

# Electromagnetic Field Momentum

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## 1 From Poynting to Poincaré

This note is in response to Question #26 concerning electromagnetic field momentum, Am. J. Phys. **63**, 777 (1995),

[http://puhep1.princeton.edu/~mcdonald/examples/EM/romer\\_ajp\\_63\\_777\\_95.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/romer_ajp_63_777_95.pdf)

The difficulty in interpreting Poynting's vector as proportional to momentum for a system that includes sources as well as fields was first pointed out by Poincaré in 1906. A relativistically consistent formalism can only be achieved by adding terms that include stresses in the sources that arise when the fields are generated.

The usual relativistic argument begins by recasting the Lorentz-force 4-vector,

$$f_\mu = F_{\mu\nu}j^\nu = \left( \frac{\mathbf{j} \cdot \mathbf{E}}{c}, \rho\mathbf{E} + \frac{\mathbf{j}}{c} \times \mathbf{B} \right), \quad (1)$$

(in Gaussian units and with metric  $\eta_{00} = 1$ ,  $\eta_{11} = \eta_{22} = \eta_{33} = -1$ ; greek indices run over 0,1,2,3 while latin indices run over 1,2,3) as the derivative of a stress tensor:

$$f_\mu = -\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\partial^\nu T_{\mu\nu}. \quad (2)$$

This leads to the result

$$T_{\mu\nu} = \frac{1}{4\pi}F_{\mu\alpha}F_\nu^\alpha + \frac{1}{16\pi}\eta_{\mu\nu}F_\alpha^\beta F_\beta^\alpha = \frac{1}{4\pi} \begin{pmatrix} (E^2 + B^2)/2 & \mathbf{E} \times \mathbf{B} = \mathbf{S}/c \\ \mathbf{E} \times \mathbf{B} & \delta_{ij}(E^2 + B^2)/2 - E_i E_j - B_i B_j \end{pmatrix}. \quad (3)$$

Next, one makes a trial definition of an energy-momentum 4-vector for the fields as

$$P_\mu = \int T_{0\mu} d\text{Vol}, \quad (4)$$

so that

$$P_0 = \int T_{00} d\text{Vol} = \frac{1}{8\pi} \int (E^2 + B^2) d\text{Vol} = U_f, \quad (5)$$

$$P_i = \int T_{0i} d\text{Vol} = \frac{1}{4\pi} \int \mathbf{E} \times \mathbf{B} d\text{Vol} = c\mathbf{P}_f, \quad (6)$$

where

$$\mathbf{P}_f = \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} d\text{Vol} \quad (7)$$

is the field 3-momentum that is the subject of Question #26. Then,  $P_\mu = (U, c\mathbf{P}_f)$  has the appearance of a familiar 4-vector.

## 2 Free Fields

If there are no sources present (free-field case), then the Lorentz-force 4-vector vanishes, the 4-divergence of  $T_{\mu\nu}$  vanishes also, and one can verify that  $P_\mu$  really transforms like a 4-vector.

The argument thus far is seconded in the books of Rohrlich and of Jackson, who don't advocate carrying it further.

## 3 Fields with Sources

Poincaré suggests we proceed to the case where sources of the fields are present. By direct application of a Lorentz transformation to the stress tensor  $T_{\mu\nu}^\star$ , where the  $\star$  indicates the rest frame of the sources, one deduces that  $P_\mu$  fails to transform like a 4-vector if there are nonzero spatial components to the stress tensor, *i.e.*, if some  $\int T_{ij}^\star \neq 0$ .

Poincaré noted that if some  $\int T_{ij}^\star$  are nonzero then the system of sources is not in mechanical equilibrium until mechanical stresses  $\int P_{ij}^\star = -\int T_{ij}^\star$  are developed to counter the electromagnetic stresses. The  $P_{ij}^\star$  can be embedded in a 4-tensor  $P_{\mu\nu}$  that includes the mechanical rest energy  $m_{\text{mech}}c^2 = \int P_{00}^\star$  and the mechanical momentum  $c\mathbf{P}_{\text{mech},i} = \int P_{0i}^\star = \int P_{i0}^\star$ .

Then when one defines

$$P_\mu = \int (T_{0\mu} + P_{0\mu}) d\text{Vol}, \quad (8)$$

one has a true 4-vector, with

$$P_0 = U + m_{\text{mech}}c^2, \quad (9)$$

$$P_i = c(\mathbf{P}_f + \mathbf{P}_{\text{mech}})_i. \quad (10)$$

This formalism does not quite succeed in providing an independent interpretation of the 'field momentum'  $\mathbf{P}_f$  when sources are present. That is, only the sum  $\mathbf{P}_f + \mathbf{P}_{\text{mech}}$  has a dynamical meaning, where  $\mathbf{P}_{\text{mech}}$  includes a contribution associated with the mechanical stress that arise in response to electromagnetic forces.

## 4 $\mathbf{P}_f^\star$ Has No Dynamical Significance

There remains the specific topic of Question #26: what interpretation should be given when  $\mathbf{P}_f^\star \neq 0$  in the 'rest frame' of the sources? In view of the difficulty of giving any independent meaning to  $\mathbf{P}_f$  when sources are present, this issue is secondary.

It is not very satisfactory to note that one can always find a frame in which  $\mathbf{P}_f$  vanishes, since in general the center of mass of the sources will be moving in this frame.

Instead, we advocate a fairly trivial solution to the problem. Simply regard the value  $\mathbf{P}_f^\star$  as a constant of the system without an interpretation of anything being in motion. This is a consistent view because the dynamical significance of momentum is in its derivative,

$$f_\mu = \frac{dP_\mu}{d\tau}, \quad (11)$$

where  $\tau$  is the proper time, and in conservation laws, both of which are unaffected by an additive constant. In this sense no dynamical meaning can be assigned to the value of  $\mathbf{P}_f^*$ , and one can consistently choose not to give it any further interpretation.

We can amplify this point by recalling the Lorentz transformation of the 4-momentum  $P_\mu = (U_f, c\mathbf{P}_f)$  in a boost by  $\vec{\beta} = \mathbf{v}/c$  from the rest ( $\star$ ) frame:

$$\mathbf{P}_f = \gamma \left( \mathbf{P}_f^* + \frac{U_f^*}{c^2} \mathbf{v} \right), \quad (12)$$

where  $\gamma = 1/\sqrt{1 - (v/c)^2}$ . Thus in a frame where the system moves with velocity  $\mathbf{v}$ , the part of the momentum that is proportional to velocity depends on the effective mass  $U_f^*/c^2$  in the rest frame, and not on the momentum  $\mathbf{P}_f^*$  in the rest frame. A nonzero value of  $\mathbf{P}_f^*$  in the rest frame has no dynamical effect on the momentum.

We have gotten used to electrons and photons having spin without being able to identify anything that rotates. So I propose that we not worry too much about a nonzero static value for the ‘field momentum’ that has no dynamical consequence. Foregoing any interpretation of  $\mathbf{P}_f^*$  is even easier than for electron spin, since that latter has dynamical significance.

## 5 An Example

I append a further argument that shows how the ‘field momentum’  $\mathbf{P}_f$  by itself does not consistently behave like a nonrelativistic momentum, whether or not its value in the rest frame of the sources is zero.

We consider a system that when at rest produces fields  $\mathbf{E}_0$  and  $\mathbf{B}_0$ . The corresponding ‘field momentum’  $\mathbf{P}_0$  may or may not be zero, but in any case is a constant vector. Only the time-dependent part of the ‘field momentum’ will have relevance to  $\mathbf{F} = d\mathbf{P}/dt$ .

Next, consider the system when it is moving with center-of-mass velocity  $\mathbf{v}$ , where  $v \ll c$ . We suppose that there is no change in the state of the system relative to its center of mass, so fields  $\mathbf{E}_0$  and  $\mathbf{B}_0$  still hold in the rest frame of the system. Then the nonrelativistic limit of the transformation of the electromagnetic fields tells us that

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{v}}{c} \times \mathbf{B}_0, \quad (13)$$

$$\mathbf{B} = \mathbf{B}_0 + \frac{\mathbf{v}}{c} \times \mathbf{E}_0, \quad (14)$$

and so the ‘field momentum’ associated with the moving system is

$$\mathbf{P}_f = \mathbf{P}_0 + \frac{1}{4\pi c^2} \int [(E_0^2 + B_0^2)\mathbf{v} + (\mathbf{E}_0 \cdot \mathbf{v})\mathbf{E}_0 + (\mathbf{B}_0 \cdot \mathbf{v})\mathbf{B}_0] d\text{Vol}, \quad (15)$$

neglecting a term in  $(v/c)^2$ . The rate of change of this momentum is

$$\frac{d\mathbf{P}_f}{dt} = \frac{2U_0}{c^2} \mathbf{a} + \frac{1}{4\pi c^2} \int (\mathbf{E}_0 \cdot \mathbf{a})\mathbf{E}_0 + (\mathbf{B}_0 \cdot \mathbf{a})\mathbf{B}_0] d\text{Vol}, \quad (16)$$

where  $\mathbf{a} = d\mathbf{v}/dt$  is the acceleration of the system, and  $U_0$  is the rest-frame field energy:

$$U_0 = \frac{1}{8\pi} \int (E_0^2 + B_0^2) d\text{Vol}. \quad (17)$$

While, as expected, the constant value  $P_0$  does not appear in the expression for the rate of change of ‘field momentum’, this expression does not quite have the desired form,  $m_{\text{eff}}\mathbf{a}$ . I infer that this is another demonstration of the view of Poincaré that the ‘field momentum’  $\mathbf{P}_f$  cannot be interpreted by itself when sources are present.

## 6 The Example of Question #26

Regarding the specific example of nested electric and magnetic dipoles, it is easy to see that the diagonal elements of the electromagnetic stress tensor,  $T_{ii}$ , are nonvanishing. The sphere of charge and sphere of current-carrying coils would fly apart without some kind of glue. The resulting mechanical stresses change the rest mass of the system and, when it is in motion, its momentum by an amount comparable to the electromagnetic ‘mass’ and momentum contributions. Trying to interpret the electromagnetic momentum without considering the corresponding stress-induced changes in the mechanical momentum is counterproductive.

But the bottom line is that no meaningful interpretation can be given to the nonzero  $\mathbf{P}_f^*$  for that system in its rest frame.