ around a uniform external magnetic field $B$. What is the damping time of the motion due to magnetic dipole radiation?\(^1\)

This problem was suggested by Shivaji Sondhi.

2 Solution

We label the unit vector along the axis of the magnet as $\hat{1}$; the unit vector in the midplane of the magnet and in the $\hat{1}$-$B$ plane is $\hat{2}$; then the unit vector $\hat{3} = \hat{1} \times \hat{2}$ also lies in the midplane of the magnet. For the precession angular velocity $\omega$ to be in the same direction as $B$, the magnetic moment $\mu$ is along the $-\hat{1}$-axis (to be confirmed in eq. (5)), as in the figure below.

If the thin disc has mass $m$ and radius $r$, then its inertia tensor has diagonal elements $(l_{11}, l_{22}, l_{33}) = mr^2(2, 1, 1)/4$. The initial angular velocity $\omega = \omega(\cos \theta, \sin \theta, 0)$ is parallel to $B$, at angle $\theta$ to the $\hat{1}$-axis. The “mechanical” angular momentum of the precessing disc is

$$L_\omega = \hat{1} \cdot \omega = \frac{mr^2 \omega}{4} (2 \cos \theta, \sin \theta, 0). \quad (1)$$

In addition, there is angular momentum associated with the electrons whose magnetic moments comprise the total magnetic moment $\mu$,

$$L_\mu = -\frac{m_e c}{e} \mu \approx -6 \times 10^{-8} \mu(1, 0, 0), \quad (2)$$

in Gaussian units, where $m_e$ and $e > 0$ are the mass and (magnitude of the) charge of an electron, and $c$ is the speed of light in vacuum.

\(^1\)A related example concerns the spin down of magnetars [1].
2.1 \( L_\mu \ll L_\omega \)

A neodymium magnet has mass density \( \rho = m/\text{Vol} = 7 \text{ g/cm}^3 \) and magnetization density \( M = \mu/\text{Vol} \approx 1000 \text{ cgs units} \), so \( |L_\mu|/L_\omega \approx 6 \times 10^{-8} M/\rho r^2 \omega \approx 10^{-5} \) for a magnet with \( r \) and \( \omega \) of order unity, and we will neglect \( L_m \) in the following.

In this approximation, the total initial angular momentum is

\[
L \approx L_\omega = \left( \frac{mr^2 \omega \cos \theta}{2}, \frac{mr^2 \omega \sin \theta}{4}, 0 \right) \equiv L(\cos \alpha, \sin \alpha, 0),
\]

where the magnitude \( L \) of the angular momentum, and its angle \( \alpha \) to the \( \hat{1} \)-axis, are related by

\[
L = \frac{mr^2 \omega}{4} \sqrt{1 + 3 \cos^2 \theta}, \quad \tan \alpha = \frac{\tan \theta}{2}, \quad \cos \alpha = \frac{2 \cos \theta}{\sqrt{1 + 3 \cos^2 \theta}}.
\]

The (torque) equation of motion for steady precession is,

\[
\frac{dL}{dt} = \omega \times L = \tau = \mu \times B, \quad \omega L \sin \beta = \mu B \sin(\pi - \theta) = \mu B \sin \theta,
\]

where the scalar equation is for the component along the \(-\hat{3}\)-axis, \( \beta = \theta - \alpha \), and \( \omega \) and \( B \) point in the same direction for \( \mu \) along the \(-\hat{1}\)-axis. Then, recalling eq. (4),

\[
\sin \beta = \sin \theta \cos \alpha - \cos \theta \sin \alpha = \sin \theta \cos \alpha(1 - \cot \theta \tan \alpha) = \frac{\sin \theta \cos \alpha}{2} = \frac{\sin \theta \cos \theta}{\sqrt{1 + 3 \cos^2 \theta}},
\]

and the equation of motion (5) can be rewritten as

\[
\frac{mr^2 \omega^2}{4} \cos \theta = \mu B, \quad \omega^2 = \frac{4 \mu B}{mr^2 \cos \theta}.
\]

This implies that \( \mu B \) and \( mr^2 \omega \) are of the same order, and that \( \omega \) is large compared to unity. The latter relation further suppresses the ratio \( |L_m|/L_\omega \) considered previously, which reinforces that we can neglect the angular momentum \( L_m \). Equation (7) also indicates that steady precession can only exist for \( \theta < \pi/2 \).

However, when we consider the energy of the system, using eq. (7),

\[
U = T + V = \frac{\omega \cdot 1 \cdot \omega}{2} - \mu \cdot B = \frac{mr^2 \omega^2}{8}(1 + \cos^2 \theta) + \mu B \cos \theta = \frac{\mu B}{2} \left( 3 \cos \theta + \frac{1}{\cos \theta} \right),
\]

the kinetic and potential energies are comparable.

\footnote{This corresponds to residual induction (saturation remanence) of \( B = 4\pi M \approx 13,000 \text{ gauss} \).}
Assuming that the motion is always just precession of the disc about \( \mathbf{B} \), the time rate of change of the energy is,

\[
\frac{dU}{dt} = \frac{\mu B \sin \theta}{2 \cos^2 \theta} \dot{\theta} (1 - 3 \cos^2 \theta). \tag{9}
\]

The energy has a minimum at \( \cos^2 \theta = 1/3 \), \( \theta = 54.7^\circ \).

At last, we come to consideration of the radiation of energy by the precessing magnetic moment,

\[
\frac{dU}{dt} = -\frac{2\mu^2}{3c^3} = -\frac{2\mu^2 \omega^4 \sin^2 \theta}{3c^3} = -\frac{32\mu^4 B^2 \tan^2 \theta}{3m^2 r^4 c^3}, \tag{10}
\]

recalling, for example, eq. (71.5) of [2], and using our eq. (7). From eqs. (9) and (10) we have that

\[
\dot{\theta} = -\frac{64\mu^3 B \sin \theta}{3m^2 r^4 c^3 (1 - 3 \cos^2 \theta)}. \tag{11}
\]

However, there is an inconsistency in the above analysis, in that the precessing disc always radiates energy, while the assumption of precession led to the identification of an energy minimum at \( \theta = 54.7^\circ \). It appears that the (adiabatic) model of instantaneous precession of the disc about \( \mathbf{B} \) is not sufficiently accurate.

Another difficulty with this example is that the steady precession of a refrigerator magnet in, say, a magnetic field of 1 T = 10,000 gauss leads to internal stresses close to the breaking strength. As noted in [3] the maximum angular velocity or rotation of a ring of radius \( r \) and cross-sectional area \( A \) about its axis is related by

\[
\omega^2 r^2 = \frac{T_{\text{max}}}{A \rho}, \tag{12}
\]

where \( T_{\text{max}}/A \approx 10^7 \) dyne/cm\(^2\) is the breaking strength. Comparing with eq. (7), we have that

\[
\omega^2 r^2 = \frac{4\mu B}{m \cos \theta} = \frac{4MB}{\rho \cos \theta} = \frac{4\omega^2_{\text{max}} r^2}{\cos \theta} > \omega^2_{\text{max}} r^2, \tag{13}
\]

for \( M = 1000 \), \( B = 10,000 \) gauss and \( \rho = 7 \) g/cm\(^3\), so the precessing disc would break apart.

2.2 \( L_\omega \ll L_\mu \)

We now consider the case of very low angular velocity and weak external magnetic fields, in which we can neglect the angular momentum \( L_\omega \) compared to the tiny angular momentum \( L_\mu \) associated with the magnetic moment \( \mu \) of the refrigerator magnet. In this case, the angular momentum is along the \( \hat{1} \)-axis,

\[
\mathbf{L} \approx L_\mu (1, 0, 0), \tag{14}
\]

and the equation of motion (5) reduces to

\[
\omega = \frac{\mu B}{L_\mu} = \frac{eB}{m_e c}, \tag{15}
\]
recalling eq. (2), such that the precession angular velocity is independent of angle $\theta$.

The condition $L_\omega \ll L_\mu$ implies that

$$mr^2\omega \ll \frac{m_e c \mu}{e}, \quad \rho \text{Vol} r^2 eB \ll \frac{m_e c M \text{Vol}}{e}, \quad B \ll \frac{m_e c^2 M}{e^2 \rho} = 5 \times 10^{-15} \text{ gauss}. \quad (16)$$

When this holds, the kinetic energy of rotation is also negligible compared to the potential energy, so the total energy is approximately

$$U \approx \mu B \cos \theta, \quad \frac{dU}{dt} \approx -\mu B \dot{\theta} \sin \theta \quad (17)$$

The energy loss due to radiation is now, using eq. (14),

$$\frac{dU}{dt} = -\frac{2 \dot{\mu}^2}{3c^3} = -\frac{2 \mu^2 \omega^4 \sin^2 \theta}{3c^3} = -\frac{2 \mu^2 e^4 B^4 \sin^2 \theta}{3m_e^4 c^7}, \quad (18)$$

which together with eq. (17) tells us that angle $\theta$ increases at the rate

$$\dot{\theta} = \frac{2 \mu e^4 B^3 \sin \theta}{3m_e^4 c^7}, \quad (19)$$

until $\theta = \pi$ and the magnetic moment $\mu$ becomes aligned with the magnetic field $B$. This motion lasts for time

$$\Delta t \approx \frac{m_e^4 c^7}{\mu e^4 B^3} \approx 3 \times 10^{50} \text{ s}, \quad (20)$$

for $\mu = M \text{ Vol} = 100 \text{ cgs units}$ and $B = 10^{-15} \text{ gauss}$. This is much longer than the age of the Universe.

References

[1] K.T. McDonald, Magnetars (Nov. 29, 1998),


[3] K.T. McDonald, Rayleigh’s Spinning Ring (July 12, 2017),