Is Faraday’s Disk Dynamo a Flux-Rule Exception?
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1 Problem

In sec. 17.1 of [177], Feynman noted that the differential form “Faraday’s law” is,
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]
and then argued that for a fixed loop one can deduce the integral form of this “law” as,
\[ \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -d dt \int_{\text{surface of loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\text{Area} = -d dt (\text{magnetic flux through loop}), \]
which is often called the “flux rule”. In sec. 17.2, he considered an experiment of Faraday from 1831, sketched below, and claimed that this is an example of an exception to the “flux rule”, where one should instead consider the motional \( \mathcal{E}\mathcal{M}\mathcal{F} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}. \)

![Diagram of Faraday's Disk Dynamo](image)

Fig. 17-2. When the disc rotates there is an emf from \( \mathbf{v} \times \mathbf{B} \), but with no change in the linked flux.

However, a recent paper [261] expressed the view that Feynman and many others are wrong about this example, and that it is well explained by the “correct” interpretation of the “flux rule”.

[231]
What’s going on here?

2 Solution

In my view, the issue is that examples of magnetic induction can be analyzed more than one way, and whenever there is more than one way of doing anything, some people become overly enthusiastic for their preferred method, and imply that other methods are incorrect.
As will be reviewed in the historical appendix below, the interpretation of Faraday’s views on magnetic induction in a closed loop, whose shape may or may not vary with time in a magnetic field that may or may not vary with time, as,

$$\mathcal{E}\mathcal{M}\mathcal{F} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int_{\text{loop}} \mathbf{B} \cdot d\text{Area}, \quad (3)$$

originated with Maxwell. However, this relation is often tricky to apply (due to ambiguities as to where the loop is, and to the meaning of $\mathcal{E}\mathcal{M}\mathcal{F}$ and of $d/dt$), so many people, including Feynman, recommend splitting the calculation into two pieces,$^1$

$$\mathcal{E}\mathcal{M}\mathcal{F} = \mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}} + \mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}}, \quad (4)$$

where,

$$\mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}} = -\frac{\partial}{\partial t} \int_{\text{loop at time } t} \mathbf{B} \cdot d\text{Area} = -\int_{\text{loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\text{Area} = \oint_{\text{loop}} \mathbf{E} \cdot d\text{l}, \quad (5)$$

using eq. (1) and Stoke’s theorem, and

$$\mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\text{l}, \quad (6)$$

in which $\mathbf{v}$ is the velocity (in the inertial lab frame of the calculation) of an element $d\text{l}$ of the loop (which is a line, but which may or may not be inside a conductor).

Maxwell showed (although apparently not very clearly) in Arts. 598-599 of [115] that eq. (3) can also be written as,

$$\mathcal{E}\mathcal{M}\mathcal{F} = \oint_{\text{loop}} (\mathbf{v} \times \mathbf{B} - \partial A/\partial t) \cdot d\text{l}, \quad (7)$$

where the electromagnetic fields $\mathbf{B}$ and $\mathbf{E}$ can be related to a vector potential $\mathbf{A}$ and a scalar potential $V$ according to,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad (8)$$

such that,$^2$

$$\mathcal{E}\mathcal{M}\mathcal{F} = \oint_{\text{loop}} (\mathbf{v} \times \mathbf{B} + \mathbf{E}) \cdot d\text{l} = \mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}} + \mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}}, \quad (9)$$

since for a closed loop, $\oint \nabla V \cdot d\text{l} = 0$ and $\oint \mathbf{E} \cdot d\text{l} = -\oint \partial \mathbf{A}/\partial t \cdot d\text{l}$.

Thus, the two methods, eq. (3) and (eq. (5), of computing induced $\mathcal{E}\mathcal{M}\mathcal{F}$s, give the same results (when correctly computed), and it is a matter of taste which method is preferred.

$^1$The partition (4) may be due to Heaviside (1885) [119].
$^2$The first clear statement of eq. (9) may be in sec. 86 of the text of Abraham (1904) [130], which credits Hertz [122] for inspiration on this.
2.1 What Does EMF Mean?

A lingering issue is that the symbol EMF has not been defined independent of eqs. (3) and (5)-(6).

It is generally agreed that EMF means electromotive force, but what does the latter mean?³

Fechner (1831), p. 225 of [70], may have been the first to interpret the symbol E in Ohm’s law [66],

\[ E = IR, \]  

(10)

as the electromotorische Kraft, i.e., electromotive force. Ohm and Fechner studied only steady currents, in which case the electromotive “force” between two points a and b is equal to the work done by electric effects when moving a unit electric charge from between two points a and b,

\[ E(a, b) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}. \]  

(11)

Although Faraday’s first report (1831) [71] of electromagnetic induction was based on time-dependent magnetic flux in a system of two coils at rest in the lab, all of his subsequent studies involved moving elements. This led Neumann, in sec. 1 of his great paper of 1845 [94] which introduced the concept of inductance, to discuss elektromotorische Kraft in the sense of the motional EMF (6).

Thus, early in the history of electromagnetic induction, two meanings of EMF, eqs. (5) and (6), became prominent, and this tradition survives to the present day, as exemplified by Feynman’s discussion [177].

Despite Maxwell’s attempt in Arts. 598-599 of [115] to merge the two concepts of EMF into one, only after Lorentz’ clarification (1895), eq. (V), p. 21 of [126], that the electromagnetic force on charge q is,

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \]  

did some people begin to accept that EMF could/should be given by eq. (9).⁴ There, the EMF is the work done on a unit charge as it traverse a circuit according to the force law (12).⁵

Despite this clarification, analysis of magnetic induction via eqs. (4)-(6) remains more appealing to many people than use of eq. (3), whose equivalence to eq. (9) is not always evident. For many, the notion of motional EMF as eq. (6) gives a better physical understanding of examples like Faraday’s disk dynamo than does consideration of magnetic flux through a deforming circuit.

³It is claimed on p. 3 of [50] (1823, perhaps the first textbook on electromagnetism), and also in [193], that Volta (1801) [16, 17] coined this term, but I do not find it in Volta’s papers, although he did speak of a force motrice in the appareil électro-moteur (galvanic batteries) which he made famous.

⁴Lorentz (1892), p. 405 of [123], still identified the last form of eq. (5) as the force électromotrice, and wrote the “flux law” in eq. (42), p. 416, as \[ \oint_{ \text{loop} } \mathbf{E} \cdot d\mathbf{l} = -(d/dt) \oint_{ \text{loop} } \mathbf{B} \cdot d\text{Area}. \] And, in 1903, eq. 27, p. 83 of [129], Lorentz again referred to this equation, but with the proviso that the loop be at rest.

⁵However, this definitions remains in some conflict with circuit analysis, where it is assumed that a unique scalar EMF/voltage can be assigned at any junction between circuit elements, as discussed in sec. 3.2 below.
2.2 When Can an EMF Be Measured?

A disconcerting aspect of the concept of EMF is that although it is defined for imaginary paths/circuits, it is not measurable in practice unless a conductor exists along the path that defines the EMF.

When a conductor is in place, the EMF between two points on it can (generally) be measured with a voltmeter, but the voltmeter reads “nothing” when not sampling a test conductor.\(^6\) As such, an EMF is a less physical concept than electric and magnetic fields, although these are already somewhat abstract.

The greater abstraction of an EMF may contribute to the ongoing differences of opinion as how best to think about it.

2.3 Two Examples

2.3.1 Faraday’s Disk Dynamo

We illustrate different analyses of Faraday’s disk dynamo in case of a spatially uniform magnetic field, using the figures below from [160] and the various comments on that paper.

Field Constant in Time

We start by considering the fixed loop ABCDOA in the figure on the right below.

There is no magnetic flux through this loop, so \( EMF_{\text{fixed loop}} = 0 \) here. On the other hand, segment DO, of length \( a \), rotates with angular velocity \( \omega \), so a point on this segment at distance \( r \) from the axis has velocity \( v = \omega r \) perpendicular to \( B \), such that,

\[
EMF_{\text{motional}} = \oint_{\text{loop}} v \times B \cdot dl = \int_a^0 \omega r B dr = -\frac{a^2 B \omega}{2}. \tag{13}
\]

Note that if we instead considered the loop ABCDD'O'A, we again have \( EMF_{\text{fixed loop}} = 0 \), while now segment D'O rotates, and again the motional (and total) EMF is given by eq. (13).

To use the generalized “flux law”, eq. (3) for loop ABCDOA, we compare the flux through the loop at times \( t \) and \( t + dt \). At the latter time, segment DO has rotated by angle \( d\theta = \omega dt \), and we must suppose that a new segment DD' is added to the loop so that it remains closed,

\[^6\text{Strictly, a voltmeter that is not connected to a test conductor is a kind of antenna, and can give a nonzero AC-voltage reading when electromagnetic radiation is present. In such cases, it is difficult to interpret the reading of the voltmeter as an EMF [235].}\]
since eq. (3) only applies to closed loops. Then, the magnetic flux through loop ABCDD’OA is $\Phi_B(t + dt) = B a^2 d\theta/2 = a^2 B \omega dt/2$, and total $\mathcal{E}\mathcal{M}\mathcal{F}$ is,

$$\mathcal{E}\mathcal{M}\mathcal{F}_{\text{total}} = -\frac{d\Phi_B}{dt} = -\frac{a^2 B \omega}{2}. \quad (14)$$

An issue for many people concerning Faraday’s disk dynamo is the location of the “seat” of the $\mathcal{E}\mathcal{M}\mathcal{F}$ (14). In this context, the notion of the motional $\mathcal{E}\mathcal{M}\mathcal{F}$ is appealing in suggesting that the Lorentz force $e v \times B$ on moving charges in the disk identifies the “seat” of the $\mathcal{E}\mathcal{M}\mathcal{F}$ as being in the copper disk, while this is left ambiguous by the more abstract relation (3).

Practical homopolar generators are more often constructed with a “drum” geometry, as sketched below (from [246]).

Here, a copper-coated iron cylinder rotates inside a cylindrical electromagnet, and a motional $\mathcal{E}\mathcal{M}\mathcal{F} v Bl_{cd}$, where $v$ is the azimuthal velocity of the copper cylinder, is generated along the rotating copper between sliding contacts at $c$ and $d$ (and elsewhere). One can also consider the circuit abcdea as having a moving segment bc, and use the generalized flux law (3) to compute the $\mathcal{E}\mathcal{M}\mathcal{F}$. But again, the computation of the motional $\mathcal{E}\mathcal{M}\mathcal{F}$ is more appealing to many (particularly in the “engineering” community).

### Field Sinusoidal in Time

Suppose instead that the external magnetic field for the disk dynamo is still spatially uniform, but varies with sinusoidally with time, say $B_{\text{ext}} = B_0 \cos \omega' t \hat{z}$.

This example was used in Cohn in [160] as an argument that generalized flux rule eq. (3) does not always work. It was then pointed out by Bewley [163] (who may have been the first after Maxwell to demonstrate eq. (4) in English [149]) that the time-dependent magnetic field induces eddy currents, which induce additional magnetic fields, which induce yet more electric fields, ... That is, this example cannot be well analyzed without first solving for the total electric and magnetic fields, and total currents, in the absence of the external circuit elements ABCD, which depend on the conductivity of the copper disk.

For very high conductivity, the induced magnetic field of the eddy currents cancels the external magnetic field inside (and at the flat surfaces of) the copper disk,\(^7\) in which case no $\mathcal{E}\mathcal{M}\mathcal{F}$ is developed in the circuit.

\(^7\)Recall that a superconductor expels an external magnetic field [153].
2.3.2 Circular Loop that Expands or Contracts Radially

This example was mentioned by Franklin (1908) on p. 1357 of [134] as part of the discussion following a presentation by Hering that the existence of a motional $\mathcal{EMF}$ does not always imply that the total $\mathcal{EMF}$ is nonzero.\footnote{For discussion by the author of Hering’s example, see [260].}

We consider a circular loop of time-dependent radius $a(t)$ in a uniform magnetic field $B$ perpendicular to the plane of the loop. In case that the loop is not a conductor, the electric field $E$ can be zero everywhere.

Then, a naïve use of eq. (2) would lead to the inference that

$$
\oint_{\text{loop}} E \cdot d\mathbf{l} = -B \cdot \frac{d\text{Area}}{dt} = -2\pi a B \frac{da}{dt},
$$

which is nonzero, and hence the electric field must be nonzero also, in contradiction to our assumption. However, this is a misuse of the “flux law”.

A better approach is to consider eq. (3) or eqs. (5)-(6). Equation (5) tells us that,

$$
\mathcal{EMF}_{\text{fixed loop}} = -\int_{\text{loop}} \frac{\partial B}{\partial t} \cdot d\text{Area} = 0,
$$

and eq. (6) gives,

$$
\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times B \cdot d\mathbf{l} = -2\pi a B \frac{da}{dt}.
$$

while eq. (3) implies,

$$
\mathcal{EMF}_{\text{total}} = \frac{d\Phi_B}{dt} = -B \cdot \frac{d\text{Area}}{dt} = -2\pi a B \frac{da}{dt} = \mathcal{EMF}_{\text{fixed loop}} + \mathcal{EMF}_{\text{motional}}.
$$

The story is consistent, that an imaginary circular loop has an imaginary $\mathcal{EMF}$ if it expands or contracts while in a uniform magnetic field and zero electric field.

There now exist conducting rubber bands, so this example could be realized with a physical, conducting loop, in which case the $\mathcal{EMF}$ (18) would become physical, and a current would flow in the contracting or expanding rubber band.

3 Additional Comments

The discovery of magnetic induction of electric effects by Faraday and others ($\approx 1831$) was a major addition to the unification of electricity and magnetism by Ørsted and Ampère and others ($\approx 1820$), and set the stage for the unification of electromagnetism and optics by Maxwell. Comments on these rich phenomena can be made from many perspectives, and in this spirit we offer some additional remarks in this section, as well as a historical review in the Appendix. These comments are, of course, not entirely independent of the previous discussion in this note.
3.1 Differing Views of “Engineers” and “Physicists”

The “flux rule” issue is complicated in that the “flux rule” means different things to different people, who might loosely be characterized as “engineers” or “physicists”.

When considering electromagnetic effects, “physicists” tend to emphasize the “field theory” of it, as championed by Maxwell [111], in which the abstract concept of electric and magnetic fields play a major role. For “physicists”, the “flux rule” is often considered to be eq. (2) which is a relation between the abstract fields $E$ and $B$, independent of such “engineering” details as “wires”, which might or might not be in motion. In this version of the “flux rule”, there is no mention of an $\mathcal{EMF}$.

In contrast, “engineers” are more interested in “practical” effects such as the behavior of electric currents in (electrical) conductors, which may well be in motion. Such currents are said to be driven by an $\mathcal{EMF}$, which can be considered to exist even when no current is flowing.

For example, “physicists” consider Ohm’s law [66] for steady currents in conductors to be,

$$ J = \sigma E, $$  \hspace{1cm} (19)

where the current density $J$ is the electric current per unit area and $\sigma$ is the electrical conductivity. Equation (19) describes a “local” behavior in some small region of an electrical conductor. In contrast, “engineers” consider Ohm’s law (for steady currents) $I$ to be a “global” relation for a closed loop of conducting material,

$$ \mathcal{E} = I \sum R, \hspace{1cm} (20) $$

where the electric resistance $R$ of a wire of conductivity $\sigma$, length $l$ and cross sectional area $A$ is given by $R = l/\pi A \sigma$, the magnitude of the current density is $J = I/A$, and $\mathcal{E}$ is the electromotive force ($\mathcal{EMF}$). The loop of conductor might consist of segments of different conductivities, and hence different resistances, and the “law” (20) involves the total resistance $\sum R$ of these segments.

The electromotive force $\mathcal{E}$ is difficult to define in general, which leads some people to say that an electromotive force is the same thing as the “voltage” measured by a “voltmeter” [245].

In electrostatics, electromotive force $\mathcal{E}$ can be well defined for a loop, whether conducting or not, by the (“physicist”) relation,

$$ \mathcal{E} = \oint_{\text{loop}} E \cdot dl. $$ \hspace{1cm} (21)

Furthermore, in electrostatics the $\mathcal{EMF}$ can be uniquely defined for any two point $b$, with respect to point $a$, as,

$$ \mathcal{E}_a(b) = -\int_a^b E \cdot dl, $$ \hspace{1cm} (22)

independent of the path of integration from $a$ to $b$.  

7
Recall that the electric field $\mathbf{E}$ at a point $\mathbf{x}$, as introduced by W. Thomson and by Maxwell, was called the force, meaning the force that would be experienced by a unit electric charge if placed at that point (and if such placement did not perturb the sources of the field $\mathbf{E}$).

The electrostatic $\mathcal{EMF}$ between two points is also called the voltage difference, or, more loosely, just the voltage.

A difficulty for considerations of the “flux rule” in time-dependent situations is that there is no generally accepted definition of $\mathcal{EMF}$ here.

In general, the integral in eq. (22) depends on the path between $a$ and $b$ in a time-dependent situation. Of course, this integral is well defined for a particular path, so when considering an electrical circuit where a path is evident, many people suppose that eq. (21) applies (at least for that path).

In time-dependent examples, an element $d\mathbf{l}$ of a path (whether or not the path lies inside a conductor) might be associated with a velocity $\mathbf{v}$, and the force, due to the fields $\mathbf{E}$ and $\mathbf{B}$ whose sources are external to that element, on an electric charge $q$ placed on that element is the Lorentz force,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

(23)

Then, one might define the $\mathcal{EMF}$ associated with a particular (possibly time dependent) loop as,

$$\mathcal{E} = \oint_{\text{loop}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l},$$

(24)

and a path-dependent $\mathcal{EMF}$ at point $b$, with respect to a reference point $a$ can be taken as,

$$\mathcal{E}_a(b) = \int_a^b (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l},$$

(25)

### 3.2 $\mathcal{EMF}$ and Circuit Analysis

A prominent “engineering” application involving $\mathcal{EMF}$ is the analysis of time-dependent electrical circuits, most of which are entirely at rest in the lab frame. The classical analysis of Kirchhoff [93, 98]) of circuits at rest with steady currents was based on the notion that a scalar voltage could be assigned to any point in a circuit. This was taken to be the electric scalar potential $V = \int \rho d\text{Vol} / 4\pi \epsilon_0$, which had been introduced by Green (1828), p. 9 of [114] (who first used the term potential), following use of a scalar potential for gravity by Lagrange (1773), p. 348 of [12], by Laplace (1799), p. 25 of [15], by Poisson (1813), p. 390 of [19], and to “permanent” magnetism by Poisson (1824), p. 493 of [51], and (1826), p. 463 of [63].

In applications of Kirchhoff’s law to time-dependent circuits, a scalar “voltage” is assumed to exist at any point in the circuit.\(^9\) However, the concept (25) of $\mathcal{EMF}$ at point $b$,\(^9\)The use of “voltmeters” to measure the “voltage” in circuits at rest, but with time-dependent magnetic fields, can exhibit counterintuitive behavior, as reviewed in [240]. Some of this behavior is so baffling that Lewin has claimed that “Kirchhoff’s Loop Rule is for the Birds,”

https://www.youtube.com/watch?v=LzT_YZ0xCFY.

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with respect to point $a$ is ambiguous if the circuit contains multiple loops such that there is more than one path for current to flow from $a$ to $b$, so this definition does not well translate into a scalar function at each point in the circuit (even when $v = 0$). It is this author’s view that the scalar voltage ($\mathcal{E} \cdot \mathcal{F}$) to be used in analysis of time-dependent circuits is best considered to be the electric scalar potential in the Lorenz gauge, as discussed in [245].

This usage provides a tension between the notion of $\mathcal{E} \cdot \mathcal{F}$ as used in analysis of circuits at rest and that for circuits with moving parts (as is the theme of the present note). However, in most practical circuits at rest, eq. (22) provides a useful approximation to a unique scalar “voltage” at points at the junction between different circuit elements, so we can be optimistic that for moving circuits the definition (25) will also provide a useful approximation.

3.3 Digression on Moving Frames of Reference

We digress to consider a related issue that has led to ongoing confusion in discussion of moving circuits, namely the relation between the electromagnetic field, potentials and $\mathcal{E} \cdot \mathcal{F}$s in different frames of reference.

Thus far, we have only discussed these quantities in an inertial lab frame, which includes a “voltmeter” to measure the $\mathcal{E} \cdot \mathcal{F}$.

The homopolar generator involves a rotating disk, so one could consider the electrodynamics as observed in the rotating frame of the disk, although in no variant of the homopolar generator considered thus far is any measurement made in that frame. As reviewed in [237], electrodynamics in a rotating frame involves many nonintuitive features, so we consider it best to leave them out of the present discussion.

However, the relation between electrodynamics in two inertial systems that have a relative velocity is well described by special relativity (which is generally considered to be implicit in Maxwell’s theory). In particular, the electric field $E'$ observed in an inertial frame with velocity $v$ with respect to the inertial lab frame is given by,

$$E' \approx E + v \times B,$$

when $v$ is small compared to the speed $c$ of light in vacuum. In this same approximation there is no Lorentz contraction of length, such that $dl = dl'$, so we can rewrite eq. (9) as,

$$\mathcal{E} \cdot \mathcal{F}_{\text{total}} = \oint_{\text{loop}} (E + v \times B) \cdot dl \approx \oint_{\text{loop}} E' \cdot dl = \mathcal{E} \cdot \mathcal{F}'_{\text{flux rule}},$$

where $\mathcal{E} \cdot \mathcal{F}'_{\text{flux rule}}$ would have to be measured by apparatus at rest in the moving frame (which is therefore different than the apparatus that measures $\mathcal{E} \cdot \mathcal{F}$ in the lab frame).

The relation (27) is valid, but not very relevant for any practical experiment on induction in a moving circuit. Nonetheless, papers such as [166, 181, 190, 207, 211], try to suggest that eq. (27) somehow changes our understanding of magnetic induction in the lab frame. These suggestions are misguided in the view of the present author, and we do not discuss them further.\(^\text{10}\)

\(^{10}\)Faraday’s law, plus Maxwell’s version of Ampère’s law, can be used to deduce the Lorentz transformations of the fields $E$ and $B$, as discussed in Appendix C of [251].
A Appendix: Historical Review

A.1 Peregrinus

In 1269, Peregrinus [1] concluded from experiment that a (permanent) magnet has two “poles”, and that like poles of different magnets repel, while unlike poles attract. Peregrinus’ methods were later notably extended by Gilbert.

A.2 Gilbert

In 1600, Gilbert published a treatise [2], which includes qualitative notions of magnetic energy and lines of force. These seem to have been inspired by experiments (suggested by Peregrinus) in which magnetized needles (or steel filings, p. 162) were used to probe the space around larger magnets, particularly spherical magnets called terrellas. The observed directions of the needle (or filings) suggested the existence of lines of force throughout space, and the ability of the magnet to deflect the needle into alignment with them suggested (to Gilbert) that some kind of magnetic energy exists outside the magnet itself.

The figures below are from p. 122 and 247 of [2].

A.3 Descartes

In 1644, Descartes published a qualitative treatise on physics science [3]. Among its notable features is perhaps the earliest conception of momentum (mass times velocity, p. 59 ff), and that light could be due to static pressure in a kind of elastic medium that fills all space, later called the æther (pp. 94-104, part III). And, on p. 271, part IV, he presented a figure based on use of iron filings near a magnet which illustrates Gilbert’s lines of magnetic force.

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11Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166, “May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!”

12The next use of momentum may be by Wallis and Wren in 1668 [4], and by Huyghens in 1669 [5].
A.4 Gray

The phenomenon of conduction of electricity may have been first reported by Gray in 1731 [6], who described experiments on the “conveyance” and “communication” of electricity (including evidence that people conduct electricity).\textsuperscript{13}

In 1739, Desaguliers [8] (a coworker of Gray) coined the term “conductor”, as well as the term “insulator”.

A.5 Michell

In 1750, Michell published a treatise on the manufacture of magnets [9], including on p. 19 the statement that the repulsive force between like poles of two magnets fall off as $1/r^2$.\textsuperscript{14}

A.6 Priestly

In 1766, Priestly [11] deduced that the static force between electric charges varies as $1/r^2$, similar to the force of gravity except that like charges repel rather than attract.

A.7 Coulomb

In 1785, Coulomb confirmed (and made widely known) that the static force between pairs of electric charges $q_1$ and $q_2$ varies as $q_1 q_2 / r^2$ [13], and that the force between idealized magnetic poles $p_1$ and $p_2$ at the ends of long, thin magnets varies as $p_1 p_2 / r^2$ [14]. The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form.

Coulomb also noted that magnetic poles appear not to be isolatable, conjecturing (p. 306 of [116]) that the fundamental constituent of magnetism, a molécule de fluide magnétique, is a dipole, such that effective poles appear at the ends of a long, thin magnet.

\textsuperscript{13}For a historical survey of research into electricity in this era, see, for example, [191].

\textsuperscript{14}Michell is also credited with being the first to discuss what are now called black holes [10].
Coulomb’s argument is much superior to that by Gilbert, p. 247 of [2].

A.8 Poisson

In 1812, Poisson [18] extended the use (by Lagrange and Laplace) of a potential $V = -\frac{Gm}{r}$ (= energy per unit mass) for the gravitational force between of a mass $m$ and a unit test mass to the case of static electrical forces, and in 1824-26 for static magnetic forces [51, 63].

A.9 Ørsted

In 1820, Ørsted [20]-[23],[36] published decisive evidence that electric currents exert forces on permanent magnets and vice versa, indicating that electricity and magnetism are related. Ørsted’s term “electric conflict”, used in his remarks on p. 276 of [21], is a precursor of the later concept of the magnetic field:

It is sufficiently evident from the preceding facts that the electric conflict is not confined to the conductor, but dispersed pretty widely in the circumjacent space. From the preceding facts we may likewise infer that this conflict performs circles.

A.10 Biot and Savart

Among the many rapid responses in 1820 to Ørsted’s discovery was an experiment by Biot and Savart [25, 28] on the force due to an electric current $I$ in a wire on one pole, $p$, of a long, thin magnet. The interpretation given of the result was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [33, 54] with a form that can be written in vector notation (and in Gaussian units, where $c$ is the speed of light in vacuum) as

$$\mathbf{F} = p \oint \frac{I \, dl \times \hat{r}}{cr^2}, \quad (28)$$

15Following the precedent from gravity, Poisson did not appear to ask where the configuration energy, such as $q_1q_2/r$ and $p_1p_2/r$, resided, nor did he consider the quantity $-\nabla V$ to be a force field in the space outside the relevant charges or poles.

16Reports have existed since at least the 1600’s that lightning can affect ship’s compasses (see, for example, p. 179 of [113]), and an account of magnetization of iron knives by lightning was published in 1735 [7]. In 1797, von Humboldt conjectured that certain patterns of terrestrial magnetism were due to lighting strikes (see p. 13 of [233], a historical review of magnetism). A somewhat indecisive experiment involving a voltaic pile and a compass was performed by Romagnosi in 1802 [222].

Historical commentaries on Ørsted’s work include [168, 170, 255].
where \( \mathbf{r} \) is the distance from a current element \( I \, dl \) to the magnetic pole. There was no immediate interpretation of eq. (28) in terms of a magnetic field,\(^{17} \) \( \mathbf{B} = \mathbf{F}/p, \)

\[
\mathbf{B} = \oint \frac{I \, dl \times \hat{\mathbf{r}}}{c r^2},
\]

(29)

which expression is now commonly called the Biot-Savart law.

Biot and Savart did not discuss the force on an electric current, but the expression,

\[
\mathbf{F} = \oint \frac{I \, dl \times \mathbf{B}}{c},
\]

(30)

is now also often called the Biot-Savart law.\(^{18}\)

\[\text{A.11 Ampère}\]

Between 1820 and 1825 Ampère made extensive studies \([26, 27, 29, 31, 32, 34, 35, 41, 42, 45, 46, 52, 53, 59]\) of the magnetic interactions of electrical currents.\(^{19}\) Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents.\(^{20,21}\)

In 1822-1823 (pp. 21-24 of \([65]\)), Ampère examined the force between two circuits, carrying currents \( I_1 \) and \( I_2 \), and inferred that this could be written (here in vector notation) as

\[
\mathbf{F}_{on \, 1} = \oint_1 \oint_2 d^2 \mathbf{F}_{on \, 1}, \quad d^2 \mathbf{F}_{on \, 1} = I_1 I_2 [3(\hat{\mathbf{r}} \cdot dl_1)(\hat{\mathbf{r}} \cdot dl_2) - 2 d l_1 \cdot dl_2] \frac{\hat{\mathbf{r}}}{c^2 r^2} = -d^2 \mathbf{F}_{on \, 2},
\]

(31)

where \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) is the distance from a current element \( I_2 \, dl_2 \) at \( \mathbf{r}_2 \) to element \( I_1 \, dl_1 \) at \( \mathbf{r}_1 \).\(^{22}\)

The integrand \( d^2 \mathbf{F}_{on \, 1} \) of eq. (31) has the appeal that it changes sign if elements 1 and 2 are

\[\text{17Although the concept of the magnetic field is latent in discussions of magnetic force by Michell, Coulomb, Poisson and Ørsted (and many other in the years 1820-45), the first use of the term “magnetic field” seems to be due to Faraday, Art. 2147 of [97].}\]

\[\text{18The earliest description of eq. (30) as the Biot-Savart law may be in sec. 2 of [132], and in English, sec. 7-6 of [169].}\]

\[\text{19Discussion in English of Ampère’s attitudes on the relation between magnetism and mechanics is given in [179, 216, 253]. Historical surveys of 19th-century electrodynamics are given in [136, 221], and studies with emphasis on Ampère include [175, 178, 194, 197, 199, 202, 204, 205, 255]. See also sec. IIA of [224].}\]

\[\text{20See, for example, [199].}\]

\[\text{21The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampérien (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (\( e^+ e^- \) “atoms”) in the 1940’s [188, 252].}\]

\[\text{22Ampère noted that}\]

\[
dl_1 = \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1, \quad \mathbf{r} \cdot dl_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1 = \frac{1}{2} \frac{\partial r^2}{\partial l_1} \, dl_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1, \quad dl_2 = -\frac{\partial \mathbf{r}}{\partial l_2} \, dl_2, \quad \mathbf{r} \cdot dl_2 = -\frac{\partial \mathbf{r}}{\partial l_2} \, dl_2,
\]

(32)

where \( l_1 \) and \( l_2 \) measure distance along the corresponding circuits in the directions of their currents. Then,

\[
dl_1 \cdot dl_2 = -dl_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2} \, dl_2 = -\frac{\partial \mathbf{r}}{\partial l_2} (\mathbf{r} \cdot dl_1) \, dl_1 \, dl_2 = -\frac{\partial \mathbf{r}}{\partial l_1} \frac{\partial \mathbf{r}}{\partial l_2} dl_1 \, dl_2 = -\left( \frac{\partial \mathbf{r}}{\partial l_1} \frac{\partial \mathbf{r}}{\partial l_2} + r \frac{\partial^2 \mathbf{r}}{\partial l_1 \partial l_2} \right) dl_1 \, dl_2,
\]

(33)

and eq. (31) can also be written in forms closer to that used by Ampère,

\[
d^2 \mathbf{F}_{on \, 1} = I_1 I_2 dl_1 dl_2 \left[ 2r \frac{\partial^2 \mathbf{r}}{\partial l_1 \partial l_2} - \frac{\partial \mathbf{r}}{\partial l_1} \frac{\partial \mathbf{r}}{\partial l_2} \right] \frac{\hat{\mathbf{r}}}{c^2 r^2} = 2I_1 I_2 dl_1 dl_2 \frac{\partial^2 \mathbf{r}}{\partial l_1 \partial l_2} \frac{\hat{\mathbf{r}}}{c^2 \sqrt{r}} = -d^2 \mathbf{F}_{on \, 2}.
\]

(34)
interchanged, and so suggests a force law for current elements that obeys Newton’s third law. However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton’s gravitational force, and Coulomb’s law for the static force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, i.e., a pair of moving electric charges.23

Ampère performed an experiment in 1821-22 [42, 47, 73] that showed a weak effect of electromagnetic induction, which was largely disregarded at the time.24

A.12 The First Electric Motors

That electromagnetic interactions could lead to rotary motion (= electric motor) was demonstrated by Faraday in 1821 [38, 44], as in the left two figures below. This was shortly followed by the device of Barlow [43] (1822), right figure below, which has much the form of Faraday’s later disk dynamo. The rapid proliferation of electromechanical devices thereafter is illustrated, for example, in [91] (1842).25

A.13 Arago

A first step towards the inverse effect, that motion of a conductor in a magnetic field produces electrical effects, was made by Arago [55, 56, 62], who reported in 1824: the results of some experiments that he has conducted on the influence that metals and many other substances exert on a magnetic needle, which has the effect of rapidly reducing the amplitude of the oscillations without altering significantly their duration.26

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23 If we follow Ampère in defining a “current element” as being electrically neutral, which is a good (but not exact [241]) approximation for currents in electrical circuits, then a moving charge is not a “current element”, and such elements cannot exist except in closed circuits (contrary to remarks such as in [208]).

24 Reviews of this experiment include [127, 178, 202].

25 However, Faraday did little further work on magnetism until after the death of his boss, Davy, in 1829.

26 As reported in [58], The curious experiments of M. Arago described by M. Gay Lussac during his visit to London in the spring of the present year (1825), in which plates of copper and other substances set in rapid rotation beneath a magnetized needle, caused it to deviate from its direction, and finally dragged it round with them, naturally excited much attention.
We now understand that Arago observed the effect of eddy currents due to the electric field induced by a time-dependent magnetic field inside a conductor, where the energy dissipated by Joule heating damped the kinetic energy/motion of the system.

A.14 Ohm

In 1827, Ohm published a treatise [66] containing his famous law, in a form closer to

\[ J = \sigma E, \]  

(35)

where \( J \) is the electric current density and \( E \) is the electric field, both inside the rest frame of a medium with electrical conductivity \( \sigma \), than to more familiar form, \( V = IR \), where \( V \) is the potential difference across an electrical resistance \( R \) that carries electric current \( I \).

Ohm did not define a conductor so much a provide a model for it, with a flavor that electric current is related to the motion of particles. This view became characteristic of the German school in the mid 1800’s, but was not taken up by the English or French until much later.

A.15 Fechner

Fechner (1831), p. 225 of [70], may have been the first to interpret the symbol \( \mathcal{E} \) in Ohm’s law [66],

\[ \mathcal{E} = IR, \]  

(36)

as the electromotorische Kraft, \textit{i.e.}, electromotive force. Ohm and Fechner studied only steady currents in fixed loops, in which case the electromotive “force” between two points \( a \) and \( b \) is equal to the work done by electric effects when moving a unit electric charge from between two points \( a \) and \( b \),

\[ \mathcal{E}(a, b) = - \int_a^b E \cdot dl, \]  

(37)

and the sum of the \( \mathcal{E}\mathcal{M}\mathcal{F} \)s around a closed loop is zero.

A.16 Faraday

In this review of Faraday’s studies of electromagnetic induction, we seek to understand to what extent Faraday anticipated a version of the “flux rule”\textit{i}.

A.16.1 Evolution of Electricity from Magnetism

Faraday’s first report, Arts. 27-28 of [71] (1831),\textsuperscript{27} of an effect of magnetic induction was via an iron ring with two coils wound upon it, as in Fig. 1 on the next page. On connecting\textsuperscript{27}Faraday’s paper [71] was one of the first to have been reviewed by referees in the modern sense [212]. The historical context of this paper is reviewed in [178, 250].
or disconnecting one coil to/from a battery, a transient current was observed in the other coil.\footnote{It is difficult to keep tracks of the signs of the currents in such examples, and Faraday struggled with these in the draft of his paper [71], as recounted in [213].}

This was a “transformer” effect, and involved no motion of conductors relative to magnetic fields.

Faraday made no further studies of transformers, and mainly studied the induction of current by conductors moving in a magnetic field over the next 20 years.\footnote{A paper [86] from 1836 features a version of Faraday’s iron-ring, used as a voltage step-up transformer. While the may be only the second published paper to discuss transformer action, its wording suggests that this effect was already well known.} He gave no explanation of transformer action as an effect of time-dependent magnetic flux (or number of magnetic field lines).

In 1834 [81, 82, 83], Faraday returned a theme mentioned briefly in Art. 32 of [71], that a spark occurs when contact is made or broken between a battery and a wire, particularly if the wire forms a coil. In Art. 1077 of [83] he noted that the spark cannot be simply related to the mechanical momentum of the electric current, since the strength of the spark depends on the shape of the loop, for a fixed length of wire.

Art. 1108 may be as close as Faraday came to relating electromagnetic induction in fixed circuits to a variation in the magnetic flux (number of magnetic field lines) through a circuit: From the facility of transference to neighbouring wires, and from the effects generally, the inductive forces appear to be lateral, i.e. exerted in a direction perpendicular to the direction of the originating and produced currents: and they also appear to be accurately represented by the magnetic curves, and closely related to, if not identical with, magnetic forces.

However, Art. 1114 indicates that Faraday’s views on this were not very clear.

\section*{A.16.2 Extensions of the Arago Effect}

In 1831, Faraday discovered that the effect of Arago could drive an electric current in an external circuit with sliding contacts to a copper disk that rotated between the poles of a permanent magnet, as sketched below (from Art. 99, p. 381, Oct. 28, 1831, of [152]). His results from studies of variants of Arago’s experiment were reported in Arts. 81-139 of [71]. Art. 88 describes the figure below (the version on the right is from [248]).
We now embark on discussion of physical explanations of these and other related studies to be considered below, noting a comment on p. 75 of [147] that:

In the history of the development of the subject there has been a singular freedom from differences of opinion as to the experimental results, but at the same time a singular lack of agreement as to the way these results were to be interpreted.

A.16.3 Anticipation of the Biot-Savart/Lorentz Force Law

In Art. 99 of [71], Faraday gave a first interpretation of the behavior he had observed:

The relation of the current of electricity produced, to the magnetic pole, to the direction of rotation of the plate, &c. &c., may be expressed by saying, that when the unmarked (south) pole is beneath the edge of the plate, and the latter revolves horizontally, screw-fashion, the electricity which can be collected at the edge of the plate nearest to the pole is positive. As the pole of the earth may mentally be considered the unmarked pole, this relation of the rotation, the pole, and the electricity evolved, is not difficult to remember. Or if, in fig. 15 (below), the circle represent the copper disk revolving in the direction of the arrows, and a the outline of the unmarked pole placed beneath the plate, then the electricity collected at b and the neighbouring parts is positive, whilst that collected at the centre c and other parts is negative. The currents in the plate are therefore from the centre by the magnetic poles towards the circumference.

We recognize this as a version of the Biot-Savart law the force $d\mathbf{F}$ on an electric current element $Idl$ in a magnetic field $\mathbf{B}$ is given by vector relation,$^{30}$

$\mathbf{dF} = pIdl \times \hat{r}/r^2,$

where $\hat{r}$ is the unit vector from the current element to the pole. The first statement of eq. (38), the force of a magnetic pole on an electric current, may be by Maxwell (1861) in eqs. (12)-(14), p. 172 of [109] (see also sec. A.2.1 of [251]). This result was stated more crisply in Arts. 602-603 of Maxwell’s *Treatise* [115]. The earliest description of eq. (38) as the Biot-Savart law may be in sec. 2 of

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*$^{30}$Biot and Savart [25, 28, 33, 54] actually discussed the force of an electric current element on a magnetic pole $p$ in the form $d\mathbf{F} = pIdl \times \hat{r}/r^2,$ where $\hat{r}$ is the unit vector from the current element to the pole. The first statement of eq. (38), the force of a magnetic pole on an electric current, may be by Maxwell (1861) in eqs. (12)-(14), p. 172 of [109] (see also sec. A.2.1 of [251]). This result was stated more crisply in Arts. 602-603 of Maxwell’s *Treatise* [115]. The earliest description of eq. (38) as the Biot-Savart law may be in sec. 2 of
\[ d\mathbf{F} = I\, dl \times \mathbf{B}. \quad (38) \]

A.16.4 Introduction of Magnetic Curves aka Field Lines

Faraday continued his discussion of the generation of electric currents in Arts. 114-116, referring to the magnetic curves in Fig. 25 above:

*By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.*

This is Faraday’s first representation of magnetic field lines,\(^{31}\) perhaps following Gilbert (1600) \(^{2}\) and Descartes (1644) \(^{3}\). In Fig. 25, \(A\) is the north pole of the magnet, and \(B\) is the south. When, the tip of a knife blade is rotated up/out of the page, with its base remaining on the magnet, Faraday noted that an electric current flows from the tip to the base, \(i.e.,\) from \(N\) to \(P\), in agreement with eq. (38) and the verbal statement thereof in Art. 99.

A.16.5 Effect of “Cutting” the Magnetic Curves

In Art. 114 Faraday spoke of such action as involving a conductor cutting the magnetic curves, which notion has come to be regarded as a central feature of Faraday’s vision of magnetic induction of electric currents. Now, it seems better to de-emphasize the (appealing) notion of “cutting of field lines”, and rather to emphasize the interpretation of a motional EMF due to the Lorentz force on moving conduction electrons in a magnetic field.

That Faraday’s view was close to that of motional \(\mathcal{EMF}\) is illustrated in Art. 3192 of \([104]\). In the case of a rectangular wire loop, rotated about a median line that is perpendicular to a uniform, constant magnetic field, as sketched in Fig. 3 below, Faraday remarked:

*In the first 180° of revolution round the axis \(a-b\), the contrary direction in which the two parts \(c-d\) and \(e-f\) intersect the lines of magnetic force within the area \(c-e-d-f\), will cause them to conspire in producing one current, tending to run round the rectangle. The parts \(c-e\) and \(d-f\) of the rectangle may be looked upon simply as conductors; for as they do not in their motion intersect any of the lines of force, so they do not tend to produce any current.*

\(^{132}\) Faraday showed that the magnetic curves associated with a current-carrying wire are circles, Arts. 232-233 of \([72]\) and Fig. 40 on the previous page.

\(^{32}\) Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166: *May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!*

\(^{33}\) See, for example, secs. A.2-3 of \([256]\).
A delicacy is in the interpretation of the term *intersect*, as wire segments *c-e* and *d-f* do touch lines of force, and the number of lines they touch varies with time. One might well say that these moving wires do “cut” lines of force. However, the effect of the $\mathbf{v} \times \mathbf{B}$ is transverse to the wires, and does not drive any current. This was noted by Faraday, who must have had a good intuition as to the vector-cross-product character of the cause of the induced current.

Thus, Art. 3192 provided a clarification to earlier statements by Faraday, such as that in Art. 256 of [72]: *If a terminated wire move so as to cut a magnetic curve, a power is called into action which tends to urge an electric current through it*, which downplays the cross-product character of the “urge”.

### A.16.6 Studies Involving Cylindrical Symmetry

In 1832, Arts. 217-229 of [72], Faraday considered apparatus that was axially symmetric, and he returned to this theme in 1851, Arts. 3084-3122 of [103]. An advantage of this configuration, shown in Fig. 8 below, is that no eddy currents are induced, as belatedly remarked by Faraday (1854) in Art. 3339 of [108].

Our present view, derived in part from these studies by Faraday, is that the magnetic field of an axially symmetric magnet (with magnetization parallel to the symmetry axis) is the same in the lab frame for any value of the angular velocity of the magnet about its axis, *i.e.*, the magnetic field does not rotate along with a rotating, axially symmetric magnet.$^{34,35}$

In Arts. 3093-3094, Faraday showed that when a loop of wire whose plane includes the axis of the magnet, as shown in Fig. 5 above, is rotated about the magnet, or if the loop is fixed in the lab and the magnet is rotated about its axis, then no current is induced in

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$^{34}$In Art. 220 of [72], Faraday summarized the studies of Arts. 218-220 (repeated and extended in Arts. 3084-3122 of [103]) as: *Thus a singular independence of the magnetism and the (rotating) bar in which it resides is rendered evident.*

$^{35}$For a review of this theme, see [195].
the loop. The latter observation is consistent with Faraday’s interpretation that an induced current is generated when a conductor “cuts” lines of the magnetic field – if those lines do not rotate when the (axially symmetric) magnet rotates. Then, the former observation follows if one supposes, along with Faraday, that physics of the rotating loop and fixed magnet is the same as that of a fixed loop and rotating magnet.\textsuperscript{36}

In Arts. 3095-3096, Faraday considered a planar loop of wire, part of which was inside a slot in the otherwise cylindrical magnet, as sketched in Fig. 6 above. No current was observed when the loop and the magnet were rotated together. In Art. 3091, Faraday reported on a variant of this configuration in which a current was generated when the part of the loop not inside the magnet was rotated (about the line joining the points where the loop entered and exited the magnet) while the magnet (and the part of the loop inside it) remained at rest.

Art. 3097 considered external wire segments that made sliding contact with a cylindrical magnet, as in Fig. 8 below (and in Figs. 34-36 below, from Arts. 217-227 of [72], where Faraday had earlier studied this configuration), often called a homopolar or unipolar generator.\textsuperscript{37} A current was generated when the external wires were rotated about the axis of the magnet when the latter remained at rest, and also when the external wires were at rest but the magnet rotated. No current was observed when the external wires and the magnet were rotated together, consistent with the null results reported in Arts. 3092 and 3095-3096.

\textsuperscript{36}Electrodynamics in a rotating frame is actually not quite the same as that in an inertial lab frame, as reviewed in [237] and references therein. Hence, it is generally preferable to emphasize arguments in the inertial lab frame.

In case of a planar loop of wire with uniform linear density of conduction electrons, whose plane includes the axis of an axially symmetric magnetic field $B$, and which plane rotates with angular velocity $\omega$ about that axis, the effective/motional $\mathcal{E}\mathcal{M}\mathcal{F}$ in the loop associated with the Biot-Savart force (38) can be computed in the lab frame, noting that the axial symmetry of $B$ implies that it can be deduced from a vector potential $A$ which is purely azimuthal, $B = \nabla \times A$, where $A = A_\phi(r, z) \hat{\phi}$ in a cylindrical coordinate system $(r, \phi, z)$. Then, noting that $v = \omega \times r = \omega r \hat{\phi}$, such that $(v \cdot \nabla)A = 0$, and that $d\text{Area} = d\text{Area} \hat{\phi}$,

\begin{equation}
\mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}} = \oint_{\text{loop}} v \times B \cdot dl = \oint_{\text{loop}} v \times (\nabla \times A) \cdot dl = \oint_{\text{loop}} [\nabla(v \cdot A) - (v \cdot \nabla)A] \cdot dl \\
= \int_{\text{surface of loop}} \nabla \times (v \cdot A) \cdot d\text{Area} = \int_{\text{surface}} \nabla \times (r A_\phi) \cdot d\text{Area} \\
= \omega \int_{\text{surface}} \nabla \times \left( A_\phi \hat{\mathbf{r}} + r \frac{\partial A_\phi}{\partial r} \hat{\mathbf{r}} + r \frac{\partial A_\phi}{\partial z} \hat{\mathbf{z}} \right) \cdot d\text{Area} \hat{\phi} \\
= \omega \int_{\text{surface}} \left( \frac{\partial A_\phi}{\partial z} + r \frac{\partial^2 A_\phi}{\partial z \partial r} - r \frac{\partial A_\phi}{\partial r} \frac{\partial^2 A_\phi}{\partial r \partial z} \right) d\text{Area} = 0. \tag{39}
\end{equation}

However, this “Maxwellian” argument is much more intricate than Faraday’s.

\textsuperscript{37}The term unipolar is due to Weber (1841) [90].
For the case that the external part of the loop was held fixed in the lab while the magnet rotated, it is important to note that part of the circuit is inside the magnet, which part is thereby rotating. In the view that the magnetic field lines do not rotate with the magnet, one might say that the part of the circuit inside the rotating magnet is also rotating, and so does “cut” field lines, inducing a current. This leads to the view expressed in [261] and elsewhere that a homopolar generator with fixed external circuit involves a loop whose shape inside the rotating conductor is changing with time, although the path of the current in the lab frame is actually time independent. It is better to say that the conduction electrons in the rotating part of the circuit experience a $v \times B$ Lorentz force, which drives the observed current in the loop.\footnote{This view avoids the issue of whether the lines of the magnetic field are rotating or not, since the value of the vector $B$ for a cylindrical magnet is the same whether or not the magnet is rotating.}

**A.16.7 Anticipation That $\nabla \cdot B = 0$**

Art. 3117 of [103] included the important conclusion that:

... there exist lines of force within the magnet, of the same nature as those without. What is more, they are exactly equal in amount to those without. They have a relation in direction to those without; and in fact are continuations of them, absolutely unchanged in their nature, so far as the experimental test can be applied to them. Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.

We recognize this as the first statement that $\nabla \cdot B = 0$.

**A.16.8 A Cylindrical Permanent Magnet is Equivalent to a Solenoidal Electromagnet**

Art. 3120 of [103] remarked that the magnetic field of a uniformly magnetized cylinder is the same as that of a solenoid electromagnet of the same dimensions.

In 1832, Arts. 217-227 of [72], Faraday replaced the copper disk and the external magnet by a conducting, rotating magnet, whose self field acting on the “free” charges in the magnet also produced a current in the circuit, which effect is often called a homopolar generator.
A.16.9 Electric Lines of Force

The notion of electric lines of force, with tension along them and repulsion between them, appears in Art. 1297:

The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (1224.); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction (1304.).

His summary in Art. 1304 includes the statements:

I have used the phrases lines of inductive force and curved lines of force (1231. 1297. 1298. 1302.) in a general sense only, just as we speak of the lines of magnetic-force. The lines are imaginary, and the force in any part of them is of course the resultant of compound forces, every molecule being related to every other molecule in all directions by the tension and reaction of those which are contiguous.

A.16.10 The Magnetic Field

We have already noted that Faraday used the term lines of magnetic force in a footnote to Art. 114 of [71] (1831).

In 1845, Art. 2147 of [97], the term magnetic field appears for the first time in print:

The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.

A.16.11 Magnetic Power

In 1850, Art. 2806 of [102], Faraday wrote:

Any portion of space traversed by lines of magnetic power, may be taken as such a (magnetic) field, and there is probably no space without them. The condition of the field may vary in intensity of power, from place to place, either along the lines or across them...

2807. When a paramagnetic conductor, as for instance, a sphere of oxygen, is introduced into such a magnetic field, considered previously as free from matter, it will cause a concentration of the lines of force on and through it, so that the space occupied by it transmits more magnetic power than before (fig. 1). If, on the other hand, a sphere of diamagnetic matter be placed in a similar field, it will cause a divergence or opening out of the lines in the equatorial direction (fig. 2); and less magnetic power will be transmitted through the space it occupies than if it were away.
Here, one can identify Faraday’s usage of the term magnetic power with the magnetic flux \( \Phi_B = \int \mathbf{B} \cdot d\text{Area} \).

A further consequence of his interaction with Thomson appears to be that in 1852, beginning in sec. 3070 of [103], Faraday wrote about lines of force more abstractly, but without full commitment to their physical existence independent of matter. Thus, in Art. 3075 he stated:

I desire to restrict the meaning of the term line of force, so that it shall imply no more than the condition of the force in any given place, as to strength and direction; and not to include (at present) any idea of the nature of the physical cause of the phenomena...

A few sentences later he continued:

...for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the ether; for it is not at all unlikely that, if there be an ether, it should have other uses than simply the conveyance of radiations (2591. 2787.).

In Art. 3175, at the end of [103], he added:

...wherever the expression line of force is taken simply to represent the disposition of the forces, it shall have the fullness of that meaning; but that wherever it may seem to represent the idea of the physical mode of transmission of the force, it expresses in that respect the opinion to which I incline at present.

This has led many to infer that Faraday then believed in the physical existence of the lines of force even though he could not “prove” that.

Faraday’s famous notion, that induced electrical currents are associated with wires “cutting” lines of magnetic force, is presented in Art. 3104, and a version of what is now called Faraday’s law,

\[
\mathcal{E} \mathcal{M} \mathcal{F} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot \text{Area},
\]

is given verbally in Art. 3115,\(^\text{39}\)

The quantity of electricity thrown into a current is directly as the amount of curves intersected.

\(^{39}\)One should not infer from this that Faraday had an explicit notion of the magnetic field \( \mathbf{B} \) as a measure of the density of lines of magnetic force. Rather, he emphasized the total number of lines within some area (the magnetic flux) as the amount of magnetic force (Art. 3109).
In sec. 3117 Faraday noted that magnetic lines of force form closed circuits: *Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.* However, the last phrase indicates that Faraday did not have a clear view of what we call the strength of a magnetic field.

In sec. 3118 Faraday (re)affirmed that magnetic field lines do not rotate with a rotating magnet, and performs various experiments with what is now called a unipolar (or homopolar) generator to demonstrate this, which experiments are an early investigation of the relativity of rotating frames.

In 1852, Faraday also published a set of more speculative comments [105] in the Phil. Mag. (rather than Phil. Trans. Roy. Soc. London, the usual venue for his Experimental Researches), arguing more strongly for the physical reality of the lines of force.

In Art. 3258 he considered the effect of a magnet in vacuum, concluding (perhaps for the first time) that the lines of force have existence independent of a material medium: *A magnet placed in the middle of the best vacuum we can produce...acts as well upon a needle as if it were surrounded by air, water or glass; and therefore these lines exist in such a vacuum as well as where there is matter.*

Faraday used examples of magnets and iron filings in various configurations to reinforce his vision of a tension along the lines of forces, and in sec. 3295 added the insight that there is a lateral repulsion between adjacent lines, referring to Fig. 5 below.

Faraday’s last published comments on lines of force are in [108].
A.17 Henry

Henry began his studies of electromagnetism in 1827 [67], and in 1831 he demonstrated an electrical machine/motor of an unusual type [69]. In 1832 he was inspired by a brief report [75] of Faraday’s discovery of electromagnetic induction to perform a series of experiments on mutual and self induction in circuits at rest [80, 84, 85, 88, 89]. Three illustrations (from [88]) of these experiments are shown below.

In 1840, sec. 56 of [89], Henry stated:

*During the time a galvanic current is increasing in quantity in a conductor, it induces, or tends to induce, a current in an adjoining parallel conductor in an opposite direction to itself.*

In secs. 72-73 he suggested that Ohm’s law could be applied to a secondary loop of resistance $R$ in manner equivalent to $\mathcal{E}_{\text{induced}} = IR$, where the induced $\mathcal{E}\mathcal{M}\mathcal{F}$ is proportional to the rate of change of the current in the primary loop. We now call the proportionality constant the mutual inductance of the two loops. Thus, Henry gave the first sense of circuit analysis for circuits containing inductance.\(^{42}\)

A.18 Weber

The term unipolar is due to Weber (1841) [90].

A.19 W. Thomson (Lord Kelvin)

A.19.1 Force Fields

In 1842 (at age 18!), W. Thomson [92] noted an analogy between the (vector) flow of heat and the “attractive force” of electricity. At that time he was concerned with electrostatics, for which it is natural to consider the force only at the locations of charges and not in the space between them. In contrast, the flow of heat exists in the space between sources and sinks of heat, so Thomson’s analogy perhaps started him thinking about possible significance of electrical forces away from the location of electric charges.

Thomson appears to have become aware of Faraday’s work in 1845, and soon published initial comments [95] about transcribing Faraday’s notions into mathematical form. He noted the contrast between Coulomb’s action-at-distance view of electrical forces, and Faraday’s

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\(^{40}\) A replica of Henry’s motor is discussed in [238].

See also [https://www.princeton.edu/ssp/joseph-henry-project/](https://www.princeton.edu/ssp/joseph-henry-project/)

\(^{41}\) A review of Henry’s work on electromagnetic induction is given in [257].

\(^{42}\) The first detailed analysis of circuits with inductance may have been given by Maxwell in 1868 [112].

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view (reminiscent of Descarte’s) that these forces are transmitted via some kind of “action of contiguous particles of some intervening medium”, and proceeded to argue that these are what might now be called “dual” explanations of electricity. We see in this discussion the beginning of Thomson’s lifelong vision of a mechanical ether supporting electricity and magnetism.\footnote{I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand; and that is why I cannot get the electro-magnetic theory. P. 270 of [118].}

In 1846, Thomson [99], p. 63, described the electrical force due to a unit charge at the origin “exerted at the point \((x, y, z)\)” as \(r/r^3\), without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term “electric field” were used by Thomson.

He immediately continued with the example of a “point” magnetic dipole \(m\), whose scalar potential is \(\Phi = m \cdot r/r^3\), noting that the magnetic force \(-\nabla \Phi\) on a unit magnetic pole \(p\) can also be written as \(\nabla \times A\) where \(A = m \times r/r^3\) (although Thomson did not assign a symbol to the vector \(A\)). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [94] implied this result, but was not explicit about its application to Thomson’s example).

In a major paper on magnetism in 1849 [100], Thomson still did not use the term “field”, but wrote in sec. 48:

\begin{quote}
The resultant force at a point in space, void of magnetized matter, is the force that the north pole of a unit-bar (or a positive unit of imaginary magnetic matter), if placed at this point, would experience.
\end{quote}

The term “magnetic field” in the contemporary sense first appears in 1851 on p. 179 of [101], where Thomson wrote:

\begin{quote}
Definition.—Any space at every point of which there is a finite magnetic force is called “a field of magnetic force;” or, magnetic being understood, simply “a field of force;” or, sometimes, “a magnetic field.”

Definition.—A “line of force” is a line drawn through a magnetic field in the direction of the force at each point through which it passes; or a line touched at each point of itself by the direction of the magnetic force.
\end{quote}

\textbf{A.19.2 B and H}

In sec. 78 of [100], Thomson considered the magnetic-field vector \((X, Y, Z)\) that we now identify with \(B\). In considerations of the effect of the magnetic field on hypothetical magnetic poles inside small cavities in a medium with magnetization density \(M\),

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