The Fractal Dimension of a Ball of Aluminum Foil

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(Mar. 6, 1997)

1 Problem

If a sheet of aluminum foil is crumpled into a ball, the mass at radius less than \( r \) can be taken as \( kr^D \) where \( D \) will lie between 2 and 3. We may call \( D \) the fractal (Haussdorff) dimension of the crumpled aluminum ball. (In practice the above relation could only hold for \( r \) larger than the thickness of the foil.)

Explain how the fractal dimension \( D \) could be determined from knowledge of the velocity \( v \) attained by the ball upon rolling without slipping down an incline of height \( h \). Ignore air resistance, rolling friction, etc.

(A standard model of crumpling predicts \( D = 2.5 \).)

2 Solution

We need some other property of the ball that depends on the fractal dimension \( D \). The moment of inertia about a diameter suggests itself.

\[
I = 2\pi \int_0^R \int_0^\pi \rho(r)r^2 \sin \theta dr d\theta r^2 \sin^2 \theta,
\]

where the density \( \rho(r) \) is related by

\[
\rho = \frac{1}{4\pi r^2} \frac{dm}{dr} = \frac{Dkr^{D-3}}{4\pi}.
\]

Hence the moment of inertia is

\[
I = \frac{2}{3} \frac{DMR^2}{D+2}.
\]

For the rolling experiment, conservation of energy tells us

\[
Mgh + \frac{1}{2} I \omega^2 + \frac{1}{2} Mv^2,
\]

where the condition of rolling without slipping is \( \omega = v/r \). Combining things, we find

\[
\frac{D}{D+2} = \frac{3}{2} \left( \frac{2gh}{v^2} - 1 \right).
\]