Is Faraday’s Disk Dynamo a Flux-Rule Exception?

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1 Problem

In sec. 17.1 of [270], Feynman noted that the differential form “Faraday’s law” is,
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},
\]
and then argued that for a fixed loop one can deduce the integral form of this “law” as,
\[
\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\text{surface of loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\text{Area} = -\frac{d}{dt} (\text{magnetic flux through loop}),
\]
which is often called the “flux rule”. In sec. 17.2, he considered an experiment of Faraday from 1831, sketched below, and claimed that this is an example of an exception to the “flux rule”, where one should instead consider the motional \(\mathcal{E}\mathcal{M}\mathcal{F} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}\).

However, a recent paper [398] expressed the view that Feynman and many others are wrong about this example, and that it is well explained by the “correct” interpretation of the “flux rule”.

What’s going on here?

2 Solution

In my view, the issue is that examples of magnetic induction can be analyzed more than one way, and whenever there is more than one way of doing anything, some people become overly enthusiastic for their preferred method, and imply that other methods are incorrect.
The interpretation of Faraday’s views on magnetic induction in a closed loop,\(^1\) whose shape may or may not vary with time in a magnetic field that may or may not vary with time, as,

\[
\mathcal{EMF} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int_{\text{loop}} B \cdot d\text{Area},
\]

originated with Maxwell.\(^2\) However, this relation is often tricky to apply (due to ambiguities as to where the loop is, and to the meaning of \(\mathcal{EMF}\) and of \(d/dt\)), so many people, including Feynman, recommend splitting the calculation into two pieces,\(^3\)\(^4\)

\[
\mathcal{EMF} = \mathcal{EMF}_{\text{fixed loop}} + \mathcal{EMF}_{\text{motional}},
\]

where,

\[
\mathcal{EMF}_{\text{fixed loop}} = -\frac{\partial}{\partial t} \int_{\text{loop at time } t} B \cdot d\text{Area} = -\int_{\text{loop}} \frac{\partial B}{\partial t} \cdot d\text{Area} = \oint_{\text{loop}} E \cdot dl,
\]

using eq. (1) and Stoke’s theorem, and

\[
\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} v \times B \cdot dl,
\]

in which \(v\) is the velocity (in the inertial lab frame of the calculation) of an element \(dl\) of the loop (which is a line, but which may or may not be inside a conductor).

Maxwell showed (although apparently not very clearly) in Arts. 598-599 of [188] that eq. (3) can also be written as,

\[
\mathcal{EMF} = \oint_{\text{loop}} \left( v \times B - \frac{\partial A}{\partial t} \right) \cdot dl,
\]

\(^1\)The term “loop” or “circuit” has the implication to “mathematicians” of being closed, so the adjective “closed” is often omitted in discussions of eq. (3). Furthermore, in the “mathematical” sense, a loop or circuit need not be associated with matter, i.e., electrical conductors.

However, eq. (3) does not apply to a line that connects two different points, which is often called an “open circuit”.

\(^2\)In Art. 530 of his Treatise [188], Maxwell considered electromagnetic induction in four different configurations, and then stated in Art. 531:

The whole of these phenomena may be summed up in one law. When the number of lines of magnetic induction which pass through the secondary circuit in the positive direction is altered, an electromotive force acts round the circuit, which is measured by the rate of decrease of the magnetic induction through the circuit.

\(^3\)As will be reviewed the historical Appendix below, Faraday’s initial discovery of electromagnetic induction was between two fixed loops, but almost all of his subsequent effort involved moving conductors where the induced \(\mathcal{EMF}\) is motional. As such, the first efforts at a theory of electromagnetic induction, by Neumann [105] and Weber [109], emphasized only the motional \(\mathcal{EMF}\), and arrived at equivalents of eq. (6), but without use of the concept of the magnetic field.

\(^4\)The partition (4) may be due to Heaviside (1885) [170]. It was advocated by Steinmetz (1908), p. 1352 of [209], where he indicated that the “flux rule” should be applied only to fixed loops, and that the Biot-Savart/Lorentz force law should be considered in cases of moving circuits/conductors.
where the electromagnetic fields \( B \) and \( E \) can be related to a vector potential \( A \) and a scalar potential \( V \) according to,
\[
B = \nabla \times A, \quad E = -\nabla V - \frac{\partial A}{\partial t},
\] (8)
such that,
\[
\mathcal{E}_M \mathcal{F} = \oint (\mathbf{v} \times \mathbf{B} + \mathbf{E}) \cdot d\mathbf{l} = \mathcal{E}_M \mathcal{F}_{\text{motional}} + \mathcal{E}_M \mathcal{F}_{\text{fixed loop}},
\] (9)
since for a closed loop, \( \oint \nabla V \cdot d\mathbf{l} = 0 \) and \( \oint \mathbf{E} \cdot d\mathbf{l} = -\oint \frac{\partial A}{\partial t} \cdot d\mathbf{l} \).

Thus, the two methods, eq. (3) and (eq. (5), of computing induced \( \mathcal{E}_M \mathcal{F} \)s, give the same results (when correctly computed), and it is a matter of taste which method is preferred.

### 2.1 What Does \( \mathcal{E}_M \mathcal{F} \) Mean?

A lingering issue is that the symbol \( \mathcal{E}_M \mathcal{F} \) has not been defined independent of eqs. (3) and (5)-(6).

It is generally agreed that \( \mathcal{E}_M \mathcal{F} \) means **electromotive force**, but what does the latter mean?\(^6\)

Fechner (1831), p. 225 of [72], may have been the first to interpret the symbol \( \mathcal{E} \) in Ohm’s law [68],
\[
\mathcal{E} = IR,
\] (10)
as the **electromotorische Kraft**, *i.e.*, electromotive force. Ohm and Fechner studied only steady currents, in which case the electromotive “force” between two points \( a \) and \( b \) is equal to the work done by electric effects when moving a unit electric charge from between two points \( a \) and \( b \),\(^7\)
\[
\mathcal{E}(a, b) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}.
\] (12)
\(^5\)The first clear statement of the equivalence of eqs. (3) and (9) may be in sec. 86 of the text of Abraham (1904) [205], which credits Hertz (1890) [185] for inspiration on this. An early statement of this in the American literature was by Bewley (1929) in Appendix I of [228]. Textbook discussions include that by Becker, pp. 139-142 of [230], by Sommerfeld, pp. 286-288 of [254], by Panofsky and Phillips, pp. 160-163 of [266], and by Zangwill, sec. 14.4 of [379].

\(^6\)It is claimed on p. 3 of [51] (1823, perhaps the first textbook on electromagnetism), and also in [298], that Volta (1801) [16, 17] coined this term, but I do not find it in Volta’s papers, although he did speak of a **force motrice** in the apparatus électro-moteur (galvanic batteries) which he made famous.

\(^7\)One can consider a “field theory of batteries,” in which the \( \mathcal{E}_M \mathcal{F} \) of a battery is associated with a nonelectrostatic field \( \mathbf{E}' \), that is nonzero only inside the battery. Then, the current density \( \mathbf{J} \) inside a material (at rest) of conductivity \( \sigma \), can be written as \( \mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}') \). If the battery is, say, a cylinder of length \( L_B \) and cross sectional area \( A_B \), and is not connected to anything, \( \mathbf{J}_B = 0 \), such that \( \mathbf{E}'_B = -\mathbf{E}_B \), where the subscript \( B \) means inside the battery. Approximating the fields inside the battery as uniform, the magnitudes of the fields inside it are \( E'_B = E_B = \mathcal{E}/L_B \), where \( \mathcal{E} \) is the \( \mathcal{E}_M \mathcal{F} \) of the battery.

If the battery is connected to an external resistance \( R_R \), where the subscript \( R \) refers to the external resistor, then current \( I = \mathcal{E}/(R_R + R_B) \) flows in the circuit, where \( R_B = L_B/\sigma_B A_B \) is the internal resistance of the battery. The \( \mathcal{E}_M \mathcal{F} \) between the terminals of the battery, along the external wires/resistor has magnitude \( \mathcal{E}_R = IR_R = \mathcal{E} R_R/(R_R + R_B) \). Since the electrostatic field \( \mathbf{E} \) obeys \( \oint \mathbf{E} \cdot d\mathbf{l} = 0 \), we infer that the electric field inside the battery now is \( E_B = -\mathcal{E}_R/L_B = -\mathcal{E} R_R/L_B(R_R + R_B) \) (and points from
Although Faraday’s first report (1831) [73] of electromagnetic induction was based on time-dependent magnetic flux in a system of two coils at rest in the lab, all of his subsequent studies involved moving elements. This led Neumann, in sec. 1 of his great paper of 1845 [105] which introduced the concept of inductance, to discuss *elektromotorische Kraft* in the sense of the motional $\mathcal{E}\mathcal{M}\mathcal{F}$ (6).

Thus, early in the history of electromagnetic induction, two meanings of $\mathcal{E}\mathcal{M}\mathcal{F}$, eqs. (5) and (6), became prominent, and this tradition survives to the present day, as exemplified by Feynman’s discussion [270].

Despite Maxwell’s attempt in Arts. 598-599 of [188] to merge the two concepts of $\mathcal{E}\mathcal{M}\mathcal{F}$ into one, only after Lorentz’ clarification (1895), eq. (V), p. 21 of [195], that the electromagnetic force on charge $q$ is,

$$ F = q(E + v \times B), $$

(13)
did some people begin to accept that $\mathcal{E}\mathcal{M}\mathcal{F}$ could/should be given by eq. (9).\(^8\) There, the $\mathcal{E}\mathcal{M}\mathcal{F}$ is the work done on a unit charge as it traverse a circuit according to the force law (13).\(^9\)

Despite this clarification, analysis of magnetic induction via eqs. (4)-(6) remains more appealing to many people than use of eq. (3), whose equivalence to eq. (9) is not always evident. For many, the notion of motional $\mathcal{E}\mathcal{M}\mathcal{F}$ as eq. (6) gives a better physical understanding of examples like Faraday’s disk dynamo than does consideration of magnetic flux through a deforming circuit.

### 2.2 When Can an $\mathcal{E}\mathcal{M}\mathcal{F}$ Be Measured?

A disconcerting aspect of the concept of $\mathcal{E}\mathcal{M}\mathcal{F}$ is that although it is defined for imaginary paths/circuits, it is not measurable in practice unless a conductor exists along the path that defines the $\mathcal{E}\mathcal{M}\mathcal{F}$.

When a conductor is in place, the $\mathcal{E}\mathcal{M}\mathcal{F}$ between two points on in can (generally) be measured with a voltmeter, but the voltmeter reads “nothing” when not sampling a test conductor.\(^10\) As such, an $\mathcal{E}\mathcal{M}\mathcal{F}$ is a less physical concept than electric and magnetic fields,

$$ E' = I/\sigma_B A_B - E_B = \frac{\mathcal{E}}{R_R + R_B} \frac{R_B}{L_B} + \frac{\mathcal{E} R_R}{L_B(R_R + R_B)} = \frac{\mathcal{E}}{L_B}, $$

(11)
as was also the case when the battery was not connected to the resistor.

Whether this “battery field theory” is of much use is debatable.

\(^8\)Lorentz (1892), p. 405 of [186], still identified the last form of eq. (5) as the *force électromotrice*, and wrote the “flux law” in eq. (42), p. 416, as $\oint_{\text{loop}} E \cdot dl = -(d/dt) \oint_{\text{loop}} B \cdot d\text{Area}$. And, in 1903, eq. 27, p. 83 of [204], Lorentz again referred to this equation, but with the proviso that the loop be at rest.

The equivalence of eqs. (3) and (9) was discussed in the influential (German) text of Abraham (1904) [205].

\(^9\)However, this definitions remains in some conflict with circuit analysis, where it is assumed that a unique scalar $\mathcal{E}\mathcal{M}\mathcal{F}$/voltage can be assigned at any junction between circuit elements, as discussed in sec. 3.2 below.

\(^10\)Strictly, a voltmeter that is not connected to a test conductor is a kind of antenna, and can give a nonzero AC-voltage reading when electromagnetic radiation is present. In such cases, it is difficult to interpret the reading of the voltmeter as an $\mathcal{E}\mathcal{M}\mathcal{F}$ [364].
although these are already somewhat abstract.

The greater abstraction of an $\mathcal{E}\mathcal{M}\mathcal{F}$ may contribute to the ongoing differences of opinion as how best to think about it.

2.3 Two Examples

2.3.1 Faraday’s Disk Dynamo

We illustrate different analyses of Faraday’s disk dynamo in case of a spatially uniform magnetic field, using the figures below from [246] and [247].

Field Constant in Time

We start by considering the fixed loop ABCDOA in the figure on the right below.

There is no magnetic flux through this loop, so $\mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}} = 0$ here. On the other hand, segment DO, of length $a$, rotates with angular velocity $\omega$, so a point on this segment at distance $r$ from the axis has velocity $v = \omega r$ perpendicular to $B$, such that,

$$\mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}} = \oint_{\text{loop}} v \times B \cdot dl = \int_0^a \omega r B \, dr = -\frac{a^2 B \omega}{2}.$$  \hspace{1cm} (14)

Note that if we instead considered the loop ABCD'D'O, we again have $\mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}} = 0$, while now segment D'O rotates, and again the motional (and total) $\mathcal{E}\mathcal{M}\mathcal{F}$ is given by eq. (14).

To use the generalized “flux law”, eq. (3) for loop ABCDOA, we compare the flux through the loop at times $t$ and $t + dt$. At the latter time, segment DO has rotated by angle $d\theta = \omega \, dt$, and we must suppose that a new segment DD' is added to the loop so that it remains closed, since eq. (3) only applies to closed loops. Then, the magnetic flux through loop ABCD'D'O is $\Phi_B(t + dt) = B a^2 d\theta/2 = a^2 B \omega \, dt/2$, and total $\mathcal{E}\mathcal{M}\mathcal{F}$ is,

$$\mathcal{E}\mathcal{M}\mathcal{F}_{\text{total}} = -\frac{d\Phi_B}{dt} = -\frac{a^2 B \omega}{2}.$$ \hspace{1cm} (15)

An issue for many people concerning Faraday’s disk dynamo is the location of the “seat” of the $\mathcal{E}\mathcal{M}\mathcal{F}$ (15). In this context, the notion of the motional $\mathcal{E}\mathcal{M}\mathcal{F}$ is appealing in suggesting that the Lorentz force $e v \times B$ on moving charges in the disk identifies the “seat” of the $\mathcal{E}\mathcal{M}\mathcal{F}$ as being in the copper disk, while this is left ambiguous by the more abstract relation (3).

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11 The earliest use of the generalized flux rule (3) for Faraday’s disk dynamo via a deforming circuit that I have found is in [247] (1949), although it was implied in [249] that this had been done previously.
Practical homopolar generators are more often constructed with a “drum” geometry, as sketched below (from [378]).

Here, a copper-coated iron cylinder rotates inside a cylindrical electromagnet, and a motional $\mathcal{E}\mathcal{M}\mathcal{F} \nu B l_c d$, where $\nu$ is the azimuthal velocity of the copper cylinder, is generated along the rotating copper between sliding contacts at $c$ and $d$ (and elsewhere). One can also consider the circuit $abcdea$ as having a moving segment $bc$, and use the generalized flux law (3) to compute the $\mathcal{E}\mathcal{M}\mathcal{F}$. But again, the computation of the motional $\mathcal{E}\mathcal{M}\mathcal{F}$ is more appealing to many (particularly in the “engineering” community).

Field Sinusoidal in Time

Suppose instead that the external magnetic field for the disk dynamo is still spatially uniform, but varies with sinusoidally with time, say $B_{\text{ext}} = B_0 \cos \omega' t \hat{z}$.

This example was used in Cohn in [246] as an argument that generalized flux rule eq. (3) does not always work. It was then pointed out by Bewley [250] (who may have been the first after Maxwell to demonstrate eq. (4) in English [228]) that the time-dependent magnetic field induces eddy currents, which induce additional magnetic fields, which induce yet more electric fields, ... That is, this example cannot be well analyzed without first solving for the total electric and magnetic fields, and total currents, in the absence of the external circuit elements ABCD, which depend on the conductivity of the copper disk.

For very high conductivity, the induced magnetic field of the eddy currents cancels the external magnetic field inside (and at the flat surfaces of) the copper disk, in which case no $\mathcal{E}\mathcal{M}\mathcal{F}$ is developed in the circuit.

2.3.2 Circular Loop that Expands or Contracts Radially

This example was mentioned by Franklin (1908) on p. 1357 of [209] as part of the discussion following a presentation by Hering that the existence of a motional $\mathcal{E}\mathcal{M}\mathcal{F}$ does not always imply that the total $\mathcal{E}\mathcal{M}\mathcal{F}$ is nonzero.

We consider a circular loop of time-dependent radius $a(t)$ in a uniform magnetic field $\mathbf{B}$ perpendicular to the plane of the loop. In case that the loop is not a conductor, the electric field $\mathbf{E}$ can be zero everywhere.

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12Recall that a superconductor expels an external magnetic field [232].
13For discussion by the author of Hering’s example, see [397].
Then, a naïve use of eq. (2) would lead to the inference that
\[ \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\mathbf{B} \cdot \frac{d\text{Area}}{dt} = -2\pi aB \frac{da}{dt}, \]
which is nonzero, and hence the electric field must be nonzero also, in contradiction to our assumption. However, this is a misuse of the “flux law”.

A better approach is to consider eq. (3) or eqs. (5)-(6). Equation (5) tells us that,
\[ \mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}} = -\int_{\text{loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\text{Area} = 0, \]
and eq. (6) gives,
\[ \mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = -2\pi aB \frac{da}{dt}. \]

while eq. (3) implies,
\[ \mathcal{E}\mathcal{M}\mathcal{F}_{\text{total}} = -\frac{d\Phi_B}{dt} = -\mathbf{B} \cdot \frac{d\text{Area}}{dt} = -2\pi aB \frac{da}{dt} = \mathcal{E}\mathcal{M}\mathcal{F}_{\text{fixed loop}} + \mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}}. \]

The story is consistent, that an imaginary circular loop has an imaginary \( \mathcal{E}\mathcal{M}\mathcal{F} \) if it expands or contracts while in a uniform magnetic field and zero electric field.

There now exist conducting rubber bands, so this example could be realized with a physical, conducting loop, in which case the \( \mathcal{E}\mathcal{M}\mathcal{F} \) (19) would become physical, and a current would flow in the contracting or expanding rubber band.

3 Additional Comments

The discovery of magnetic induction of electric effects by Faraday and others (≈ 1831) was a major addition to the unification of electricity and magnetism by Ørsted and Ampère and others (≈ 1820), and set the stage for the unification of electromagnetism and optics by Maxwell. Comments on these rich phenomena can be made from many perspectives, and in this spirit we offer some additional remarks in this section, as well as a historical review in the Appendix. These comments are, of course, not entirely independent of the previous discussion in this note.

3.1 Differing Views of “Engineers” and “Physicists”

The “flux rule” issue is complicated in that the “flux rule” means different things to different people, who might loosely be characterized as “engineers” or “physicists”.

When considering electromagnetic effects, “physicists” tend to emphasize the “field theory” of it, as championed by Maxwell [137], in which the abstract concept of electric and magnetic fields play a major role. For “physicists”, the “flux rule” is often considered to be eq. (2) which is a relation between the abstract fields \( \mathbf{E} \) and \( \mathbf{B} \), independent of such
“engineering” details as “wires”, which might or might not be in motion. In this version of
the “flux rule”, there is no mention of an $\mathcal{EMF}$.

In contrast, “engineers” are more interested in “practical” effects such as the behavior
of electric currents in (electrical) conductors, which may well be in motion. Such currents
are said to be driven by an $\mathcal{EMF}$, which can be considered to exist even when no current is
flowing.

For example, “physicists” consider Ohm’s law [68] for steady currents in conductors to be,

$$ J = \sigma E, $$  \hspace{1cm} (20)

where the current density $J$ is the electric current per unit area and $\sigma$ is the electrical
conductivity. Equation (20) describes a “local” behavior in some small region of an electrical
conductor. In contrast, “engineers” consider Ohm’s law (for steady currents) $I$ to be a
“global” relation for a closed loop of conducting material,

$$ \mathcal{E} = I \sum R, $$  \hspace{1cm} (21)

where the electric resistance $R$ of a wire of conductivity $\sigma$, length $l$ and cross sectional
area $A$ is given by $R = l/\pi A \sigma$, the magnitude of the current density is $J = I/A$, and
$\mathcal{E}$ is the electromotive force ($\mathcal{EMF}$). The loop of conductor might consist of segments of
different conductivities, and hence different resistances, and the “law” (21) involves the total
resistance $\sum R$ of these segments.

The electromotive force $\mathcal{E}$ is difficult to define in general, which leads some people to say
that an electromotive force is the same thing as the “voltage” measured by a “voltmeter” [377].

In electrostatics, electromotive force $\mathcal{E}$ can be well defined for a loop, whether conducting
or not, by the (“physicist”) relation,

$$ \mathcal{E} = \oint_{\text{loop}} E \cdot dl. $$  \hspace{1cm} (22)

Furthermore, in electrostatics the $\mathcal{EMF}$ can be uniquely defined for any two point $b$, with
respect to point $a$, as,

$$ \mathcal{E}_a(b) = - \int_a^b E \cdot dl, $$  \hspace{1cm} (23)

independent of the path of integration from $a$ to $b$.

Recall that the electric field $\mathbf{E}$ at a point $\mathbf{x}$, as introduced by W. Thomson and by
Maxwell, was called the $\textbf{force}$, meaning the force that would be experienced by a unit electric
charge if placed at that point (and if such placement did not perturb the sources of the field
$\mathbf{E}$).

The electrostatic $\mathcal{EMF}$ between two points is also called the $\textbf{voltage difference}$, or, more
loosely, just the $\textbf{voltage}$.

A difficulty for considerations of the “flux rule” in time-dependent situations is that
there is no generally accepted definition of $\mathcal{EMF}$ here.
In general, the integral in eq. (23) depends on the path between \( a \) and \( b \) in a time-dependent situation. Of course, this integral is well defined for a particular path, so when considering an electrical circuit where a path is evident, many people suppose that eq. (22) applies (at least for that path).

In time-dependent examples, an element \( dl \) of a path (whether or not the path lies inside a conductor) might be associated with a velocity \( v \), and the force, due to the fields \( E \) and \( B \) whose sources are external to that element, on an electric charge \( q \) placed on that element is the Lorentz force,

\[
F = q(E + v \times B).
\]

Then, one might define the \( \mathcal{EMF} \) associated with a particular (possibly time dependent) loop as,

\[
\mathcal{E} = \oint_{\text{loop}} (E + v \times B) \cdot dl,
\]

and a path-dependent \( \mathcal{EMF} \) at point \( b \), with respect to a reference point \( a \) can be taken as,

\[
\mathcal{E}_a(b) = \int_a^b (E + v \times B) \cdot dl,
\]

### 3.2 \( \mathcal{EMF} \) and Circuit Analysis

A prominent “engineering” application involving \( \mathcal{EMF} \) is the analysis of time-dependent electrical circuits, most of which are entirely at rest in the lab frame. The classical analysis of Kirchhoff [104, 110]) of circuits at rest with steady currents was based on the notion that a scalar voltage could be assigned to any point in a circuit. This was taken to be the electric scalar potential \( V = \int \rho d\text{Vol}/4\pi\varepsilon_0 \), which had been introduced by Green (1828), p. 9 of [144] (who first used the term potential), following use of a scalar potential for gravity by Lagrange (1773), p. 348 of [12], by Laplace (1799), p. 25 of [15], by Poisson (1813), p. 390 of [19], and to “permanent” magnetism by Poisson (1824), p. 493 of [52], and (1826), p. 463 of [66].

In applications of Kirchhoff’s law to time-dependent circuits, a scalar “voltage” is assumed to exist at any point in the circuit.\(^{14}\) However, the concept (26) of \( \mathcal{EMF} \) at point \( b \), with respect to point \( a \) is ambiguous if the circuit contains multiple loops such that there is more than one path for current to flow from \( a \) to \( b \), so this definition does not well translate into a scalar function at each point in the circuit (even when \( v = 0 \)). It is this author’s view that the scalar voltage (\( \mathcal{EMF} \)) to be used in analysis of time-dependent circuits is best considered to be the electric scalar potential in the Lorenz gauge, as discussed in [377].

This usage provides a tension between the notion of \( \mathcal{EMF} \) as used in analysis of circuits at rest and that for circuits with moving parts (as is the theme of the present note). However,

\(^{14}\)The use of “voltmeters” to measure the “voltage” in circuits at rest, but with time-dependent magnetic fields, can exhibit counterintuitive behavior, as reviewed in [371]. Some of this behavior is so baffling that Lewin has claimed that “Kirchhoff’s Loop Rule is for the Birds,” https://www.youtube.com/watch?v=LzT_YZ0xCFY.
in most practical circuits at rest, eq. (23) provides a useful approximation to a unique scalar “voltage” at points at the junction between different circuit elements, so we can be optimistic that for moving circuits the definition (26) will also provide a useful approximation.

### 3.3 Digression on Moving Frames of Reference

We digress to consider a related issue that has led to ongoing confusion in discussion of moving circuits, namely the relation between the electromagnetic field, potentials and $\mathcal{EMF}$s in different frames of reference.

Thus far, we have only discussed these quantities in an inertial lab frame, which includes a “voltmeter” to measure the $\mathcal{EMF}$.

The homopolar generator involves a rotating disk, so one could consider the electrodynamics as observed in the rotating frame of the disk, although in no variant of the homopolar generator considered thus far is any measurement made in that frame. As reviewed in [367], electrodynamics in a rotating frame involves many nonintuitive features, so we consider it best to leave them out of the present discussion.\(^{15}\)

However, the relation between electrodynamics in two inertial systems that have a relative velocity is well described by special relativity (which is generally considered to be implicit in Maxwell’s theory). In particular, the electric field $E'$ observed in an inertial frame with velocity $v$ with respect to the inertial lab frame is given by,

$$E' \approx E + v \times B,$$

when $v$ is small compared to the speed $c$ of light in vacuum. In this same approximation there is no Lorentz contraction of length, such that $dl = dl'$, so we can rewrite eq. (9) as,

$$\mathcal{EMF}_{\text{total}} = \oint_{\text{loop}} (E + v \times B) \cdot dl \approx \oint_{\text{loop}} E' \cdot dl = \mathcal{EMF}'_{\text{flux rule}},$$

where $\mathcal{EMF}'_{\text{flux rule}}$ would have to be measured by apparatus at rest in the moving frame (which is therefore different than the apparatus that measures $\mathcal{EMF}$ in the lab frame).

The relation (28) is valid, but not very relevant for any practical experiment on induction in a moving circuit. Nonetheless, papers such as [253, 276, 295, 318, 324], try to suggest that eq. (28) somehow changes our understanding of magnetic induction in the lab frame. These suggestions are misguided in the view of the present author, and we do not discuss them further.\(^{16}\)

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\(^{15}\)In several papers [233, 236, 240, 243], Kron argued that if one chooses to analyze problems of rotating electrical machinery in the rotating frame, then one must employ much of the mathematical apparatus of general relativity, including use of the Christoffel symbols. For additional comments by the present author on this theme, see [382].

\(^{16}\)Faraday’s law, plus Maxwell’s version of Ampère’s law, can be used to deduce the Lorentz transformations of the fields $E$ and $B$, as discussed in Appendix C of [387].
A Appendix: Historical Review

A.1 Peregrinus

In 1269, Peregrinus [1] concluded from experiment that a (permanent) magnet has two “poles”, and that like poles of different magnets repel, while unlike poles attract. Peregrinus’ methods were later notably extended by Gilbert.

A.2 Gilbert

In 1600, Gilbert published a treatise [2], which includes qualitative notions of magnetic energy and lines of force. These seem to have been inspired by experiments (suggested by Peregrinus) in which magnetized needles (or steel filings, p. 162) were used to probe the space around larger magnets, particularly spherical magnets called terrellas. The observed directions of the needle (or filings) suggested the existence of lines of force throughout space, and the ability of the magnet to deflect the needle into alignment with them suggested (to Gilbert) that some kind of magnetic energy exists outside the magnet itself.

The figures below are from p. 122 and 247 of [2].

A.3 Descartes

In 1644, Descartes published a qualitative treatise on physics science [3]. Among its notable features is perhaps the earliest conception of momentum (mass times velocity, p. 59 ff), and that light could be due to static pressure in a kind of elastic medium that fills all space, later called the æther (pp. 94-104, part III). And, on p. 271, part IV, he presented a figure based on use of iron filings near a magnet which illustrates Gilbert’s lines of magnetic force.

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17Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166, “May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!”

18The next use of momentum may be by Wallis and Wren in 1668 [4], and by Huyghens in 1669 [5].
A.4 Gray

The phenomenon of conduction of electricity may have been first reported by Gray in 1731 [6], who described experiments on the “conveyance” and “communication” of electricity (including evidence that people conduct electricity).\(^{19}\)

In 1739, Desaguliers [8] (a coworker of Gray) coined the term “conductor”, as well as the term “insulator”.

A.5 Michell

In 1750, Michell published a treatise on the manufacture of magnets [9], including on p. 19 the statement that the repulsive force between like poles of two magnets fall off as \(1/r^2\).\(^{20}\)

A.6 Priestly

In 1766, Priestly [11] deduced that the static force between electric charges varies as \(1/r^2\), similar to the force of gravity except that like charges repel rather than attract.

A.7 Coulomb

In 1785, Coulomb confirmed (and made widely known) that the static force between pairs of electric charges \(q_1\) and \(q_2\) varies as \(q_1q_2/r^2\) [13], and that the force between idealized magnetic poles \(p_1\) and \(p_2\) at the ends of long, thin magnets varies as \(p_1p_2/r^2\) [14]. The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form.

Coulomb also noted that magnetic poles appear not to be isolatable, conjecturing (p. 306 of [165]) that the fundamental constituent of magnetism, a *molécule de fluide magnétique*, is a dipole, such that effective poles appear at the ends of a long, thin magnet.

\(^{19}\)For a historical survey of research into electricity in this era, see, for example, [296].

\(^{20}\)Michell is also credited with being the first to discuss what are now called black holes [10].
Coulomb’s argument is much superior to that by Gilbert, p. 247 of [2].

A.8 Poisson

In 1812, Poisson [18] extended the use (by Lagrange and Laplace) of a potential $V = -Gm/r$ (= energy per unit mass) for the gravitational force between of a mass $m$ and a unit test mass to the case of static electrical forces, and in 1824-26 for static magnetic forces [52, 66].

A.9 Ørsted

In 1820, Ørsted [20]-[23],[36] published decisive evidence that electric currents exert forces on permanent magnets and *vice versa*, indicating that electricity and magnetism are related. Ørsted’s term “electric conflict”, used in his remarks on p. 276 of [21], is a precursor of the later concept of the magnetic field:

*It is sufficiently evident from the preceding facts that the electric conflict is not confined to the conductor, but dispersed pretty widely in the circumjacent space. From the preceding facts we may likewise infer that this conflict performs circles.*

A.10 Biot and Savart

Among the many rapid responses in 1820 to Ørsted’s discovery was an experiment by Biot and Savart [25, 28] on the force due to an electric current $I$ in a wire on one pole, $p$, of a long, thin magnet. The interpretation given of the result was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [33, 55] with a form that can be written in vector notation (and in Gaussian units, where $c$ is the speed of light in vacuum) as

$$\mathbf{F} = p \oint \frac{I \, dl \times \hat{r}}{cr^2},$$

(29)

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21Following the precedent from gravity, Poisson did not appear to ask where the configuration energy, such as $q_1q_2/r$ and $p_1p_2/r$, resided, nor did he consider the quantity $-\nabla V$ to be a force field in the space outside the relevant charges or poles.

22Reports have existed since at least the 1600’s that lightning can affect ship’s compasses (see, for example, p. 179 of [143]), and an account of magnetization of iron knives by lightning was published in 1735 [7]. In 1797, von Humboldt conjectured that certain patterns of terrestrial magnetism were due to lighting strikes (see p. 13 of [362], a historical review of magnetism). A somewhat indecisive experiment involving a voltaic pile and a compass was performed by Romagnosi in 1802 [343].

Historical commentaries on Ørsted’s work include [255, 258, 391].
where \( \mathbf{r} \) is the distance from a current element \( I \, dl \) to the magnetic pole. There was no immediate interpretation of eq. (29) in terms of a magnetic field, \(^{23}\) \( \mathbf{B} = \mathbf{F}/p, \)

\[
\mathbf{B} = \oint \frac{I \, dl \times \hat{\mathbf{r}}}{c r^2},
\]

which expression is now commonly called the Biot-Savart law.

Biot and Savart did not discuss the force on an electric current, but the expression,

\[
\mathbf{F} = \oint \frac{I \, dl \times \mathbf{B}}{c},
\]

is now also often called the Biot-Savart law. \(^{24}\)

### A.11 Ampère

Between 1820 and 1825 Ampère made extensive studies \([26, 27, 29, 31, 32, 34, 35, 41, 42, 43, 46, 47, 53, 54, 60, 62, 67]\) of the magnetic interactions of electrical currents. \(^{25}\) Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents. \(^{26, 27}\)

In 1822-1823 (pp. 21-24 of [67]), Ampère examined the force between two circuits, carrying currents \( I_1 \) and \( I_2 \), and inferred that this could be written (here in vector notation) as

\[
\mathbf{F}_{\text{on 1}} = \oint_1 \oint_2 d^2 \mathbf{F}_{\text{on 1}}, \quad d^2 \mathbf{F}_{\text{on 1}} = I_1 I_2 [3(\mathbf{r} \cdot d\mathbf{l}_1)(\mathbf{r} \cdot d\mathbf{l}_2) - 2 d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\mathbf{r}}{c^2 r^2} = -d^2 \mathbf{F}_{\text{on 2}}, (32)
\]

where \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) is the distance from a current element \( I_2 \, dl_2 \) at \( \mathbf{r}_2 \) to element \( I_1 \, dl_1 \) at \( \mathbf{r}_1 \). \(^{28}\)

The integrand \( d^2 \mathbf{F}_{\text{on 1}} \) of eq. (32) has the appeal that it changes sign if elements 1 and 2 are

\[23\]Although the concept of the magnetic field is latent in discussions of magnetic force by Michell, Coulomb, Poisson and Örsted (and many other in the years 1820-45), the first use of the term “magnetic field” seems to be due to Faraday, Art. 2147 of [108].

\[24\]The earliest description of eq. (31) as the Biot-Savart law may be in sec. 2 of [207], and in English, sec. 7-6 of [257].

\[25\]Discussion in English of Ampère’s attitudes on the relation between magnetism and mechanics is given in [272, 333, 389]. Historical surveys of 19th-century electrodynamics are given in [213, 345], and studies with emphasis on Ampère include [265, 271, 300, 304, 306, 310, 313, 314, 391]. See also sec. IIA of [350].

\[26\]See, for example, [306].

\[27\]The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampérien (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (\(e^+e^-\) “atoms”) in the 1940’s [293, 388].

\[28\]Ampère noted that

\[
dl_1 = \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1, \quad \mathbf{r} \cdot d\mathbf{l}_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1 = \frac{1}{2} \frac{\partial r^2}{\partial l_1} \, dl_1 = r \frac{\partial r}{\partial l_1} \, dl_1, \quad d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2} \, dl_2, \quad \mathbf{r} \cdot d\mathbf{l}_2 = -r \frac{\partial r}{\partial l_2} \, dl_2, \quad (33)
\]

where \( l_1 \) and \( l_2 \) measure distance along the corresponding circuits in the directions of their currents. Then,

\[
dl_1 \cdot d\mathbf{l}_2 = -dl_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2} \, dl_2 = -\frac{\partial \mathbf{r}}{\partial l_2} (\mathbf{r} \cdot d\mathbf{l}_1) \, dl_1 \, dl_2 = -\frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \, dl_1 \, dl_2 = -\left( \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} + r \frac{\partial^2 r}{\partial l_1 \partial l_2} \right) \, dl_1 \, dl_2, \quad (34)
\]

and eq. (32) can also be written in forms closer to that used by Ampère,

\[
d^2 \mathbf{F}_{\text{on 1}} = I_1 I_2 d\mathbf{l}_1 d\mathbf{l}_2 \left[ 2r \frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \right] \frac{\mathbf{r}}{c^2 r^2} = 2I_1 I_2 d\mathbf{l}_1 d\mathbf{l}_2 \frac{\partial^2 r}{\partial l_1 \partial l_2} \frac{\mathbf{r}}{c^2 r^2} = -d^2 \mathbf{F}_{\text{on 2}}. \quad (35)
\]
interchanged, and so suggests a force law for current elements that obeys Newton’s third law. However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton’s gravitational force, and Coulomb’s law for the static force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, i.e., a pair of moving electric charges.²⁹

Around 1825, Ampère noted, p. 214 of [62], p. 29 of [67], p. 366 of the English translation in [389], that for a closed circuits eq. (32) can be rewritten as,

\[
\mathbf{F}_{\text{on} \ 1} = \oint_{\mathcal{L}_1} \oint_{\mathcal{L}_2} I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{c^2 r^2},
\]

(36)
in vector notation (which, of course, Ampère did not use). Ampère made very little comment on this result,³⁰ and certainly did not factorize it into the forms now related to the Biot-Savart law(s) (30)-(31).

Ampère performed an experiment in 1821-22 [42, 48, 75] that showed a weak effect of electromagnetic induction, which was largely disregarded at the time.³¹

A.12 The First Electric Motors

That electromagnetic interactions could led to rotary motion (= electric motor) was demonstrated by Faraday in 1821 [38, 45], as in the left two figures below. This was shortly followed by the device of Barlow [44] (1822), right figure below, which has much the form of Faraday’s later disk dynamo. The rapid proliferation of electromechanical devices thereafter is illustrated, for example, in [99] (1842).³²

²⁹If we follow Ampère in defining a “current element” as being electrically neutral, which is a good (but not exact [373]) approximation for currents in electrical circuits, then a moving charge is not a “current element”, and such elements cannot exist except in closed circuits (contrary to remarks such as in [319].

³⁰As a consequence, the form (36) is generally attributed to Grassmann [102], as in [350], for example.

³¹Reviews of this experiment include [196, 271, 309, 310].

³²However, Faraday did little further work on magnetism until after the death of his boss, Davy, in 1829.
A.13 Arago

A first step towards the inverse effect, that motion of a conductor in a magnetic field produces electrical effects, was made by Arago [56, 57, 65], who reported in 1824: the results of some experiments that he has conducted on the influence that metals and many other substances exert on a magnetic needle, which has the effect of rapidly reducing the amplitude of the oscillations without altering significantly their duration.\footnote{As reported by Babbage and Herschel (1825) [59]: The curious experiments of M. Arago described by M. Gay Lussac during his visit to London in the spring of the present year (1825), in which plates of copper and other substances set in rapid rotation beneath a magnetized needle, caused it to deviate from its direction, and finally dragged it round with them, naturally excited much attention. They extended Arago’s study by showing that if a magnet is moved with respect to a copper disk initially at rest, the disk can be set in motion.}

We now understand that Arago observed the effect of eddy currents due to the electric field induced by a time-dependent magnetic field inside a conductor, where the energy dissipated by Joule heating damped the kinetic energy/motion of the system. A step towards this understanding was made by Christie (1826) [64], who noted that if the copper disk is cut into two or more concentric rings, the effect of a magnet on its rotation is greatly reduced.

A.14 Ohm

In 1827, Ohm published a treatise [68] containing his famous law, in a form closer to

$$J = \sigma E,$$  \hspace{1cm} (37)

where $J$ is the electric current density and $E$ is the electric field, both inside the rest frame of a medium with electrical conductivity $\sigma$, than to more familiar form, $V = IR$, where $V$ is the potential difference across an electrical resistance $R$ that carries electric current $I$.

Ohm did not define a conductor so much a provide a model for it, with a flavor that electric current is related to the motion of particles. This view became characteristic of the German school in the mid 1800’s, but was not taken up by the English or French until much later.

A.15 Fechner

Fechner (1831), p. 225 of [72], may have been the first to interpret the symbol $E$ in Ohm’s law [68],

$$E = IR,$$ \hspace{1cm} (38)

as the electromotorische Kraft, i.e., electromotive force. Ohm and Fechner studied only steady currents in fixed loops, in which case the electromotive “force” between two points $a$ and $b$ is equal to the work done by electric effects when moving a unit electric charge from between two points $a$ and $b$,

$$E(a, b) = -\int_a^b E \cdot dl,$$ \hspace{1cm} (39)

and the sum of the $E\mathcal{M}\mathcal{F}$s around a closed loop is zero.
A.16 Faraday

In this review of Faraday’s studies of electromagnetic induction, we seek to understand to what extent Faraday anticipated a version of the “flux rule”.

A.16.1 Evolution of Electricity from Magnetism

Faraday’s first report, Arts. 27-28 of [73] (1831), of an effect of magnetic induction was via an iron ring with two coils wound upon it, as in Fig. 1 on the next page. On connecting or disconnecting one coil to/from a battery, a transient current was observed in the other coil.

This was a “transformer” effect, and involved no motion of conductors relative to magnetic fields.

Faraday made no further studies of transformers, and mainly studied the induction of current by conductors moving in a magnetic field over the next 20 years. He gave no explanation of transformer action as an effect of time-dependent magnetic flux (or number of magnetic field lines).

In 1834 [85, 86, 87], Faraday returned a theme mentioned briefly in Art. 32 of [73], that a spark occurs when contact is made or broken between a battery and a wire, particularly if the wire forms a coil. In Art. 1077 of [87] he noted that the spark cannot be simply related to the mechanical momentum of the electric current, since the strength of the spark depends on the shape of the loop, for a fixed length of wire.

Art. 1108 may be as close as Faraday came to relating electromagnetic induction in fixed circuits to a variation in the magnetic flux (number of magnetic field lines) through a circuit: From the facility of transference to neighbouring wires, and from the effects generally, the inductive forces appear to be lateral, i.e. exerted in a direction perpendicular to the direction of the originating and produced currents: and they also appear to be accurately represented by the magnetic curves, and closely related to, if not identical with, magnetic forces.

However, Art. 1114 indicates that Faraday’s views on this were not very clear.

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34 Faraday’s paper [73] was one of the first to have been reviewed by referees in the modern sense [326]. The historical context of this paper is reviewed in [271, 386].

35 It is difficult to keep tracks of the signs of the currents in such examples, and Faraday struggled with these in the draft of his paper [73], as recounted in [328].

36 A paper [91] from 1836 features a version of Faraday’s iron-ring, used as a voltage step-up transformer. While the may be only the second published paper to discuss transformer action, its wording suggests that this effect was already well known.
A.16.2 Extensions of the Arago Effect

In 1831, Faraday discovered that the effect of Arago could drive an electric current in an external circuit with sliding contacts to a copper disk that rotated between the poles of a permanent magnet, as sketched below (from Art. 99, p. 381, Oct. 28, 1831, of [231]). His results from studies of variants of Arago’s experiment were reported in Arts. 81-139 of [73]. Art. 88 describes the figure below (the version on the right is from [383]).

We now embark on discussion of physical explanations of these and other related studies to be considered below, noting a comment on p. 75 of [226] that:
In the history of the development of the subject there has been a singular freedom from differences of opinion as to the experimental results, but at the same time a singular lack of agreement as to the way these results were to be interpreted.

A.16.3 Eddy Currents

In Art. 131 of [73], Faraday stated:
Future investigations will no doubt ... decide the point whether the retarding or dragging action spoken of (by Arago) is always simultaneous with electric currents, i.e., eddy currents. This statement might have been clearer if Art. 131 had been followed by Art. 123, where it was stated:
These currents are discharged or return in the parts of the plate on each side of and more distant from the place of the pole, where, of course, the magnetic induction is weaker.
One can reasonably infer that Faraday had a vision of eddy currents as shown below, although Faraday himself never made such a sketch.
A.16.4 Anticipation of the Biot-Savart/Lorentz Force Law

In Art. 99 of [73], Faraday gave a first interpretation of the behavior he had observed using a galvanometer connected to a rotating copper disk via sliding contacts:

*The relation of the current of electricity produced, to the magnetic pole, to the direction of rotation of the plate, &c. &c., may be expressed by saying, that when the unmarked (south) pole is beneath the edge of the plate, and the latter revolves horizontally, screw-fashion, the electricity which can be collected at the edge of the plate nearest to the pole is positive. As the pole of the earth may mentally be considered the unmarked pole, this relation of the rotation, the pole, and the electricity evolved, is not difficult to remember. Or if, in fig. 15 (below), the circle represent the copper disk revolving in the direction of the arrows, and a the outline of the unmarked pole placed beneath the plate, then the electricity collected at b and the neighbouring parts is positive, whilst that collected at the centre c and other parts is negative. The currents in the plate are therefore from the centre by the magnetic poles towards the circumference.*

We recognize this as a version of the Biot-Savart law the force \( d\mathbf{F} \) on an electric current element \( I\,dl \) in a magnetic field \( \mathbf{B} \) is given by vector relation,\(^{37}\)

\[
d\mathbf{F} = I\,dl \times \mathbf{B}. \tag{40}
\]

A.16.5 Magnetic Curves aka Field Lines

Faraday continued his discussion of the generation of electric currents in Arts. 114-116, referring to the magnetic curves in Fig. 25 above:

*By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.*

This is Faraday’s first representation of magnetic field lines,\(^{38}\) perhaps following Gilbert

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\(^{37}\)Biot and Savart [25, 28, 33, 55] actually discussed the force of an electric current element on a magnetic pole \( p \) in the form \( d\mathbf{F} = pI\,dl \times \hat{r}/r^2 \), where \( \hat{r} \) is the unit vector from the current element to the pole. The first statement of eq. (40), the force of a magnetic pole on an electric current, may be by Maxwell (1861) in eqs. (12)-(14), p. 172 of [132] (see also sec. A.2.1 of [387]). This result was stated more crisply in Arts. 602-603 of Maxwell’s *Treatise* [188]. The earliest description of eq. (40) as the Biot-Savart law may be in sec. 2 of [207].

\(^{38}\)Faraday showed that the magnetic curves associated with a current-carrying wire are circles, Arts. 232-233 of [74] and Fig. 40 on the previous page.
In Fig. 25, \( A \) is the north pole of the magnet, and \( B \) is the south. When, the tip of a knife blade is rotated up/out of the page, with its base remaining on the magnet, Faraday noted that an electric current flows from the tip to the base, \( i.e., \) from \( N \) to \( P \), in agreement with eq. (40) and the verbal statement thereof in Art. 99.

**A.16.6 Effect of “Cutting” the Magnetic Curves**

In Art. 114 Faraday spoke of such action as involving a conductor cutting the magnetic curves, which notion has come to be regarded as a central feature of Faraday’s vision of magnetic induction of electric currents. Now, it seems better to de-emphasize the (appealing) notion of “cutting of field lines”, and rather to emphasize the interpretation of a motional \( E\mathcal{M}F \) due to the Lorentz force on moving conduction electrons in a magnetic field.

That Faraday’s view was close to that of motional \( E\mathcal{M}F \) is illustrated in Art. 3192 of [118]. In the case of a rectangular wire loop, rotated about a median line that is perpendicular to a uniform, constant magnetic field, as sketched in Fig. 3 below, Faraday remarked:

*In the first 180° of revolution round the axis \( a-b \), the contrary direction in which the two parts \( c-d \) and \( e-f \) intersect the lines of magnetic force within the area \( c-e-d-f \), will cause them to conspire in producing one current, tending to run round the rectangle. The parts \( c-e \) and \( d-f \) of the rectangle may be looked upon simply as conductors; for as they do not in their motion intersect any of the lines of force, so they do not tend to produce any current.*

A delicacy is in the interpretation of the term *intersect*, as wire segments \( c-e \) and \( d-f \) do touch lines of force, and the number of lines they touch varies with time. One might well say that these moving wires do “cut” lines of force. However, the effect of the \( \mathbf{v} \times \mathbf{B} \) is transverse to the wires, and does not drive any current. This was noted by Faraday, who must have had a good intuition as to the vector-cross-product character of the cause of the induced current.

Thus, Art. 3192 provided a clarification to earlier statements by Faraday, such as that in Art. 256 of [74]: *If a terminated wire move so as to cut a magnetic curve, a power is called into action which tends to urge an electric current through it*, which downplays the cross-product character of the “urge”.

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39 Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166: *May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!*

40 See, for example, secs. A.2-3 of [393].
A.16.7 Studies Involving Cylindrical Symmetry

In 1832, Arts. 217-229 of [74], Faraday considered apparatus that was axially symmetric, and he returned to this theme in 1851, Arts. 3084-3122 of [117]. An advantage of this configuration, shown in Fig. 8 below, is that no eddy currents are induced, as belatedly remarked by Faraday (1854) in Art. 3339 of [126].

Our present view, derived in part from these studies by Faraday, is that the magnetic field of an axially symmetric magnet (with magnetization parallel to the symmetry axis) is the same in the lab frame for any value of the angular velocity of the magnet about its axis, i.e., the magnetic field does not rotate along with a rotating, axially symmetric magnet.\textsuperscript{41,42}

In Arts. 3093-3094, Faraday showed that when a loop of wire whose plane includes the axis of the magnet, as shown in Fig. 5 above, is rotated about the magnet, or if the loop is fixed in the lab and the magnet is rotated about its axis, then no current is induced in the loop. The latter observation is consistent with Faraday’s interpretation that an induced current is generated when a conductor “cuts” lines of the magnetic field – if those lines do not rotate when the (axially symmetric) magnet rotates. Then, the former observation follows if one supposes, along with Faraday, that physics of a the rotating loop and fixed magnet is

\textsuperscript{41}In Art. 220 of [74], Faraday summarized the studies of Arts. 218-220 (repeated and extended in Arts. 3084-3122 of [117]) as:

Thus a singular independence of the magnetism and the (rotating) bar in which it resides is rendered evident.

\textsuperscript{42}For a review of this theme, see [302].
the same as that of a fixed loop and rotating magnet.\textsuperscript{43}

In Arts. 3095-3096, Faraday considered a planar loop of wire, part of which was inside a slot in the otherwise cylindrical magnet, as sketched in Fig. 6 above. No current was observed when the loop and the magnet were rotated together. In Art. 3091, Faraday reported on a variant of this configuration in which a current was generated when the part of the loop not inside the magnet was rotated (about the line joining the points where the loop entered and exited the magnet) while the magnet (and the part of the loop inside it) remained at rest.

Art. 3097 considered external wire segments that made sliding contact with a cylindrical magnet, as in Fig. 8 below (and in Figs. 34-36 below, from Arts. 217-227 of [74], where Faraday had earlier studied this configuration), often called a homopolar or unipolar generator.\textsuperscript{44} A current was generated when the external wires were rotated about the axis of the magnet when the latter remained at rest, and also when the external wires were at rest but the magnet rotated. No current was observed when the external wires and the magnet were rotated together, consistent with the null results reported in Arts. 3092 and 3095-3096.

For the case that the external part of the loop was held fixed in the lab while the magnet rotated, it is important to note that part of the circuit is inside the magnet, which part is thereby rotating. In the view that the magnetic field lines do not rotate with the magnet,\textsuperscript{43}Electrodynamics in a rotating frame is actually not quite the same as that in an inertial lab frame, as reviewed in [367] and references therein. Hence, it is generally preferable to emphasize arguments in the inertial lab frame.

In case of a planar loop of wire with uniform linear density of conduction electrons, whose plane includes the axis of an axially symmetric magnetic field $\mathbf{B}$, and which plane rotates with angular velocity $\omega$ about that axis, the effective/motional $\mathcal{E}\mathcal{M}\mathcal{F}$ in the loop associated with the Biot-Savart force (40) can be computed in the lab frame, noting that the axial symmetry of $\mathbf{B}$ implies that it can be deduced from a vector potential $\mathbf{A}$ which is purely azimuthal, $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A} = A_\phi(r, z) \hat{\phi}$ in a cylindrical coordinate system $(r, \phi, z)$. Then, noting that $\mathbf{v} = \omega \times \mathbf{r} = \omega r \hat{\phi}$, such that $(\mathbf{v} \cdot \nabla)\mathbf{A} = 0$, and that $d\text{Area} = d\text{Area} \hat{\phi}$,

\[
\mathcal{E}\mathcal{M}\mathcal{F}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \oint_{\text{loop}} \mathbf{v} \times (\nabla \times \mathbf{A}) \cdot d\mathbf{l} = \oint_{\text{loop}} [\nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}] \cdot d\mathbf{l} \\
= \int_{\text{surface}} \nabla \times \nabla(\mathbf{v} \cdot \mathbf{A}) \cdot d\text{Area} = \omega \int_{\text{surface}} \nabla \times (\mathbf{v} \times (\hat{\mathbf{r}} A_\phi \hat{\mathbf{r}} + r \frac{\partial A_\phi}{\partial r} \hat{\mathbf{r}} + r \frac{\partial A_\phi}{\partial z} \hat{\mathbf{z}})) \cdot d\text{Area} \hat{\phi} \\
= \omega \int_{\text{surface}} \left( \frac{\partial A_\phi}{\partial z} + r \frac{\partial^2 A_\phi}{\partial z \partial r} - r \frac{\partial A_\phi}{\partial r} - r \frac{\partial^2 A_\phi}{\partial r \partial z} \right) d\text{Area} = 0.
\]

However, this “Maxwellian” argument is much more intricate than Faraday’s.

\textsuperscript{44}The term unipolar is due to Weber (1839) [98].
one might say that the part of the circuit inside the rotating magnet is also rotating, and so does “cut” field lines, inducing a current. This leads to the view expressed in [398] and elsewhere that a homopolar generator with fixed external circuit involves a loop whose shape inside the rotating conductor is changing with time, although the path of the current in the lab frame is actually time independent. It is better to say that the conduction electrons in the rotating part of the circuit experience a $\mathbf{v} \times \mathbf{B}$ Lorentz force, which drives the observed current in the loop.\footnote{This view avoids the issue of whether the lines of the magnetic field are rotating or not, since the value of the vector $\mathbf{B}$ for a cylindrical magnet is the same whether or not the magnet is rotating.}

A.16.8 Anticipation That $\nabla \cdot \mathbf{B} = 0$

Art. 3117 of [117] included the important conclusion that: 
... there exist lines of force within the magnet, of the same nature as those without. What is more, they are exactly equal in amount to those without. They have a relation in direction to those without; and in fact are continuations of them, absolutely unchanged in their nature, so far as the experimental test can be applied to them. Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.

We recognize this as the first statement that $\nabla \cdot \mathbf{B} = 0$.

A.16.9 A Cylindrical Permanent Magnet is Equivalent to a Solenoidal Electromagnet

Art. 3120 of [117] remarked that the magnetic field of a uniformly magnetized cylinder is the same as that of a solenoid electromagnet of the same dimensions.

In 1832, Arts. 217-227 of [74], Faraday replaced the copper disk and the external magnet by a conducting, rotating magnet, whose self field acting on the “free” charges in the magnet also produced a current in the circuit, which effect is often called a homopolar generator.

A.16.10 Electric Lines of Force

The notion of electric lines of force, with tension along them and repulsion between them, appears in Art. 1297:
The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (1224.); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction (1304.).

His summary in Art. 1304 includes the statements:
I have used the phrases lines of inductive force and curved lines of force (1231. 1297. 1298. 1302.) in a general sense only, just as we speak of the lines of magnetic-force. The lines are imaginary, and the force in any part of them is of course the resultant of compound forces,
every molecule being related to every other molecule in all directions by the tension and reaction of those which are contiguous.

A.16.11 The Magnetic Field

We have already noted that Faraday used the term lines of magnetic force in a footnote to Art. 114 of [73] (1831).

In 1845, Art. 2147 of [108], the term magnetic field appears for the first time in print: The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.

A.16.12 Magnetic Power

In 1850, Art. 2806 of [116], Faraday wrote:

Any portion of space traversed by lines of magnetic power, may be taken as such a (magnetic) field, and there is probably no space without them. The condition of the field may vary in intensity of power, from place to place, either along the lines or across them...

2807. When a paramagnetic conductor, as for instance, a sphere of oxygen, is introduced into such a magnetic field, considered previously as free from matter, it will cause a concentration of the lines of force on and through it, so that the space occupied by it transmits more magnetic power than before (fig. 1). If, on the other hand, a sphere of diamagnetic matter be placed in a similar field, it will cause a divergence or opening out of the lines in the equatorial direction (fig. 2); and less magnetic power will be transmitted through the space it occupies than if it were away.

Here, one can identify Faraday’s usage of the term magnetic power with the magnetic flux $\Phi_B = \int B \cdot d\text{Area}$.

A further consequence of his interaction with Thomson appears to be that in 1852, beginning in sec. 3070 of [117], Faraday wrote about lines of force more abstractly, but without full commitment to their physical existence independent of matter. Thus, in Art. 3075 he stated:

I desire to restrict the meaning of the term line of force, so that it shall imply no more than the condition of the force in any given place, as to strength and direction; and not to include (at present) any idea of the nature of the physical cause of the phenomena...

A few sentences later he continued:

...for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion
that in the transmission of the force there is such an action, external to the magnet, than
that the effects are merely attraction and repulsion at a distance. Such an action may be a
function of the ether; for it is not at all unlikely that, if there be an ether, it should have
other uses than simply the conveyance of radiations (2591. 2787.).

In Art. 3175, at the end of [117], he added:

...wherever the expression line of force is taken simply to represent the disposition of the
forces, it shall have the fullness of that meaning; but that wherever it may seem to represent
the idea of the physical mode of transmission of the force, it expresses in that respect the
opinion to which I incline at present.

This has led many to infer that Faraday then believed in the physical existence of the lines
of force even though he could not “prove” that.

Faraday’s famous notion, that induced electrical currents are associated with wires “cut-
ting” lines of magnetic force, is presented in Art. 3104, and a version of what is now called
Faraday’s law,

\[ \mathcal{E} = - \frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot \text{Area}, \]  

(42)
is given verbally in Art. 3115.\textsuperscript{46}

The quantity of electricity thrown into a current is directly as the amount of curves inter-
sected.

In sec. 3117 Faraday noted that magnetic lines of force form closed circuits:

Every line of force therefore, at whatever distance it may be taken from the magnet, must
be considered as a closed circuit, passing in some part of its course through the magnet, and
having an equal amount of force in every part of its course.

However, the last phrase indicates that Faraday did not have a clear view of what we call
the strength of a magnetic field.

In sec. 3118 Faraday (re)affirmed that magnetic field lines do not rotate with a rotating
magnet, and performs various experiments with what is now called a unipolar (or homopolar)
generator to demonstrate this, which experiments are an early investigation of the relativity
of rotating frames.

In 1852, Faraday also published a set of more speculative comments [119] in the Phil.
Mag. (rather than Phil. Trans. Roy. Soc. London, the usual venue for his Experimental
Researches), arguing more strongly for the physical reality of the lines of force.

In Art. 3258 he considered the effect of a magnet in vacuum, concluding (perhaps for the
first time) that the lines of force have existence independent of a material medium:

A magnet placed in the middle of the best vacuum we can produce...acts as well upon a
needle as if it were surrounded by air, water or glass; and therefore these lines exist in such
a vacuum as well as where there is matter.

Faraday used examples of magnets and iron filings in various configurations to reinforce
his vision of a tension along the lines of forces, and in sec. 3295 added the insight that there
is a lateral repulsion between adjacent lines, referring to Fig. 5 below.

\textsuperscript{46}One should not infer from this that Faraday had an explicit notion of the magnetic field \( \mathbf{B} \) as a measure
of the density of lines of magnetic force. Rather, he emphasized the total number of lines within some area
(the magnetic flux) as the amount of magnetic force (Art. 3109).
Faraday’s last published comments on lines of force are in [126].

A.17 Henry

Henry began his studies of electromagnetism in 1827 [69], and in 1831 he demonstrated an electrical machine/motor of an unusual type [71]. In 1832 he was inspired by a brief report [77] of Faraday’s discovery of electromagnetic induction to perform a series of experiments on mutual and self induction in circuits at rest [82, 88, 90, 95, 97]. Three illustrations (from [95]) of these experiments are shown below.

In 1840, sec. 56 of [97], Henry stated:

*During the time a galvanic current is increasing in quantity in a conductor, it induces, or tends to induce, a current in an adjoining parallel conductor in an opposite direction to itself.*

47 A replica of Henry’s motor is discussed in [369].

See also https://www.princeton.edu/ssp/joseph-henry-project/

48 A review of Henry’s work on electromagnetic induction is given in [394].

26
In secs. 72-73 he suggested that Ohm’s law could be applied to a secondary loop of resistance $R$ in manner equivalent to $E_{\text{induced}} = IR$, where the induced $\mathcal{E}\mathcal{M}\mathcal{F}$ is proportional to the rate of change of the current in the primary loop. We now call the proportionality constant the mutual inductance of the two loops. Thus, Henry gave the first sense of circuit analysis for circuits containing inductance.\(^{49}\)

### A.18 Wheatstone

### A.19 Gauss

In 1867 Gauss published an analysis that he dated to 1835 (p. 609 of [89]), in which he stated that a time-dependent electric current leads to an electric field which is the time derivative of what we now called the vector potential. English translation from [322]:

> **The Law of Induction**
> Found out Jan. 23, 1835, at 7 a.m. before getting up.
> 1. The electricity producing power, which is caused in a point $P$ by a current-element $\gamma$, at a distance from $P$, = $r$, is during the time $dt$ the difference in the values of $\gamma/r$ corresponding to the moments $t$ and $d$t, divided by $dt$. where $\gamma$ is considered both with respect to size and direction. This can be expressed briefly and clearly by
> $$\frac{-d(\gamma/r)}{dt}. \tag{43}$$

On p. 612 (presumably also from 1835), Gauss noted a relation (here transcribed into vector notation) between the vector $\mathbf{A} = \oint d\mathbf{l}/r$ and the magnetic scalar potential $\Omega$ of a circuit with unit electrical current (which he related to the solid angle subtended by the circuit on p. 611),

$$-\nabla\Omega = \nabla \times \mathbf{A}. \tag{44}$$

While we would identify eq. (44) with the magnetic field $\mathbf{H}$, Gauss called it the “electricity-generating force”.

In any case, eq. (44) is the earliest appearance of the curl operator (although published later than MacCullagh’s use of this, p. 22 of [96]).

### A.20 Grassmann

In 1845, Grassmann [102] remarked that although Ampère claimed [67] that his force law was *uniquement déduite de l’expérience*, it included the assumption that it obeyed Newton’s third law. He noted that Ampère’s law (32) implies that the force is zero for parallel current elements whose lie of centers makes angle $\cos^{-1} \sqrt{2/3}$ to the direction of the currents, which seemed implausible to him. Grassmann claimed that, unlike Ampère, he would make no “arbitrary” assumptions, but in effect he assumed that there is no magnetic force between collinear current elements, which leads to a force law,

$$\mathbf{F}_{\text{on} \ 1} = \int_1 \int_2 d^2\mathbf{F}_{\text{on} \ 1}, \quad d^2\mathbf{F}_{\text{on} \ 1} = I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{r}}{c^2 r^2}, \tag{45}$$

\(^{49}\)The first detailed analysis of circuits with inductance may have been given by Maxwell in 1868 [139].
in vector notation (which Grassmann did not use in [102], although he invented the notion of an exterior product of vectors in an n-dimensional space [101]). While \( d^2 \mathbf{F}_{on1} \) is not equal and opposite to \( d^2 \mathbf{F}_{on2} \), Grassmann showed that the total force on circuit 1 is equal and opposite to that on circuit 2, \( \mathbf{F}_{on1} = -\mathbf{F}_{on2} \).

Grassmann’s result is now called the Biot-Savart force law, eqs. (30)-(31),

\[
\mathbf{F}_{on1} = \oint_1 \frac{I_1 d\mathbf{l}_1 \times \mathbf{B}_2}{c}, \quad \mathbf{B}_2 = \oint_2 \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{cr^2},
\]

although Grassmann did not identify the quantity \( \mathbf{B}_2 \) as the magnetic field.

In 1844, Grassmann [101] invented linear algebra, which went largely unrecognized for 30 years. In 1877 [153], he related this to the algebra of Hamilton’s quaternions [138], which together with a paper by Clifford (1878) [154], form the basis of contemporary geometric algebra.

### A.21 Neumann

In 1845, Neumann [105] independently arrived at (what is now written as) the form (45), and verified that it gives the same total force between closed circuits as does Ampère’s eq. (32).

In sec. 11 of [105], Neumann also discussed the magnetic energy of two circuits, calling this the “potential”, \(^{50}\)

\[
U = \oint_1 \oint_2 \frac{I_1 d\mathbf{l}_1 \cdot I_2 d\mathbf{l}_2}{c^2 r}.
\]

We now also write this as

\[
U = \oint_i I_i d\mathbf{l}_i \cdot \mathbf{A}_i, \quad \mathbf{A}_j = \oint_j \frac{I_j d\mathbf{l}_j}{c r},
\]

such that Neumann is often credited in inventing the vector potential \( \mathbf{A} \), although he appears not to have factorized his eq. (47) into eq. (48).

Add discussion of Neumann’s views on motional EMF.

### A.22 Weber

The term unipolar induction for Faraday’s homopolar dynamo is due to Weber (1839) [98].

Weber was perhaps the last major physicist who did not use electric and magnetic fields to describe electromagnetism, preferring an (instantaneous) action-at-a-distance formulation for the forces between charges (1846 [109], p. 144 of [190]), \(^{51}\)

\[
F = \frac{ee'}{r^2} \left[ 1 - a^2 \left( \frac{dv}{dt} \right)^2 + 2a^2r \frac{d^2r}{dt^2} \right].
\]

This was the first published force law for moving charges (which topic Ampère refused to speculate upon). \(^{52}\) The constant \( a \) has dimensions of velocity\(^{-1} \), and was later (1856) written

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\(^{50}\)If we write eq. (47) as \( U = I_1 I_2 M_{12} \), then \( M_{12} \) is the mutual inductance of circuits 1 and 2. Neumann included a factor of 1/2 in his version of eq. (47), associated with his choice of units.

\(^{51}\)For an extensive discussion of Weber’s electrodynamics, see [330]. Maxwell gave a review of the German school of electrodynamics of the mid 19th century in the final chapter 23 of his Treatise [188].

\(^{52}\)A comparison of the theories of Neumann and Weber is given in sec. 2 of [327].
by Weber and Kohlsrausch [130] as $1/C$, who noted that their $C$ is the ratio of the magnetic units to electrical units in the description of static phenomenon, which they determined experimentally to have a value close to $4.4 \times 10^8$ m/s. Apparently, they regarded it as a coincidence that their $C$ is roughly $\sqrt{2}$ times the speed $c$ of light.

A.23 W. Thomson (Lord Kelvin)

A.23.1 Force Fields

In 1842 (at age 18!), W. Thomson [100] noted an analogy between the (vector) flow of heat and the “attractive force” of electricity. At that time he was concerned with electrostatics, for which it is natural to consider the force only at the locations of charges and not in the space between them. In contrast, the flow of heat exists in the space between sources and sinks of heat, so Thomson’s analogy perhaps started him thinking about possible significance of electrical forces away from the location of electric charges.

Thomson appears to have become aware of Faraday’s work in 1845, and soon published initial comments [106] about transcribing Faraday’s notions into mathematical form. He noted the contrast between Coulomb’s action-at-distance view of electrical forces, and Faraday’s view (reminiscent of Descarte’s) that these forces are transmitted via some kind of “action of contiguous particles of some intervening medium”, and proceeded to argue that these are what might now be called “dual” explanations of electricity. We see in this discussion the beginning of Thomson’s lifelong vision of a mechanical ether supporting electricity and magnetism.53

In 1846, Thomson [111], p. 63, described the electrical force due to a unit charge at the origin “exerted at the point $(x, y, z)$” as $r/r^3$, without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term “electric field” were used by Thomson.

He immediately continued with the example of a “point” magnetic dipole $\mathbf{m}$, whose scalar potential is $\Phi = \mathbf{m} \cdot \mathbf{r}/r^3$, noting that the magnetic force $-\nabla \Phi$ on a unit magnetic pole $p$ can also be written as $\nabla \times \mathbf{A}$ where $\mathbf{A} = \mathbf{m} \times \mathbf{r}/r^3$ (although Thomson did not assign a symbol to the vector $\mathbf{A}$). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [105] implied this result, but was not explicit about its application to Thomson’s example).

In a major paper on magnetism in 1849 [114], Thomson still did not use the term “field”, but wrote in sec. 48:

*The resultant force at a point in space, void of magnetized matter, is the force that the north pole of a unit-bar (or a positive unit of imaginary magnetic matter), if placed at this point, would experience.*

The term “magnetic field” in the contemporary sense first appears in 1851 on p. 179 of [115], where Thomson wrote:

*Definition.—Any space at every point of which there is a finite magnetic force is called “a field*

---

53I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand; and that is why I cannot get the electro-magnetic theory. P. 270 of [169].
of magnetic force;” or, magnetic being understood, simply “a field of force;” or, sometimes, “a magnetic field.”

Definition.—A “line of force” is a line drawn through a magnetic field in the direction of the force at each point through which it passes; or a line touched at each point of itself by the direction of the magnetic force.

A.23.2 B and H

In sec. 78 of [114], Thomson considered the magnetic-field vector \((X, Y, Z)\) that we now identify with \(B\). In considerations of the effect of the magnetic field on hypothetical magnetic poles inside small cavities in a medium with magnetization density \(M\),

A.24 Maxwell

This Appendix uses SI units, while the main text employs Gaussian units.

Maxwell published his developments of the theory of electrodynamics in four steps, *On Faraday’s Lines of Force* [127] (1856), *On Physical Lines of Force* [132, 133, 134, 135] (1861-62), *A Dynamical Theory of the Electromagnetic Field* [137] (1864), and in his *Treatise on Electricity and Magnetism* [148, 149, 187, 188] (1873). In this Appendix we review his electrodynamic arguments related, in a broad sense, to the issue of the “Lorentz” force law.

As this Appendix has grown rather long, we first preview Maxwell’s arguments most directly related to the force on a moving charge.

Appendix A.1.7 notes Maxwell’s first discussion of a moving (charge) particle on p. 64 of [127] (1856), in which he took the convective derivative of the vector potential to obtain our eq. (61), whose meaning was not very apparent. It it interesting to this author that Maxwell did not in 1856 or later convert our eq. (61) into our (63), which was subsequently claimed by Helmholtz, Larmor, Watson, and J.J. Thomson to have been what Maxwell should have done.

Appendix A.2.6 reviews Maxwell’s use (1861) [133] of his theory of molecular vortices and an energy argument to deduce our eq. (80), which is the curl of the Lorentz force law. Maxwell then integrated this by adding the term \(\nabla \Psi\) to the argument of the curl operator, which yields the “Lorentz” force law, our eq. (81), if we accept Maxwell’s interpretation of \(\Psi\) as the electric tension (electric scalar potential).

Appendix A.3.6 discusses Maxwell’s consideration (1864), sec. 63 of [137], of a circuit at rest which led him to identify the electromotive force vector on a charge at rest as our eq. (106). Then, Appendix A.3.7 recounts Maxwell’s extrapolation in sec. 64 of [137] to a moving charge (circuit element) by the addition of the single-charge version of the Biot-Savart force law, \(\mathbf{F} = q \mathbf{v} \times \mathbf{B}\), to arrive at the “Lorentz” force law, our eq. (111), in the same form, our eq. (81), as he had previously found in [133].

The “Lorentz” force law, our eqs. (81) and (111) were also deduced by Maxwell (1873) in Arts. 598-599 of [149] via a slightly different argument, as discussed above in sec. 1 above. Maxwell included mention of this force law in Art. 619 of [149], the summary of his theory of the electromagnetic field, but in a somewhat unfortunate manner, as reviewed in Appendix A.4.2.
A.24.1 In *On Faraday’s Lines of Force* [127]

For a discussion of Maxwell’s thoughts in 1855, which culminated in the publication [127], see [335].

A.24.1.1 Theory of the Conduction of Current Electricity and *On Electro-motive Forces*

On p. 46 of [127], Maxwell stated: *According to the received opinions we have here a current of fluid moving uniformly in conducting circuits, which oppose a resistance to the current which has to be overcome by the application of an electro-motive force at some part of the circuit.*

He continued on pp. 46-47: *When a uniform current exists in a closed circuit it is evident that some other forces must act on the fluid besides the pressures. For if the current were due to difference of pressures, then it would flow from the point of greatest pressure in both directions to the point of least pressure, whereas in reality it circulates in one direction constantly. We must therefore admit the existence of certain forces capable of keeping up a constant current in a closed circuit. Of these the most remarkable is that which is produced by chemical action. A cell of a voltaic battery, or rather the surface of separation of the fluid of the cell and the zinc, is the seat of an electro-motive force which can maintain a current in opposition to the resistance of the circuit. If we adopt the usual convention in speaking of electric currents, the positive current is from the fluid through the platinum, the conducting circuit, and the zinc, back to the fluid again.*

Here, Maxwell seemed to accept the *received opinions* \(^{54}\) that electric current is a fluid; and actually two counterpropagating fluids.

A.24.1.2 Ohm’s Law, Electromotive Force and the Electric Field

On p. 47 of [127], Maxwell wrote a version of Ohm’s Law as \( F = IK \) for an electrical circuit of resistance \( K \) that carries current \( I \). He calls \( F \) the *electro-motive force*, which is consistent with a more contemporary notation,

\[
\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = IR \quad (F = IK),
\]

where \( \mathcal{E} = F \) is the electromotive “force” (with dimensions of electric potential (voltage) rather than of force), \( \mathbf{E} \) is the electrical field and \( d\mathbf{l} \) is line element inside the wire of the circuit.

On p. 53, Maxwell introduced the (free/conduction) electric current-density vector \( \mathbf{J} = (a_2, b_2, c_2) \), the electric scalar potential \( \Psi = p_2 \), and the electric field \( \mathbf{E} = (\alpha_2, \beta_2, \gamma_2) \), writing in his eq. (A),

\[
\mathbf{E} = \mathbf{E}_{\text{other}} - \nabla \Psi \quad \left( \alpha_2 = X_2 - \frac{dp_2}{dx}, \text{etc.} \right),
\]

with \( \mathbf{E}_{\text{other}} = (X_2, Y_2, Z_2) \) being a possible contribution to the electric field not associated with a scalar potential. To possible confusion, Maxwell called the vector \( \mathbf{E} \) an *electro-motive force*, which term he also used for the scalar \( \mathcal{E} \).

\(^{54}\)Maxwell cited French translations of papers by Kirchhoff [113] and Quincke [128]. He seemed unaware that Kirchhoff had also published an English version of his paper [113].
Also on p. 53, Maxwell introduced the electrical resistivity \( \varrho = k_2 \) (reciprocal of the electrical conductivity \( \sigma = 1/\varrho \), so the Ohm’s Law can be written as Maxwell’s eq. (B),

\[
E = \varrho J_{\text{free}} = \frac{J_{\text{free}}}{\sigma} \quad (\alpha_2 = k_2 a_2, \ \text{etc.}) , \quad (52)
\]

On p. 54, Maxwell noted that for any closed curve eq. (51) implies,

\[
\mathcal{E} = \oint E \cdot dl = \oint E_{\text{other}} , \quad (53)
\]

He also introduced the concept of the flux (\textit{conduction}) \( E \cdot dS \) of a field \( E \) across a surface element \( dS \), and noted (Gauss’ Law) that for a closed surface,

\[
\int E \cdot dS = \int \nabla \cdot E \, d\text{Vol} , \quad (54)
\]

He indicated in his eq. (C) that he will often write the divergence of a vector field as \( 4\pi \rho \).

**A.24.1.3 The Magnetic Fields \( H \) and \( \mu H = B \)**

On p. 54 Maxwell also introduced magnetic phenomena, and emphasized a formal parallel with electric phenomena. He labeled the magnetic field \( H \) as \((\alpha_1, \beta_1, \gamma_1)\) and the magnetic (induction) field \( B \) as \((a_1, b_1, c_1)\),\(^{55}\) called the (relative) magnetic permeability \( \mu \) the reciprocal of the resistance to magnetic induction, \( k_1 \), and noted (in words) that the parallel to our eq. (52), his eq. (B), is,

\[
H = \frac{B}{\mu} \quad (\alpha_1 = k_1 a_1, \ \text{etc.}) , \quad (55)
\]

and that in the relation \( \nabla \cdot B = \mu \rho_m \), \( \rho_m \) is the density of \textit{real magnetic matter}.

**A.24.1.4 Ampère’s Law**

On p. 56, Maxwell stated Ampère’s Law in the form,\(^{56}\)

\[
\nabla \times H = J_{\text{free}} \quad \left( a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy} , \ \text{etc.} \right) , \quad (56)
\]

and on p. 57 he noted that the divergence of eq. (56) is zero, so that his discussion is limited to \textit{closed currents} that obey \( \nabla \cdot J = 0 \) (\textit{i.e.}, to magnetostatics). Indeed, he added: \textit{in fact we know little of the magnetic effects of any current that is not closed}.\(^ {57}\)

\(^{55}\)Maxwell did not use the symbol \( B \) for the magnetic (induction) field until 1873, in his \textit{Treatise} [149], when he followed W. Thomson (1871), eq. (r), p. 401 of [168], who first defined \( B = \mu_0 H + M \), where \( M \) is the density of magnetization.

\(^{56}\)This is probably the first statement of Ampère’s Law as a differential equation. See A.11 above for discussion of Ampère’s statement of his law.

\(^{57}\)This statement can be regarded as a precursor to Maxwell’s later vision (first enunciated in eq. (112), p. 19, of [134]) that all currents are closed if one considers the “displacement-current” (density) \( dD/dt \) in addition to the conduction-current density \( J_{\text{free}} \). But it also indicates that Maxwell chose not to consider the notion of moving charged particle as elements of an electrical current, as advocated by Weber (1846) [338] (see p. 88 of the English translation) as a way of understanding Ampère’s expression for the force between two current loops.

For discussion of the “displacement current” of a uniformly moving charge, see [372].
A.24.1.5 Helmholtz’ Theorem

There followed an interlude on various theorems, some due to Green [144], and also Helmholtz’ theorem [131] that “any” vector field \( \mathbf{E} \) can be related to a scalar potential \( \Psi \) and a vector potential \( \mathbf{A} \) as \( \mathbf{E} = \nabla \Psi + \nabla \times \mathbf{A} \), where \( \Psi = 0 \) if \( \nabla \cdot \mathbf{E} = 0 \) and \( \mathbf{A} = 0 \) if \( \nabla \times \mathbf{E} = 0 \).\(^{58}\)

A.24.1.6 Magnetic Field Energy and the Electric Field Induced by a Changing Current

After this, Maxwell considered the energy stored in the magnetic field. On p. 63, he first argued that if the magnetic field were due to a density \( \rho_m \) of magnetic charges, the field could be deduced from a magnetic scalar potential \( \Psi_m \) \((= p_1)\) and the energy stored in the field during the assembly of this configuration could be written,\(^{59}\)

\[
U_m = \frac{1}{2} \int \rho_m \Psi_m \, d\text{Vol} \quad \left( Q = \int \int \int \rho_1 p_1 \, dx \, dy \, dz \right). \quad (57)
\]

He then noted that this form can be transformed to,

\[
U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{2} \, d\text{Vol} \quad \left( Q = \frac{1}{4\pi} \int \int \int (a_1 \alpha_1 + b_1 \beta_1 + c_1 \gamma_1) \, dx \, dy \, dz \right). \quad (58)
\]

He next argued that since this form does not include any trace of the origin of the magnetic field, it should also hold if the field is due to electrical currents, it can be transformed to,

\[
U_m = \int \frac{\mathbf{J} \cdot \mathbf{A}}{2} \, d\text{Vol} \quad \left( Q = \frac{1}{4\pi} \int \int \int \left\{ p_1 \rho_1 - \frac{1}{4\pi} \left( \alpha_0 \alpha_2 + \beta_0 \beta_2 + \gamma_0 \gamma_2 \right) \right\} \, dx \, dy \, dz \right). \quad (59)
\]

where \( \mathbf{B} = \nabla \times \mathbf{A} \).\(^{60,61,62}\)

\(^{58}\)We also write that \( \mathbf{E} = \mathbf{E}_{\text{irr}} + \mathbf{E}_{\text{rot}} \) where \( \mathbf{E}_{\text{irr}} = \nabla \Psi \) obeys \( \nabla \times \mathbf{E}_{\text{irr}} = 0 \) and \( \mathbf{E}_{\text{rot}} = \nabla \times \mathbf{A} \) obeys \( \nabla \cdot \mathbf{E}_{\text{rot}} = 0 \). Many people write \( \mathbf{E}_{\text{irr}} = \mathbf{E}_\parallel \) and \( \mathbf{E}_{\text{rot}} = \mathbf{E}_\perp \).

\(^{59}\)It seems to this author that Maxwell omitted a factor of 1/2 in throughout his discussion on pp. 63-64.

\(^{60}\)Maxwell seems to have made a sign error in his integration by parts of the integrand \( \mathbf{H} \cdot \mathbf{B} = \mathbf{H} \cdot \nabla \times \mathbf{A} \).

Note that \( \nabla(\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{H} \).

\(^{61}\)Maxwell did not seem to suppose in [127] that there is no real magnetic matter, so his \( \mathbf{B} \) was also related to a scalar potential, and his version of eq. (59) has an additional term related to possible magnetic charges and the scalar potential.

\(^{62}\)W. Thomson, p. 63 of [111] (1846), described the electrical force due to a unit charge at the origin exerted at the point \((x, y, z)\) as \( r/4\pi \epsilon r^3 \), without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term “electric field” were used by Thomson.

Thomson immediately continued with the example of a small magnet, i.e., a “point” magnetic dipole \( \mathbf{m} \), whose scalar potential is \( \Phi = \mu \mathbf{m} \cdot r/4\pi r^3 \), noting that the magnetic force \((X, Y, Z) = -\nabla \Phi = \mathbf{B} \) on a unit magnetic pole \( p \) can also be written as \( \nabla \times \mathbf{A} \) (although Thomson did not assign a symbol to the vector \( \mathbf{A} \)), where \( \mathbf{A} = (\alpha, \beta, \gamma) = \mu \mathbf{m} \times r/4\pi r^3 \), with \( \nabla \cdot \mathbf{A} = 0 \), his eq. (2). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [105] implied this result, and is generally credited with the invention of the vector potential although the relation \( \mathbf{B} = \nabla \times \mathbf{A} \) is not evident in this paper).
The time rate of change of energy in the magnetic field is \(-\mathbf{J} \cdot \mathbf{E}\), so, by taking the time derivative of eq. (59), and noting that \(\mathbf{A} = (\alpha_0, \beta_0, \gamma_0)\) scales linearly with \(\mathbf{J}\), he infers (on p. 64) that the electro-motive force due to the action of magnets and currents is,

\[
\mathbf{E}_{\text{induced}} = -\frac{\partial \mathbf{A}}{\partial t} \left( \alpha_2 = -\frac{1}{4\pi} \frac{d\alpha_0}{dt} \right),
\]

(60)

On p. 66, he stated this equation in words as: Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction. Here, Maxwell describes the vector potential \(\mathbf{A}\) as the electro-tonic intensity, following Faraday, Art. 60 of [73].

A.24.1.7 The Electro-Motive Force on a Moving Particle

At the bottom of p. 64, Maxwell made a statement that anticipated his later efforts towards the “Lorentz” force law: If \(\alpha_0\) be expressed as a function of \(x, y, z,\) and \(t,\) and if \(x, y, z\) be the co-ordinates of a moving particle, then the electro-motive force measured in the direction of \(x\) is,

\[
\alpha_2 = -\frac{1}{4\pi} \left( \frac{d\alpha_0}{dx} \frac{dx}{dt} + \frac{d\alpha_0}{dy} \frac{dy}{dt} + \frac{d\alpha_0}{dz} \frac{dz}{dt} + \frac{d\alpha_0}{dt} \right),
\]

\[
\mathbf{e} = -\left( \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} \right),
\]

(61)

where we use the symbol \(\mathbf{e} = (\alpha_2, \beta_2, \gamma_2)\) for Maxwell’s vector electromotive force, which is not necessarily the same as the lab-frame electric field \(\mathbf{E}\). Here, Maxwell claimed that the electric field experienced by a moving particle should be computed using the convective derivative of the vector potential, and not just the partial time derivative.

If Maxwell had persisted in following the consequences of this claim, he could have deduced, via a vector-calculus identity, that the electro-motive force experienced by a moving particle is,

\[
\mathbf{e} = -\left( \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) + \mathbf{v} \times (\nabla \times \mathbf{A}) \right) = \mathbf{v} \times \mathbf{B} - \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) \right).
\]

(62)

If Maxwell had further considered that the electric field can have a term deducible from a scalar potential \(\Psi\), then he might have claimed that the total electro-motive force on a moving particle is,

\[
\mathbf{e} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \nabla(\Psi + \mathbf{v} \cdot \mathbf{A}) - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.
\]

(63)

We will see below that Maxwell did not follow the path sketched above, but made important variants thereto, while others in the late 1800’s (Helmholtz [150], Larmor [166], Watson...
[180], J.J. Thomson in his Appendix to Chap. IX of [188], p. 260) argued that he should have proceeded as above.

A.24.1.8 Faraday’s Law

Faraday’s Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [127]: the electro-motive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit, as a summary of Faraday’s comments in [117].\(^\text{66}\) We express this as the equation (for a circuit at rest in the lab where \( \mathbf{E} = \mathbf{E} \)),

\[
E = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}. \tag{64}
\]

A.24.2 In On Physical Lines of Force [132, 133, 134, 135]

A.24.2.1 The Magnetic Stress Tensor, Ampère’s Law and the Biot-Savart Force Law

In [132] (1861), Maxwell considered a (linear) magnetic medium with uniform (relative) permeability \( \mu \) to be analogous to a fluid filled with vortices, which led him, on p. 168, to deduce/invent the stress tensor \( T_{ij} = p_{ij} \) of the magnetic field,

\[
T_{ij} = \mu H_i H_j - \delta_{ij} 4\pi p_m, \tag{65}
\]

where \( H = (\alpha, \beta, \gamma) \) is the magnetic field, and \( p_m = p_1 \) is a magnetic pressure. The volume force density \( f \) in the medium is then given by Maxwell’s eq (3),

\[
f = \nabla \cdot T, \tag{66}
\]

and the \( x \)-component of this force, \( f_x = X \), is given by Maxwell’s eqs. (4)-(5),

\[
f_x = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = 2\mu H_x \frac{\partial H_x}{\partial x} - 4\pi \frac{\partial p_m}{\partial x} + \mu H_x \frac{\partial H_y}{\partial y} + H_y \frac{\partial H_x}{\partial y} + \mu H_x \frac{\partial H_z}{\partial z} + H_z \frac{\partial H_x}{\partial y} = H_x \nabla \cdot \mu \mathbf{H} + \frac{\mu \partial H^2}{2 \partial x} - \mu H_x \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) - \mu H_x \left( \frac{\partial H_z}{\partial z} - \frac{\partial H_x}{\partial x} \right) - 4\pi \frac{\partial p_m}{\partial x} = H_x \nabla \cdot \mathbf{B} + \frac{\mu \partial H^2}{2 \partial x} - B_y (\nabla \times H)_z + B_z (\nabla \times H)_y - 4\pi \frac{\partial p_m}{\partial x} = H_x \nabla \cdot \mathbf{B} + \frac{\mu \partial H^2}{2 \partial x} - \mathbf{B} \times (\nabla \times \mathbf{H}) - 4\pi \frac{\partial p_m}{\partial x}, \tag{67}
\]

where Maxwell introduced \( \mathbf{B} = \mu \mathbf{H} \) as the magnetic induction field. In his eq. (6), p. 168, Maxwell stated that,

\[
\nabla \cdot \mathbf{B} = \mu \rho_m \left( \frac{d}{dx} \mu \alpha + \frac{d}{dy} \mu \beta + \frac{d}{dz} \mu \gamma = 4\pi m \right), \tag{68}
\]

\(^{66}\)The statement appears on a letter from Maxwell to W. Thomson, Nov. 23, 1854, p. 703 of [238].
where $\rho_m (= m)$ is the density of “imaginary magnetic matter”. In his eq. (9), p. 171, Maxwell stated that,

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} \left[ \frac{1}{4\pi} \left( \frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) = p, \text{ etc.} \right], \tag{69}$$

where $\mathbf{J}_{\text{free}} = (p, q, r)$ is the density of (free) electric current. He did not reference his discussion of our eq. (56) in his paper [127], but arrived at eq. (69) via an argument that involved both magnetic poles and electric currents. Thus, the force density (67) can be rewritten as, Maxwell’s eqs. (12)-(14), p. 172,

$$\mathbf{f} = \rho_m \mathbf{B} + \mathbf{J}_{\text{free}} \times \mathbf{B} + \nabla \frac{\mu H^2}{2} - 4\pi \nabla \rho_m. \tag{70}$$

The second term of eq. (70) is (this author believes) the first statement of what is now commonly called the Biot-Savart force law for a free electric-current density,

$$\mathbf{F} = \int \mathbf{J}_{\text{free}} \times \mathbf{B} \, d\text{Vol}, \tag{71}$$

in terms of a magnetic field (of which Biot and Savart [25] had no conception).\(^{67}\)

However, Maxwell did not consider an electric current to be a flow of charged particles, so he did not immediately interpret eq. (70) as a derivation of the “Lorentz” force $qv \times \mathbf{B}$ on a moving electric charge $q$.

Also, Maxwell did not note at this time that the third and fourth terms of eq. (70) cancel, in that $p_m = \mu H^2/2$ is the “magnetic pressure”,\(^{68}\) nor did he infer that $\nabla \cdot \mathbf{B} = 0 = \mu \rho_m$.

**A.24.2.2 Magnetic Field Energy**

On p. 63 of [127], Maxwell had deduced that the energy stored in the magnetic field can be computed according to our eq. (58) via an argument that supposed the existence of magnetic charges (monopoles) and a corresponding magnetic scalar potential. In Prop VI, pp. 286-288 of [133], Maxwell used his model of molecular vortices to deduce the same result (given in his eqs. (45)-(46), p. 288, and again in his eq. (51), p. 289),

$$U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{2} \, d\text{Vol} \quad \left( E = \frac{1}{2} \mu (\alpha^2 + \beta^2 + \gamma^2) V \right). \tag{72}$$

**A.24.2.3 Faraday’s Law**

In Prop. VII, pp. 288-291 of [133], Maxwell considered the time derivative of the magnetic field energy, written in his eq. (52), p. 289, as,

$$\frac{dU_m}{dt} = \int \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \, d\text{Vol} \quad \left[ \frac{dE}{dt} = \frac{1}{4\pi} \mu V \left( \alpha \frac{d\alpha}{dt} + \beta \frac{d\beta}{dt} + \gamma \frac{d\gamma}{dt} \right) \right], \tag{73}$$

\(^{67}\)ADD COMMENT about Neumann and Weber.

\(^{68}\)If the permeability $\mu$ is nonuniform, the third and fourth terms combine to yield the term $(H^2/2)\nabla \mu$, as noted by Helmholtz (1881) [160].
He also introduced the *electromotive force* \( \mathcal{E} = (P, Q, R) \) on a unit electric charge, and argued, using his model of molecular vortices, that the (electro)magnetic field does work on an electric current density \( \mathbf{J}_{\text{free}} \) at rate,

\[
\frac{dU_m}{dt} = -\int \mathbf{J}_{\text{free}} \cdot \mathbf{E} \, d\text{Vol}.
\] (74)

In cases like the present where the *electromotive force* \( \mathcal{E} \) is that on a hypothetical unit charge at rest in the lab, it is the same as the electric field \( \mathbf{E} \), whose symbol will be used in this section.\(^{70}\) Maxwell next gave an argument in his eqs. (48)-(50) equivalent to using our eq. (69) to find,

\[
\frac{dU_m}{dt} = -\int \mathbf{E} \cdot (\nabla \times \mathbf{H}) \, d\text{Vol} = -\int \mathbf{H} \cdot (\nabla \times \mathbf{E}) \, d\text{Vol}.
\] (75)

Comparing our eqs. (73) and (75), we infer,\(^{71}\) as in Maxwell’s eq. (54), p. 290, that,

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \left( \frac{dQ}{dz} - \frac{dR}{dy} = \mu \frac{d\alpha}{dt}, \text{ etc.} \right).
\] (76)

This is the first statement of Faraday’s law as a (vector) differential equation. Surprisingly, Maxwell did not give this differential form either in \([137]\) or in his *Treatise* \([188]\).

**A.24.2.4 \( \nabla \cdot \mathbf{B} = 0, \nabla \cdot \mathbf{A} = 0 \)**

Also on p. 290 of \([133]\), Maxwell discussed the relation \( \mathbf{B} = \nabla \times \mathbf{A} \), his eq. (55), *subject to the conditions*, his eqs. (56) and (57),

\[
\nabla \cdot \mathbf{B} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{A} = 0.
\] (77)

Maxwell gave no justification for these conditions, which are the first statements by him of them.\(^{72}\) While we now recognize that \( \nabla \cdot \mathbf{B} = 0 \) holds in the absence of true magnetic charges, as apparently is the case in Nature (and that if \( \mathbf{B} = \nabla \times \mathbf{A} \), then \( \nabla \cdot \mathbf{B} = 0 \) follows from vector calculus), the relation \( \nabla \cdot \mathbf{A} = 0 \) is a choice of “gauge” (in particular, the Coulomb gauge) and not a law of Nature.

**A.24.2.5 Electric Field outside a Toroidal Coil with a Time-Varying Current**

We digress slightly to note that on pp. 338-339 of \([133]\), Maxwell considered a toroidal coil in his Fig. 3. If this coil carries an electric current, there is no exterior magnetic field even in the case of a time-dependent current (if one neglects electromagnetic radiation,\(^{69}\)Strictly, Maxwell wrote in his eq. (47), p. 289, that \(dE/dt\) is a surface integral rather than a volume integral.\(^{70}\)In sec. A.2.5, which concerns moving charges, we will use the symbol \( \mathcal{E} \) for Maxwell’s *electromotive force* vector.\(^{71}\)This argument ignores a possible contribution to the field energy from the electric field.\(^{72}\)Maxwell likely followed Thomson, who considered that \( \nabla \cdot \mathbf{A} = 0 \) in eqs. (2) and (3) of \([111]\). Thomson was inspired by Stokes’ discussion \([103]\) of incompressible fluid flow where the velocity vector \( \mathbf{u} \) obeys \( \nabla \cdot \mathbf{u} = 0 \), Stokes’ eq. (13).
whose existence Maxwell reported only in the next paper, [134], in his series On Physical
Lines of Force). However, a changing current induces an external electric field, which seems
like action at a distance. Maxwell noted that while the external magnetic field is zero, the
external vector potential \( \mathbf{A} \) (electrotonic state) is not, and the external electric field is related
to the time derivative of \( \mathbf{A} \). It seems that the vector potential \( \mathbf{A} \) had “physical reality” for
Maxwell, which view was later extended to a quantum context by Aharonov and Bohm
[260, 385].

A.24.2.6 Electromotive Force in a Moving Body

In Prop. XI, pp. 340-341 of [133], Maxwell considered a body that might be deforming,
translating, and/or rotating, and discussed the resulting changes in the magnetic field \( \mathbf{H} \). On one hand, he stated in his eq. (70), p. 341, that,

\[
\delta \mathbf{H} = (\delta \mathbf{x} \cdot \nabla) \mathbf{H} + \delta t \frac{\partial \mathbf{H}}{\partial t} \left( \delta \mathbf{\alpha} = \frac{d\alpha}{dx} \delta x + \frac{d\alpha}{dy} \delta y + \frac{d\alpha}{dz} \delta z + \frac{d\alpha}{dt} \delta t \right),
\]

(78)

which uses the convective derivative. On the other hand, he stated before his eq. (69): The
variation of the velocity of the vortices in a moving element is due to two causes—the action
of the electromotive forces, and the change of form and position of the element. The whole
variation of \( \mathbf{\alpha} \) is therefore,

\[
\delta \mathbf{\alpha} = \frac{1}{\mu} \left( \frac{dQ}{dz} - \frac{dR}{dy} \right) \delta t + \alpha \frac{d}{dx} \delta x + \beta \frac{d}{dy} \delta y + \gamma \frac{d}{dz} \delta z
\]

\[
- \left( \delta \mathbf{H} = -\frac{1}{\mu} \nabla \times \mathbf{E} \delta t + (\mathbf{H} \cdot \nabla) \delta \mathbf{x} \right).
\]

(79)

If we accept this relation, then we can follow Maxwell that for an incompressible medium,
whose velocity field obeys \( \nabla \cdot \mathbf{v} = 0 \), and if \( \nabla \cdot \mathbf{H} = 0 \) (which Maxwell stated to hold in the
absence of free magnetism), then his eqs. (69)-(70) do lead to his eq. (76), p. 342,

\[
\frac{d}{dz} \left( Q + \mu \gamma \frac{dx}{dt} - \mu \alpha \frac{dz}{dt} - \mu G \right) - \frac{d}{dy} \left( R + \mu \alpha \frac{dy}{dt} - \mu \beta \frac{dx}{dt} - \mu H \right) = 0
\]

\[
\nabla \times \left( \mathbf{E} - \mathbf{v} \times \mathbf{B} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.
\]

(80)

\[73\text{I believe that our eq. (78) holds for a body that has translated by } \delta \mathbf{x}, \text{ without rotation or deformation.}\]

\[74\text{In this section, which considers the electromotive force on a moving, unit charge, we use the symbol } \mathbf{E} \text{ for this, rather than the symbol } \mathbf{E} \text{.}\]

\[75\text{The term } (\mathbf{H} \cdot \nabla) \delta \mathbf{x} \text{ was motivated by Maxwell’s Props. IX and X, pp. 340-341 of [133], but is not evident to this author.}\]
where here Maxwell wrote the vector potential \((\text{electrotonic components})\) as \(\mathbf{A} = -(F, G, H)\). This leads to Maxwell’s eq. (77),

\[
\mathcal{E} = \mathbf{v} \times \mathbf{B} - \nabla \Psi - \frac{\partial \mathbf{A}}{\partial t} \quad \left( P = \mu \gamma \frac{dy}{dt} - \mu \beta \frac{dz}{dt} + \frac{dF}{dt} - \frac{d\Psi}{dx}, \text{ etc.} \right)
\]  

(81)

where the “function of integration” \(\Psi\) was interpreted by Maxwell as the electric scalar potential \((\text{electric tension})\).

The sense of Maxwell’s derivation is that \(q \mathbf{E}\) would be the force experienced (in the lab frame) by an electric charge in the moving body, i.e.,

\[
\mathbf{F} = q \mathcal{E} = q \left( \mathbf{v} \times \mathbf{B} - \nabla \Psi - \frac{\partial \mathbf{A}}{\partial t} \right) = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (\text{“Lorentz”}),
\]  

(82)

using the relation \(\mathbf{E} = -\nabla \Psi - \frac{\partial \mathbf{A}}{\partial t}\) for the electric field in the lab frame. Then, eq. (82) is the first statement of the “Lorentz” force law. However, Maxwell’s argument seemed to have had little impact, perhaps due to the doubtful character of his argument leading to our eq. (79).

If the body were in uniform motion with velocity \(\mathbf{v}\), \(\mathcal{E}\) could be interpreted as the electric field \(\mathbf{E}'\) experienced by an observer moving with the body. Then, (in Gaussian units),

\[
\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B},
\]  

(83)

which is the low-velocity Lorentz transformation of the electric field \(\mathbf{E}\).

A.24.2.7 Faraday’s Law, Revisited

On p. 343 of [133], Maxwell considered a moving conductor, and moving circuit, in a magnetic field, with no electric field in the lab frame. In his eqs. (78)-(79) he applied his eq. (77) to a segment of a moving conductor, finding,

\[
\mathcal{E}' \cdot \mathbf{d}\mathbf{l} = \mathbf{v} \times \mathbf{B} \cdot \mathbf{d}\mathbf{l} = -\mathbf{B} \cdot \mathbf{v} \times \mathbf{d}\mathbf{l} = -\mathbf{B} \cdot \frac{d\mathbf{S}}{dt}
\]  

(84)

\[
\left[ e = S (Pl + Qm + Rn) = S \mu \alpha \left( m \frac{dz}{dt} - n \frac{dy}{dt} \right) \right],
\]

where \(\mathcal{E}\) is the electromotive force vector with respect to the moving conductor, \(d\mathbf{S}/dt = dx/dt \times d\mathbf{l}\) is the area swept out by the moving line element, \(d\mathbf{l}\) of the conductor in unit time.

In the case of a moving, closed circuit, the total (scalar) electromotive force is then,

\[
\mathcal{E} = \oint \mathcal{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{d\Phi_m}{dt} ,
\]  

(85)

\(i.e.,\) the total electromotive force in a closed conductor is measured by the change of the number of lines of force which pass through it; and this is true whether the change be produced by the motion of the conductor or by any external cause.

Since the above argument applies only to the case of a moving circuit, it does not demonstrate Maxwell’s claim that our eq. (85) also holds for a circuit at rest with the number of
lines of force which pass through it changing due to an external cause, i.e., a time variation of the magnetic field $\mathbf{B}$. Presumably, Maxwell meant for the reader to recall his discussion on pp. 338-339 of [133] (sec. A.2.4 above): This experiment shows that, in order to produce the electromotive force, it is not necessary that the conducting wire should be placed in a field of magnetic force, or that lines of magnetic force should pass through the substance of the wire or near it. All that is required is that lines of force should pass through the circuit of the conductor, and that these lines of force should vary in quantity during the experiment.

### A.24.2.8 Electric Currents in the Model of Molecular Vortices

On p. 13 of [134], Maxwell stated: According to our theory, the particles which form the partitions between the cells constitute the matter of electricity. The motion of these particles constitutes an electric current; the tangential force with which the particles are pressed by the matter of the cells is electromotive force, and the pressure of the particles on each other corresponds to the tension or potential of the electricity. Similarly, on p. 86 of [135], Maxwell stated: in this paper I have regarded magnetism as a phenomenon of rotation, and electric currents as consisting of the actual translation of particles.

Maxwell illustrated this vision in his Fig. 2, along with the description: *Let A B, P1. V. fig. 2, represent a current of electricity in the direction from A to B.*

A contemporary version of this view of electric currents in magnetic matter is that there

---

76 On p. 283 of [132], Maxwell wrote: “What is an electric current?”

I have found great difficulty in conceiving of the existence of vortices in a medium, side by side, revolving in the same direction about parallel axes. The contiguous portions of consecutive vortices must be moving in opposite directions; and it is difficult to understand how the motion of one part of the medium can coexist with, and even produce, an opposite motion of a part in contact with it. The only conception which has at all aided me in conceiving of this kind of motion is that of the vortices being separated by a layer of particles, revolving each on its own axis in the opposite direction to that of the vortices, so that the contiguous surfaces of the particles and of the vortices have the same motion.

In mechanism, when two wheels are intended to revolve in the same direction, a wheel is placed between them so as to be in gear with both, and this wheel is called an “idle wheel”. The hypothesis about the vortices which I have to suggest is that a layer of particles, acting as idle wheels, is interposed between each vortex and the next, so that each vortex has a tendency to make the neighbouring vortices revolve in the same direction with itself.
exist a “bound” current density therein, related by,

\[ \mathbf{J}_{\text{bound}} = \nabla \times \mathbf{M}, \]  

(86)

where \( \mathbf{M} \) is the density of magnetization (i.e., of Ampèrian magnetic dipoles, which are “molecular” current loops). However, this relation does not appear in On Physical Lines of Force, where Maxwell seemed to have supposed that his vision applied to all media, including “vacuum”, and not just to magnetic matter.

A.24.2.9 Displacement Current and Electromagnetic Waves

The most novel aspect of Maxwell’s paper On Physical Lines of Force was his introduction of the “displacement current”, and his deduction that the equations of electromagnetism then imply the existence of electromagnetic waves that propagate with the speed of light.

On p. 14 of [134], his discussion reads:

Electromotive force acting on a dielectric produces a state of polarization of its parts similar in distribution to the polarity of the particles of iron under the influence of a magnet, and, like the magnetic polarization, capable of being described as a state in which every particle has its poles in opposite conditions. In a dielectric under induction, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively, and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another.

The effect of this action on the whole dielectric mass is to produce a general displacement of the electricity in a certain direction. This displacement does not amount to a current, because when it has attained a certain value it remains constant, but it is the commencement of a current, and its variations constitute currents in the positive or negative direction, according as the displacement is increasing or diminishing. The amount of the displacement depends on the nature of the body, and on the electromotive force; so that if \( h \) is the displacement (in the \( z \)-direction), \( R \) the electromotive force, and \( E \) a coefficient depending on the nature of the dielectric,

\[ R = -4\pi E^2 h; \]

and if \( r \) is the value of the electric current (in the \( z \)-direction) due to displacement,

\[ r_{\text{displacement}} = \left( -\frac{dh}{dt} \right) \left( \frac{1}{4\pi E^2} \frac{dR}{dt} \right), \]

(87)

where it seems to this author that a minus sign should be inserted in Maxwell’s original version of our eq. (87).\(^{77}\)

In the above, the electromotive force vector \( (P, Q, R) = \mathbf{E} \) is the electric field, \( E^2 = 1/\epsilon \) where \( \epsilon \) is the relative permittivity (dielectric constant), the displacement vector \( (f, g, h) = -\mathbf{D}/4\pi \) is proportional to our present electric field vector \( \mathbf{D} \), and \( (p, q, r) = \mathbf{J}_{\text{free}} \) is the free current density. Maxwell’s relation \( R = -4\pi E^2 h \) (repeated in his eq. (105), p. 18, of [132]) is equivalent to,

\[ \mathbf{E} = \frac{\mathbf{D}}{\epsilon}. \]

(88)

\(^{77}\)For comments on reversals of signs in the relation between Maxwell’s electric displacement and electric field, see [274].
Then, Maxwell’s expression for the electric current due to displacement is equivalent to,

\[ J_{\text{displacement}} = \frac{dD}{dt}. \]  

(89)

On p. 19 of [134], Maxwell argued that a variation of displacement is equivalent to a current, and this current \( r_{\text{displacement}} \) of our eq. (87) must be taken into account in equations (9) [our eq. (69)] and added to \( r \). The three equations then become,

\[ p = \frac{1}{4\pi} \left( \frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) - \frac{1}{E^2} \frac{dP}{dt}, \text{ etc.} \quad \left( \nabla \times H = J_{\text{free}} + \frac{dD}{dt} \right). \]  

(90)

This is the first statement of Maxwell’s “fourth” equation as we know it today.

Maxwell next noted that the equation of continuity for free charge and current densities is, his eq. (113), p. 19 of [134],

\[ \nabla \cdot J_{\text{free}} + \frac{\partial \rho_{\text{free}}}{\partial t} = 0 \quad \left( \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} + \frac{de}{dt} = 0 \right), \]  

(91)

where \( e = \rho_{\text{free}} \) is the free charge density. On taking the divergence of eq. (90) and using eq. (91), we arrive at Maxwell’s eqs. (114)-(115),

\[ \frac{\partial}{\partial t} \nabla \cdot D = \frac{\partial \rho_{\text{free}}}{\partial t}, \quad \nabla \cdot D = \rho_{\text{free}} \quad \left[ e = \frac{1}{4\pi E^2} \left( \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \right) \right], \]  

(92)

which is the earliest statement of Maxwell’s “first” equation.78

In Prop. XVI, p. 22 of [134], Maxwell considered the rate of propagation of transverse vibrations through the elastic medium of which the cells are composed, and found the constant \( c \) in our equation (90) to have a value remarkably close to the speed of light in vacuum, and concluded that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.

A.24.2.10 Electric Tension and Poisson’s Equation

A “sidelight” of Maxwell’s discussion on p. 20 of [134] was his statement that in static electricity, the electromotive force (vector) can be related to the electric tension via his eq. (118), \( E = -\nabla \Psi \). Further, with \( E = D/\epsilon \), his eq. (119) (with a change of sign), and \( \nabla \cdot D = \rho_{\text{free}} \), his eq. (115), one has that the tension \( \Psi \) obeys Poisson’s equation,

\[ \nabla^2 \Psi = -\frac{\rho_{\text{free}}}{\epsilon}, \]  

(93)

Maxwell’s eq. (123) (with a change of sign).79

78 Nowadays it is more common to argue that Maxwell’s “first” and “fourth” equations together imply the continuity equation (91).
79 As will be noted in Appendices A.3.10 and A.4.3 below, the relation (93) strictly holds only in the Coulomb gauge.
A.24.3 In *A Dynamical Theory of the Electromagnetic Field* [137]

A.24.3.1 Scalar Electromotive Force and the Integral Form of Faraday’s Law

In sec. 24 of [137] Maxwell considered an electrical circuit A that carries current \( I_A = u \), and another circuit B that carries current \( I_B = v \), and noted that the magnetic flux, \( \Phi_m \), through circuit A is given by the first equation on p. 468,

\[
\Phi_m = \int_A \mathbf{B} \cdot d\mathbf{S} = LI_A + MI_B \quad (Lu + Mv),
\]

where \( \mathbf{B} \) is the magnetic field, \( d\mathbf{S} \) is an element of the area of a surface bounded by circuit A, \( L \) is the self inductance of circuit A, and \( M \) is the mutual inductance between circuits A and B. Maxwell called this flux the *momentum*, or the *reduced momentum* of the circuit.

In sec. 50, Maxwell gave a verbal statement of Faraday’s Law:

1st, If any closed curve be drawn in the field, the value of \( M \) for that curve will be expressed by the number of lines of force which pass through that closed curve.

2ndly. If this curve be a conducting circuit and be moved through the field, an electromotive force will act in it, represented by the rate of decrease of the number of lines passing through the curve.

We transcribe this into symbols as,

\[
E = -\frac{d\Phi_m}{dt},
\]

where \( E \) is the scalar electromotive force.

A.24.3.2 Electromagnetic Force and Displacement Current

However, in sec. 56, Maxwell used the term electromotive force in a different way, to describe a vector, \( \mathbf{E} = (P, Q, R) \): \( P \) represents the difference of potential per unit of length in a conductor placed in the direction of \( x \) at the given point. This appears to mean that,

\[
\mathbf{E} = (P, Q, R) = -\nabla \Psi,
\]

where \( \Psi \) is Maxwell’s symbol for the electric scalar potential. If so, this is the first mention of an aspect of the electric field \( \mathbf{E} \) in [137], although he had introduced the *electrical displacement* \( \mathbf{D} = (f, g, h) \) in sec. 55, along with “displacement-current” (density), \((1/4\pi) d\mathbf{D}/dt \) in his eq. (A),

\[
\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \frac{d\mathbf{D}}{dt} \quad \left( (p', q', r') = (p, q, r) + \frac{d(f, g, h)}{dt} \right),
\]

where the free current density \( \mathbf{J}_{\text{free}} = (p, q, r) \) was introduced in sec. 54, and the total motion of electricity is \( \mathbf{J}_{\text{total}} = (p', q', r') \). Maxwell did not use separate symbols for partial and total derivatives, so that there can be some ambiguity as to his meaning when his equations describe moving systems.

A.24.3.3 Vector Potential aka Electromagnetic Momentum
In sec. 57, Maxwell introduced the vector potential \( \mathbf{A} = (F, G, H) \), but called it the electromagnetic momentum. In his eq. (29), Maxwell identified \(-d\mathbf{A}/dt\) with the part of the electromotive force which depends on the motion of magnets or currents. Thus, we might now presume that Maxwell’s \( \mathbf{E} = (P, Q, R) \) of his sec. 56 is the electric field,
\[
\mathbf{E} = -\nabla \Psi - \frac{\partial \mathbf{A}}{\partial t},
\]
but this conclusion may be premature.

A.24.3.4 Magnetic Flux aka Total Electromagnetic Momentum of a Circuit

In eq. (29) of sec. 58, Maxwell gave the relation for the magnetic flux \( \Phi_m \) through a circuit, the number of lines of magnetic force which pass through it,
\[
\left( \Phi_m = \int \mathbf{B} \cdot d\mathbf{S} = \right) \oint \mathbf{A} \cdot d\mathbf{l} \left[ \int \left( F \frac{dx}{ds} + G \frac{dy}{ds} + H \frac{dz}{ds} \right) ds \right],
\]
and called this the total electromagnetic momentum (which we must remember to distinguish from the electromagnetic momentum \( \mathbf{A} \)).

He also noted in sec. 58 that,
\[
\left( \Phi_m = \oint \mathbf{A} \cdot d\mathbf{l} = \int \right) \nabla \times \mathbf{A} \cdot d\mathbf{S} \left[ \left( \frac{dH}{dy} - \frac{dG}{dz} \right) dy dz \right],
\]
is the number of lines of magnetic force which pass through the area \( dy \, dz \).

A.24.3.5 The Magnetic Fields \( \mathbf{H} \) and \( \mathbf{B} \) and Maxwell’s Fourth Equation

In sec. 59, Maxwell introduced the magnetic field \( \mathbf{H} = (\alpha, \beta, \gamma) \).

In sec. 60, Maxwell introduced the (relative) permeability \( \mu \), calling it the coefficient of magnetic induction.

In eq. (B) of sec. 61, Maxwell gave the Equations for Magnetic Force,
\[
\mu \mathbf{H} \left( = \mathbf{B} \right) = \nabla \times \mathbf{A} \left[ \mu \alpha = \left( \frac{dH}{dy} - \frac{dG}{dz} \right), \text{ etc.} \right].
\]

We use the symbol \( \mathbf{B} \) for Maxwell’s \( \mu \mathbf{H} \).\(^{80}\)

In eq. (C) of sec. 62, Maxwell gives a version of Ampère’s Law,
\[
\nabla \times \mathbf{H} = \mathbf{J}_{\text{total}} \left( \frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p', \text{ etc.} \right),
\]
recalling from our eq. (97) that Maxwell’s vector \( (p', q', r') \) is the total current density \( \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{displacement}} \).

A.24.3.6 Electromotive Force in a Circuit at Rest

\(^{80}\)The symbol \( \mathbf{B} \) for the quantity \( \mu_0 \mathbf{H} + \mathbf{M} \), where \( \mathbf{M} \) is the magnetization density, was introduced by W. Thomson in 1871, eq. (r), p. 401 of [168], and appears in Art. 399 of Maxwell’s Treatise [149].
In eq. (32), sec. 63 of [137], Maxwell stated that the *electromotive force acting round* an electrical circuit is related by,

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} \quad \left[ \xi = \int \left( P \frac{dx}{ds} + Q \frac{dy}{ds} + R \frac{dz}{ds} \right) ds \right]. \tag{103} \]

Supposing this circuit, A, carries current \( I_A = u \), and another circuit, B, carries current \( I_B = v \), Maxwell, in his eq. (33), reminded us the magnetic flux through circuit A is given by our eq. (94),

\[ \Phi_m = \oint_A \mathbf{A} \cdot d\mathbf{l} = LI_A + MI_B \quad \left[ \int \left( F \frac{dx}{ds} + G \frac{dy}{ds} + H \frac{dz}{ds} \right) ds = Lu + Mv \right], \tag{104} \]

as he had previously discussed in sec. 24. Then, his eq. (34) states that,

\[ \mathcal{E} = -\frac{d}{dt}(LI_A + MI_B) \quad \left( = -\int \frac{dA}{dt} \cdot d\mathbf{l} \right), \tag{105} \]

so comparison with our eq. (103) leads to the inference, Maxwell’s eq. (35), that *if there is no motion of the circuit A*,

\[ \mathcal{E} = -\frac{\partial A}{\partial t} - \nabla \Psi \quad \left( = -\frac{dF}{dt} - \frac{dG}{dx} \right), \tag{106} \]

where \( \Psi \) could be any scalar function. But, the discussion in his sec. 56 led Maxwell to identify \( \Psi \) of eq. (106) as the electrical scalar potential.

In eq. (106), we have written \( \frac{\partial A}{\partial t} \), while Maxwell wrote \( \frac{dA}{dt} \), in that for an observer (at rest) of a circuit at rest, use of a convective derivative is not appropriate.

**A.24.3.7 The Vector Electromotive Force on a Moving Conductor**

In sec. 64, Maxwell deduced the force on a bar the slides on a U-shaped rail, while carrying a current, with the entire system in an external magnetic field. He gave no figure in [137], but the figure below is associated with his discussion of this example in Arts. 594-597 of [188]. \( C \) represents a battery that drives the current in the circuit.

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81 This meaning of the term *electromotive force* is still in use today. However, Maxwell also used the term *electromotive force* in sec. 65 of [137] to describe the force \( \mathbf{v} \times \mu \mathbf{H} \) on a moving, unit charge in a magnetic field, referring to his eq. (D).
Maxwell desired a very general discussion in sec. 64, so he considered a circuit whose plane was not perpendicular to any of the $x$, $y$ or $z$ axes, which makes his description rather intricate. Here, we suppose the circuit lies in the $x$-$z$ plane, with the sliding piece, $AB$, of length $a$ parallel to the $x$-axis, and the long arms of the U-shaped rail parallel to the $z$-axis, at, say $x = 0$ and $a$. The velocity $v_z = dz/dt$ of the sliding bar is in the $z$-direction, and the uniform, external magnetic field is in the $y$-direction.

As in sec. 63, Maxwell considered changes in the magnetic flux through the circuit, $\oint \mathbf{A} \cdot d\mathbf{l}$, our eq. (99), to infer the strength of his vector $\mathbf{E}$. The part of the line integral over the sliding bar changes at rate,$^82$

$$a\frac{dA_x}{dz} \frac{dz}{dt},$$

(107)

as indicated in the first equation on p. 485. Because the length of the circuit in $z$ is increasing, the line integral also changes at rate,

$$\frac{dz}{dt}[A_z(x = 0) - A_z(x = a)] = -\frac{dz}{dt} \frac{dA_z}{dx} a,$$

(108)

as given in the second equation on p. 485. Hence, the total rate of change of magnetic flux, given in the third and fourth equations on p. 485, is,$^83$

$$\frac{d\Phi_m}{dt} = av_z \left( \frac{dA_x}{dz} - \frac{dA_z}{dx} \right) = av_z B_y = -\mathbf{E} = -\oint \mathbf{E} \cdot d\mathbf{l}. $$

(109)

Maxwell considered that eq. (109) describes a contribution to the electromotive force $\mathbf{E}$ beyond that in eq. (106), which additional contribution would be localized to the component $\mathbf{E}_x$ along the sliding bar (of length $a$), i.e., $\oint \mathbf{E} \cdot d\mathbf{l} = a\mathbf{E}_x$. Hence, he concluded in his eq. (36) that,

$$\mathbf{E}_x = -v_z B_y \quad \left( P = -\mu \beta \frac{dz}{dt} \right), \quad i.e., \quad \mathbf{E} = \mathbf{v} \times \mathbf{B},$$

(110)

is the part of $\mathbf{E}$ due to the motion of the sliding bar.

Finally, in sec. 65, Maxwell stated that the total electromotive force on a moving conductor is his eq. (D),

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi \quad \left[ P = \mu \left( \gamma \frac{dy}{dt} - \beta \frac{dz}{dt} \right) - \frac{dF}{dt} - \frac{d\Psi}{dx}, \text{etc.} \right],$$

(111)

recalling his eq. (35), our eq. (106). Again, we have written $\partial \mathbf{A}/\partial t$ where Maxwell wrote $d\mathbf{A}/dt$.

Note that Maxwell’s argument in sec. 65 does not address the mechanical force on the sliding bar, $I \mathbf{a} \times \mathbf{B}$, which is the subject of most present discussions of this example.

A.24.3.8 Other General Equations of the Electromagnetic Field

For completeness, we briefly record other general considerations by Maxwell in [137].

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$^82$In eqs. (107)-(110), the quantities $A_x$, $A_z$ and $B_y$ are evaluated at the location of the sliding bar.

$^83$On p. 485, the equations of Magnetic Force (8) should read: the equations of Magnetic Force (B).
In sec. 66, Maxwell’s eq. (66), \( P = kf, \) etc., would seem to be the equivalent of \( D = \varepsilon E, \) for the displacement field \( D = (f, g, h) \), the electric field \( E = (P, Q, R) \), and the (relative) permittivity \( \varepsilon = 1/k \).

However, in sec. 67 we read that Ohm’s law can be written as \( P = -\rho p, \) etc., which would seem to imply that \( E = -\varrho J \), where \( \varrho \) is the electrical resistivity \((= 1/\text{conductivity})\) and \( J = (p, q, r) \) is the electric current density.\(^{84}\) Then in sec. 68 we read that,

\[
e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad (\nabla \cdot D = -\rho),
\]

and in sec. 69 that the equation of continuity is,

\[
\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \quad \left( \frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \right),
\]

where \( e = \rho \) is the electric charge density. It appears that either \( E = -(P, Q, R) \) and \( D = -(f, g, h) \), or \( \rho = -e.\)^{85}

Fortunately, in Arts. 604-619 of his Treatise [188], Maxwell presented these relations with signs that match our present usage.

### A.24.3.9 Electromagnetic Theory of Light

Also for completeness, we include remarks on secs 91-100 of [137], where Maxwell presented his electromagnetic theory of light.

He considered plane waves in a linear, nonconducting medium with permittivity \( \varepsilon \) and permeability \( \mu \), such that \( D = \varepsilon E \) and \( B = \mu H.\)^{86} The waves propagated in direction \( \hat{n} \) with speed \( v \), such that the wave fields were only functions of the single scalar \( \varphi = \hat{n} \cdot x - vt \), as noted at the beginning of sec. 92.

For these waves, the relation between the magnetic field \( B \) and the vector potential \( A \) can be written as,

\[
B = \nabla \times A = \hat{n} \times \frac{\partial A}{\partial \varphi},
\]

Consequently, \( \hat{n} \cdot B = 0 \), and the magnetic field vector is transverse to the direction of propagation of the wave, as noted in eq. (62), sec. 92.

In sec. 93, Maxwell used his equation \( \nabla \times H = J_{\text{total}} \), noting that in the nonconducting medium the only current is the displacement current \( \partial D/\partial t = \varepsilon \partial E/\partial t \), and that the electric field can be written in terms of potentials as \( E = -\partial A/\partial t - \nabla \Psi \). Then, he wrote,

\[
\nabla \times H = \frac{\nabla \times B}{\mu} = \frac{\nabla \times (\nabla \times A)}{\mu} = \frac{\nabla (\nabla \cdot A) - \nabla^2 A}{\mu} = J_{\text{total}} = \frac{\partial D}{\partial t} = \frac{\partial E}{\partial t} = -\varepsilon \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \Psi \right),
\]  

\( ^{84}\)There seems to be a minus sign “loose” here. In a draft of [137], p. 160 of [334], Ohm’s law was written as \( P = \varrho p, \) etc., and in Art. 609 of [188], eq. (G) reads \( J = C \mu \). (\( J = \sigma E \)).

\( ^{85}\)This issue here is likely related to Maxwell’s vision of electric charge as an aspect of the displacement field \( D \), which leads to a concept of charge density that is the negative of our present view.

\( ^{86}\)In this and the following section we use SI units.
which corresponds to Maxwell’s eq. (68) of sec. 94. On taking the curl of this, and replacing \( \nabla \times \mathbf{A} \) by \( \mathbf{B} \), he found,

\[
\nabla^2 \mathbf{B} = \epsilon \mu \frac{\partial^2 \mathbf{B}}{\partial t^2}, \tag{116}
\]

his eq. (69). Hence, the wavespeed is \( v = 1/\sqrt{\epsilon \mu} \), which in air/vacuum is extremely close to the speed of light.

Maxwell did not comment on the electric field of the wave, saying (sec. 95) instead that this wave consists entirely of magnetic disturbances.\(^{87}\)

**A.24.3.10 Waves of the Potentials, Coulomb Gauge**

In secs. 98-99 of [137], Maxwell considers waves of the potentials. He reminded the reader that his discussion in sec. 94, our eq. (115), could emphasize the potentials rather than the magnetic field, resulting in the wave equation,

\[
\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\epsilon \mu \left( \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \frac{\partial \Psi}{\partial t} \right). \tag{119}
\]

Perhaps taking inspiration from sec. 82 of [114] by W. Thomson (1950), Maxwell noted that one can make a (gauge) transformation of the vector potential according to his eq. (74), p. 500,

\[
\mathbf{A}' = \mathbf{A} - \nabla \chi, \tag{120}
\]

such that \( \mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}' \), and the scalar potential should be transformed to his eq. (77), where our \( \Psi' \) is Maxwell’s \( \phi \),

\[
\Psi' = \Psi + \frac{\partial \chi}{\partial t}, \tag{121}
\]

such that \( \mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \Psi = -\partial \mathbf{A}'/\partial t - \nabla \Psi' \).

In particular, Maxwell noted that since \( \nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \nabla^2 \chi \), we can have \( \nabla \cdot \mathbf{A}' = 0 \), his eq. (75), by taking \( \nabla^2 \chi = \nabla \cdot \mathbf{A} \), his eq. (73). With this choice of the gauge function \( \chi \), the potentials \( \mathbf{A}' \) and \( \Psi' \) are in the Coulomb gauge (which Maxwell had already favored in 1861, p. 290 of [133]). The wave equation for the vector potential in the Coulomb gauge follows from eq. (119) as,

\[
\nabla^2 \mathbf{A}' - \epsilon \mu \frac{\partial^2 \mathbf{A}'}{\partial t^2} = \epsilon \mu \nabla \frac{\partial \Psi'}{\partial t} \quad \text{(Coulomb gauge).} \tag{122}
\]

\(^{87}\)Maxwell could we have added an argument for the electric field by noting that,

\[
\nabla \times \mathbf{H} = \hat{n} \times \frac{\partial \mathbf{H}}{\partial \phi} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = -\epsilon v \frac{\partial \mathbf{E}}{\partial \phi}, \quad \Rightarrow \quad \hat{n} \times \mathbf{H} = \epsilon v \mathbf{E}, \quad \text{and} \quad \hat{n} \cdot \frac{\partial \mathbf{E}}{\partial \phi} = \nabla \cdot \mathbf{E} = 0, \tag{117}
\]

and then taking the curl of Faraday’s law to find,

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mu \mathbf{H} = -\epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{118}
\]

such that \( \mathbf{E} \) is transverse to \( \hat{n} \) and \( \mathbf{B} \), and has wave propagation at speed \( v = 1/\sqrt{\epsilon \mu} \).

48
However, Maxwell stated in his eq. (78) that we find the right side of eq. (122) to be zero.

To see that $\Psi' = 0$ (and hence that $\nabla \partial \Psi / \partial t = 0$) in Maxwell’s example, one can transcribe Maxwell’s eq. (G) of sec. 65, $\nabla \cdot D = \rho_{\text{free}}$, together with eq. (E) of sec. 66 that $D = \epsilon E$, and eq. (35) of sec. 63 that $E = -\nabla \Psi - \partial A / \partial t$, as,

$$- \frac{\nabla \cdot D}{\epsilon} = -\nabla \cdot E = \nabla^2 \Psi + \frac{\partial}{\partial t} \nabla \cdot A = -\frac{\rho_{\text{free}}}{\epsilon},$$

such that in the Coulomb gauge, where $\nabla \cdot A' = 0$, we have that

$$\nabla^2 \Psi = -\frac{\rho_{\text{free}}}{\epsilon} \quad \text{(Coulomb gauge).}$$

For Maxwell’s example of plane waves, $\rho_{\text{free}} = 0$, such that $\Psi' = \int (\rho_{\text{free}} / \epsilon r) d\text{Vol} = 0$. Maxwell gave the reader little clue of this lore, although one infers that he was aware of it.

In sec. 99 Maxwell argued for a stronger result, which we now report as that for plane waves, $E = -\partial A'/\partial t$ (and $B = \nabla \times A'$) in any gauge (where the potentials are $A = A' + \nabla \chi$ and $\Psi = \Psi' - \partial \chi / \partial t$). To this author, Maxwell derivation was not convincing. However, the result follows from the condition that $\nabla \cdot E = 0$ for plane waves, and Helmholtz’ theorem (Appendix A.1.5 above), such that $E = E_{\text{rot}} = -\partial A_{\text{rot}} / \partial t$, where $\nabla \cdot A_{\text{rot}} = 0$, i.e., $A_{\text{rot}} = A'$ = the Coulomb-gauge vector potential. Then, $0 = E_{\text{irr}} = -\partial A_{\text{irr}} / \partial t - \nabla \Psi$, so while in general the potentials $A_{\text{irr}}$ and $\Psi$ can be nonzero, they do not contribute to the plane wave.

### A.24.4 In Maxwell’s Treatise [148, 149, 187, 188]

Maxwell’s discussion of the “Lorentz” force law in Arts. 598-601 of his Treatise has been reviewed in sec. 1 above. However, Maxwell’s presentation in his earlier papers of this topic (and of other aspects of his novel vision of electrodynamics) is perhaps superior to that in his Treatise.

Here, we add some comments on the electric potential, on Art. 619, and on Maxwell’s electromagnetic theory of light.

#### A.24.4.1 Articles 70-77, On Potential Functions

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88Our eq. (123) is Maxwell’s eq. (79), sec. 99, except that he had $\Psi'$ (his $\phi$) in place of $\Psi$ (and $J = \nabla \cdot A$).

89Last-minute corrections by Maxwell to sec. 99 of [137] are discussed in [256]. See also p. 203 of [278].

90In any region where $\nabla \cdot E \approx 0$, such as far from all sources of the electromagnetic fields, these fields can be deduced only from a Coulomb-gauge vector potential to a good approximation.

91On p. 300 of Whittaker’s History of the Theories of Aether and Electricity [213], one reads about Maxwell: In 1871 he returned to Cambridge as Professor of Experimental Physics; and two years later published his Treatise on Electricity and Magnetism. In this celebrated work is comprehended almost every branch of electric and magnetic theory; but the intention of the writer was to discuss the whole as far as possible from a single point of view, namely, that of Faraday; so that little or no account was given of the hypotheses which had been propounded in the two preceding decades by the great German electricians. So far as Maxwell’s purpose was to disseminate the ideas of Faraday, it was undoubtedly fulfilled; but the Treatise was less successful when considered as the exposition of its author’s own views. The doctrines peculiar to Maxwell—the existence of displacement-currents, and of electromagnetic vibrations identical with light were not introduced in the first volume, or in the first half of the second volume; and the account which was given of them was scarcely more complete, and was perhaps less attractive, than that which had been furnished in the original memoirs.

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At the end of Art. 70 of [187], Maxwell wrote: **Definition of Potential.** The Potential at a Point is the work which would be done on a unit of positive electricity by the electric forces if it were placed at the point without disturbing the electric distribution, and carried from that point to an infinite distance; or, what comes to the same thing, the work which must be done by an external agent in order to bring the unit of positive electricity from an infinite distance (or from any place where the potential is zero) to the given point.

This statement is in Vol. 1 of Maxwell’s *Treatise*, which deals only with static phenomenon. However, even in Vol. 2, Maxwell never acknowledged that this definition of potential is ill defined when time-dependent magnetic fields are involved, such that the work done depends on the path.

In Art. 73, **Potential due to any Electrical System**, Maxwell stated that the electric potential $V$ can be computed from the electric density $\rho$ (tacitly, of free charge) according to,

$$
V = \int \frac{\rho_{\text{free}}}{4\pi \epsilon r} \, d\text{Vol}.
$$

In Art. 77, **On the Equations of Laplace and Poisson**, Maxwell stated that the potential $V$ obeys Poisson’s equation,

$$
\nabla^2 V = -\frac{\rho_{\text{free}}}{\epsilon},
$$

in present notation (Maxwell defined his symbol $\nabla^2$ to be the negative of ours).

Equations (125)-(126) can be taken as a matter of definition in time-dependent examples, which corresponds to use of the Coulomb gauge. The wording of the *Treatise* is consistent throughout with these equations, i.e., with the choice of the Coulomb gauge.92

**A.24.4.2 Articles 489-490, Reaction of the Magnetic System on the Electric Circuit**

In Art. 489, Maxwell mentioned the language of Faraday, which gives importance to the number of lines of magnetic induction $\mathbf{B}$ which pass through a circuit.

In Art. 490, he considered the force on an element of an electric circuit that carries current $i$, due to a magnetic field $\mathbf{B}$, even when the circuit is flexible. He did not derive an expression for the force, but simply stated: *We may express in the language of Quaternions both the direction and the magnitude of this force by saying that it is the vector part of the result of multiplying the vector $i \, ds$, the element of the current, by the vector $\mathbf{B}$, the magnetic induction. That is, $d\mathbf{F} = i \, ds \times \mathbf{B}$.*

**A.24.4.3 Articles 502-527, Ampère’s Theory**

In Arts. 502-527, Maxwell reviewed Ampère’s theory of the forces on current elements, without mention of the view of Faraday (Art. 491). In Art. 525 he noted that Ampère’s experiments do not lead (as claimed by Ampère [62]) to a unique expression for this force,

92It may be that when Maxwell wrote in Art. 598 that $\Psi$ represents, according to a certain definition, the electric potential, he had in mind the definitions of our eqs. (125)-(126) rather than that of Art. 70.
and in Art. 526 he gave four possible forms, the first being that of Ampère, and the second he attributed to Grassmann. While we would now say that Grassmann’s form is equivalent to that given by Maxwell in Art. 490, Maxwell did not make this connection. Instead, he concluded in Art. 527: Of these four different assumptions that of Ampère is undoubtedly the best, since it is the only one with makes the forces on two elements not only equal and opposite but in the straight line which joins them.

A.24.4.4 Articles 530-531, General law of induction of currents

In Art. 530 of his Treatise [149], Maxwell considered electromagnetic induction in four different configurations, and then stated in Art. 531: The whole of these phenomena may be summed up in one law. When the number of lines of magnetic induction which pass through the secondary circuit in the positive direction is altered, an Electromotive force acts round the circuit, which is measured by the rate of decrease of the magnetic induction through the circuit.

A.24.4.5 Article 541, Faraday’s method of stating the laws of induction with reference to the lines of magnetic force

Here, Maxwell gave another statement of Faraday’s law: The total Electromotive force acting round a circuit at any instant is measured by the rate of decrease of the number of lines of magnetic force which pass through it.

Then, in Arts. 525-545 he reviewed views of Lenz, Helmholtz, W. Thomson and Weber on magnetic induction.

A.24.4.6 Articles 598-599 of Maxwell’s Treatise

In his Treatise [149], Maxwell argued that an element of a circuit (Art. 598), or a particle (Art. 599) which moves with velocity $\mathbf{v}$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ experiences an Electromotive intensity (Art. 598), i.e., a vector electromagnetic force given by eq. (B) of Art. 598 and eq. (10) of Art. 599,$^{93}$

$$\mathbf{E} = \mathbf{\Omega} \times \mathbf{B} - \mathbf{A} - \nabla \Psi,$$

(127)

where $\mathbf{E}$ is the Electromotive force, $\mathbf{\Omega}$ is the velocity $\mathbf{v}$, $\mathbf{B}$ is the magnetic field $\mathbf{B}$, $\mathbf{A}$ is the vector potential and $\Psi$ represents, according to a certain definition, the electric (scalar) potential. If we interpret Electromotive force to mean the force per charge $q$ of the particle,$^{94}$ i.e., $\mathbf{E} = \mathbf{F}/q$, then we could write eq. (127) as,

$$\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

(128)

$^{93}$This result also appeared in eq. (77), p. 343 of [133] (1861), and in eq. (D), sec. 65, p. 485, of [137] (1864), where $\mathbf{E}$ was called the electromotive force. The evolution of Maxwell’s thoughts on the “Lorentz” force are traced in Appendix XXXXXXXX below. See also [301, 327, 345].

$^{94}$In contrast to, for example, Weber [338], Maxwell did not present in his Treatise a view of an electric charge as a “particle”, but rather as a state of “displaced” aether. However, in his earliest derivation of our eq. (127), his eq. (77), p. 342 of [133], Maxwell was inspired by his model of molecular vortices in which moving particles (“idler wheels”) corresponded to an electric current (see also sec. XXXXXXXX below).

For comments on Maxwell’s various views on electric charge, see [281].

51
noting that the electric field \( \mathbf{E} \) is given (in emu) by 
\[- \frac{\partial \mathbf{A}}{\partial t} - \nabla \Psi.\]

Our equation (128) is now known as the Lorentz force,\(^{98,99}\) and it seems seldom noted that Maxwell gave this form, perhaps because he presented eq. (10) of Art. 599 as applying to an element of a circuit rather than to a charged particle. In Arts. 602-603, Maxwell discussed the Electromotive Force acting on a Conductor which carries an Electric Current through a Magnetic field, and clarified in his eq. (11), Art. 603 that the force on current density \( \mathbf{J} \) is,
\[
\mathbf{F} = \mathbf{J} \times \mathbf{B} \quad (\mathbf{F} = \mathbf{J} \times \mathbf{B}).
\]  

(129)

If Maxwell had considered that a small volume of the current density is equivalent to an electric charge \( q \) times its velocity \( \mathbf{v} \), then his eq. (11), Art. 602 could also have been written as,
\[
\frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \quad (= \mathbf{\Psi} \times \mathbf{B}),
\]  

(130)

which would have confirmed the interpretation we have given to our eq. (128) as the Lorentz force law. However, Maxwell ended his Chap. VIII, Part IV of his Treatise with Art. 603, leaving ambiguous some the meaning of that chapter.

In his Arts. 598-599, Maxwell considered a lab-frame view of a moving circuit. However, we can also interpret Maxwell’s \( \mathbf{E} \) as the electric field \( \mathbf{E}' \) in the frame of the moving circuit, such that Maxwell’s transformation of the electric field is,\(^{100}\)
\[
\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}.
\]  

(131)

The transformation (131) is compatible with both magnetic Galilean relativity, eq. (135),

\(^{95}\)This assumes that Maxwell’s \( \dot{\mathbf{A}} \) corresponds to \( \partial \mathbf{A}/\partial t \), and not to the convective derivative \( D\mathbf{A}/Dt = \partial \mathbf{A}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{A} \).

\(^{96}\)Maxwell never used the term electric field as we now do, and instead spoke of the (vector) electromotive force or intensity (see Art. 44 of [148]). The distinction is important only when discussing a moving medium, as in Arts. 598-599.

\(^{97}\)The relation \( \mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \Psi \) for the electric field holds in any gauge. However, Maxwell always worked in the Coulomb gauge, where \( \nabla \cdot \mathbf{A} = 0 \), as affirmed, for example, in Art. 619. Maxwell was aware that, in the Coulomb gauge, the electric scalar potential \( \Psi \) is the instantaneous Coulomb potential, obeying Poisson’s equation at any fixed time, as mentioned at the end of Art. 783. The discussion in Art. 783 is gauge invariant until the final comment about \( \nabla^2 \Psi \) (in the Coulomb gauge). That is, Maxwell missed an opportunity to discuss the gauge advocated by Lorenz [141], to which he was averse [368].

\(^{98}\)Lorentz actually advocated the form \( \mathbf{F} = q (\mathbf{D} + \mathbf{v} \times \mathbf{H}) \) in eq. (V), p. 21, of [195], although he seems mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [210]. That is, Lorentz considered \( \mathbf{D} \) and \( \mathbf{H} \), rather than \( \mathbf{E} \) and \( \mathbf{B} \), to be the microscopic electromagnetic fields.

\(^{99}\)It is generally considered that Heaviside first gave the Lorentz force law (128) for electric charges in [181], but the key insight is already visible for the electric case in [170] and for the magnetic case in [175].

\(^{100}\)A more direct use of Faraday’s law, without invoking potentials, to deduce the electric field in the frame of a moving circuit was made in sec. 9-3, p. 160, of [266], which argument appeared earlier in sec. 86, p. 398, of [205]. An extension of this argument to deduce the full Lorentz transformation of the electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) is given in Appendix C of [387].
and the low-velocity limit of special relativity, eq. (137). These two versions of relativity differ as to the transformation of the magnetic field. In particular, if $B = 0$ while $E$ were due to a single electric charge at rest (in the unprimed frame), then magnetic Galilean relativity predicts that the moving charge/observer would consider the magnetic field $B'$ to be zero, whereas it is nonzero according to special relativity.

These themes were considered by Maxwell in Arts. 600-601, under the heading: On the Modification of the Equations of Electromotive Intensity when the Axes to which they are referred are moving in Space, which we review in sec. XXXX below.

Details

In Art. 598, Maxwell started from the integral form of Faraday’s law, that the (scalar) electromotive force $E$ in a circuit is related to the rate of change of the magnetic flux through it by his eqs. (1)-(2),

$$E = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} \int B \cdot dS = -\frac{d}{dt} \oint A \cdot dl = -\oint \left( \frac{\partial A}{\partial t} + (v \cdot \nabla)A \right) \cdot dl,$$

(138)

where the last form, involving the convective derivative, holds for a circuit that moves with velocity $v$ with respect to the lab frame. In his discussion leading to eq. (3) of Art. 598,

\[ \tag{132} \]

\[ \tag{133} \]

\[ \tag{134} \]

\[ \tag{135} \]

For comparison, the low-velocity limit of special relativity has the transformations,

$$\rho_s' \approx \rho_s - \frac{v}{c^2} \cdot J_s, \quad J'_s \approx J_s - \rho_s v, \quad V'_s \approx V_s - v \cdot A_s, \quad A'_s \approx A_s - \frac{v}{c^2} V_s,$$

(136)

\[ \tag{137} \]

In Maxwell’s notation, $E = \mathcal{E}$, $p = \Phi_m$, $(F, G, H) = A$, $(F dx/ds + G dy/ds + H dz/ds) ds = A \cdot dl$, $(dx/dt, dy/dt, dz/dt) = v$, and $(a, b, c) = B$. 

101 The notion of Galilean electrodynamics, consistent with Galilean relativity, i.e., the coordinate transformation $x' = x - vt$, $y' = y$, $z' = z$, $t' = t$, seems to have been developed only in 1973 [285]. The term Galilean relativity was first used in 1911 [214]. In Galilean electrodynamics there are no electromagnetic waves, but only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics as a whole. In contrast, electromagnetic waves can exist in the low-velocity approximation to special relativity, and, of course, propagate in vacuum with speed $c$.

In Galilean electrodynamics the symbol $c$ does not represent the speed of light (as light does exist in this theory), but only the function $1/\sqrt{\mu_0\varepsilon_0}$ of the (static) permittivity and permeability of the vacuum.

In fact, there are two variants of Galilean electrodynamics:

1. **Electric Galilean relativity** (for weak magnetic fields) in which the transformations between two inertial frames with relative velocity $v$ are (sec. 2.2 of [285]),

\[ \tag{132} \]

\[ \tag{133} \]

where $\rho$ and $J$ are the electric charge and current densities, $V$ and $A$ are the electromagnetic scalar and vector potentials, $E = -\nabla V - \partial A/\partial t$ is the electric field, $B = \nabla \times A$ is the magnetic (induction) field.

2. **Magnetic Galilean relativity** (for weak electric fields, sec. 2.3 of [285]) with transformations,

\[ \tag{134} \]

\[ \tag{135} \]

For comparison, the low-velocity limit of special relativity has the transformations,

\[ \tag{136} \]

\[ \tag{137} \]
Maxwell argued for the equivalent of use of the vector-calculus identity,
\[ \nabla(v \cdot A) = (v \cdot \nabla)A + (A \cdot \nabla)v + v \times (\nabla \times A) + A \times (\nabla \times v), \tag{139} \]
which implies for the present case,
\[ (v \cdot \nabla)A = -v \times (\nabla \times A) + \nabla(v \cdot A) = -v \times B + \nabla(v \cdot A), \tag{140} \]
\[ \mathcal{E} = \oint \left( v \times B - \frac{\partial A}{\partial t} \right) \cdot dl = \oint \mathcal{E} \cdot dl, \tag{141} \]
since \( \oint \nabla(v \cdot A) \cdot dl = 0. \) Our eq. (141) corresponds to Maxwell’s eqs. (4)-(5), from which we infer that the vector electromotive intensity \( \mathcal{E} \) has the form,
\[ \mathcal{E} = v \times B - \frac{\partial A}{\partial t} - \nabla V, \tag{142} \]
for some scalar field \( V \) (Maxwell’s \( \Psi \)), that Maxwell identified with the electric scalar potential.

If it were clear that \( V(\Psi) \) is indeed the electric scalar potential, then Maxwell should be credited with having “discovered” the “Lorentz” force law. However, Helmholtz [eq. (5d), p. 309 of [150] (1874)], Larmor [p. 12 of [166] (1884)], Watson [p. 273 of [180] (1888)], and J.J. Thomson [in his editorial note on p. 260 of [188] (1892)] argued that our eq. (140) leads to,
\[ \mathcal{E} = \oint \left[ v \times B - \frac{\partial A}{\partial t} - \nabla (v \cdot A) \right] \cdot dl, \tag{143} \]
so Maxwell’s eq. (D) of Art. 598 and eq. (10) of Art. 599 should really be written as,
\[ \mathcal{E} = v \times B - \frac{\partial A}{\partial t} - \nabla (\Psi + v \cdot A), \tag{144} \]
where \( \Psi \) is the electric scalar potential.\(^{103} \) It went unnoticed by these authors that use of eq. (144) rather than (142) would destroy the elegance of Maxwell’s argument in Arts. 600-601 (discussed in sec. XXXXX below), as well as that Maxwell’s earlier derivations of our eq. (142), on pp. 340-342 of [133] and in secs. 63-65 of [137],\(^{104} \) used different methods which did not suggest the possible presence of a term \(-\nabla(v \cdot A)\) in our eq. (142). However, the practical effect of these doubts by illustrious physicists was that Maxwell has not been credited for having deduced the “Lorentz” force law, which became generally accepted only in the 1890’s.

The view of this author is that Maxwell did deduce the “Lorentz” force law, although in a manner that was “not beyond a reasonable doubt”.

\(^{103} \)A possible inference from eq. (144) is that the Lorentz force law should actually be,
\[ \mathbf{F} = q \left[ \mathbf{E} + \mathbf{v} \times \mathbf{B} - \nabla (\mathbf{v} \cdot \mathbf{A}) \right] = q \left[ \mathbf{E} + (\mathbf{v} \cdot \nabla) \mathbf{A} \right] = -q \left( \nabla V + \frac{dA}{dt} \right), \tag{145} \]
Some debate persists on this issue, as discussed, for example, in [351] and references therein.
\(^{104} \)These derivations of Maxwell are reviewed in Appendix XXXXX below.
A.24.4.7 Articles 600-601 of Maxwell’s *Treatise*

In Art. 600, Maxwell considered a moving point with respect to two coordinate systems, the lab frame where \( x = (x, y, z) \), and a frame moving with uniform velocity \( \mathbf{v} \) respect to the lab in which the coordinates of the point are \( x' = (x', y', z') \), with quantities in the two frames related by Galilean transformations. Noting that a force has the same value in both frames, Maxwell deduced that the “Lorentz” force law has the same form in both frames, provided the electric scalar potential \( V' \) in the moving frame is related to lab-frame quantities by,

\[
V' = V - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A}.
\] (146)

This is the form according to the low-velocity Lorentz transformation (137), and also to the transformations of magnetic Galilean electrodynamics (135), which latter is closer in spirit to Maxwell’s arguments in Arts. 600-601.

**Details**

In Art. 600, Maxwell consider both translations and rotations of the moving frame, but we restrict our discussion here to the case of translation only, with velocity \( \mathbf{v} = (u, v, w) = (\delta x/\delta t, \delta y/\delta t, \delta z/\delta t) \) with respect to the lab.\(^{105}\) Maxwell labeled the velocity of the moving point with respect to the moving frame by \( \mathbf{u}' = d\mathbf{x}'/dt' \), while he called labeled its velocity with respect to the lab frame by \( \mathbf{u} = dx/dt \). Then, Maxwell stated the velocity transformation to be, eq. (1) of Art. 600,\(^{106}\)

\[
\mathbf{u}' = \mathbf{u} - \mathbf{v}, \quad \text{i.e.,} \quad \mathbf{u} = \mathbf{v} + \mathbf{u}' \quad \left( \frac{dx}{dt} = \frac{\delta x}{dt} + \frac{dx'}{dt'} \right),
\] (147)

which corresponds to the Galilean coordinate transformation,

\[
x' = x - vt, \quad t' = t, \quad \nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.
\] (148)

Maxwell next considered the transformation of the time derivative of the vector potential \( \mathbf{A} = (F, G, H) \) in his eq. (3), Art. 600,

\[
\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} \quad \left( \frac{dF'}{dt'} = \frac{dF}{dx} \frac{\delta x}{\delta t} + \frac{dF}{dy} \frac{\delta y}{\delta t} + \frac{dF}{dz} \frac{\delta z}{\delta t} + \frac{dF}{dt} \right),
\] (149)

which tacitly assumed that \( \mathbf{A}' = \mathbf{A} \), and hence that \( \mathbf{B}' = \mathbf{B} \).\(^{107}\) In eqs. (4)-(7) of Art. 600, Maxwell argued for the equivalent of use of the vector-calculus identity (139), which implies

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\(^{105}\)For discussion of electrodynamics in a rotating frame (in which one must consider “fictitious” charges and currents, see, for example, [367].

\(^{106}\)Equation (2) of Art. 600 refers to rotations of a rigid body about the origin of the moving frame.

\(^{107}\)While this assumption does not correspond to the low-velocity Lorentz transformation of the field between inertial frames, it does hold for the transformation from an inertial frame to a rotating frame. Faraday considered rotating magnets in [117], and in sec. 3090, p. 31, concluded that No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it. That is, \( \mathbf{B}' = \mathbf{B} \) relates the magnetic field in an inertial and a rotating frame. Possibly, this might have led Maxwell to infer a similar result for a moving inertial frame as well.
eq. (140), and hence that,

\[
\frac{\partial A'}{\partial t'} = \frac{\partial A}{\partial t} - v \times B + \nabla (v \cdot A).
\]  

(150)

Then, in eqs. (8)-(9) of Art. 600, Maxwell combined his eq. (B) of Art. 598 with our eqs. (147) and (150) to write the electromotive force \( \mathbf{E} \) as, in the notation of the present section,

\[
\mathbf{E} = \frac{u}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial A}{\partial t} - \nabla V = \frac{u'}{c} \times \mathbf{B} - \frac{1}{c} \frac{\partial A'}{\partial t'} - \nabla \left( V - \frac{v}{c^2} \cdot \mathbf{A} \right).
\]  

(151)

Finally, since a force has the same value in two frames related by a Galilean transformation, Maxwell inferred that the electromotive force \( \mathbf{E}' \) in the moving frame can be written as

\[
\mathbf{E}' = u' \times \mathbf{B} - \frac{\partial A'}{\partial t'} - \nabla' \left( \Psi + \Psi' \right) = u' \times \mathbf{B}' + \mathbf{E}',
\]  

(152)

where the electric scalar potential \( V' \) in the moving frame is related to lab-frame quantities by,

\[
V' = V - \frac{v}{c^2} \cdot \mathbf{A} \quad (= \Psi + \Psi').
\]  

(153)

This is the form according to the low-velocity Lorentz transformation (137), and also to the transformations of magnetic Galilean electrodynamics (135), which latter is closer in spirit to Maxwell’s arguments in Arts. 600-601.

Further, the force \( \mathbf{F}' \) on a moving electric charge \( q \) in the moving frame is given by the “Lorentz” form,

\[
\mathbf{F}' = q \left( \mathbf{E}' + u' \times \mathbf{B}' \right),
\]  

(154)

which has the same form eq. (128) in the lab frame. As Maxwell stated at the beginning of Art. 601: It appears from this that the electromotive intensity is expressed by a formula of the same type, whether the motions of the conductors be referred to fixed axes or to axes moving in space.\(^{108}\)

**A.24.4.8 Articles 602-603, The “Biot-Savart” Force Law**

In articles 602-603, Maxwell considered the force on a current element \( I \, d\mathbf{l} \) in a circuit at rest in a magnetic field \( \mathbf{B} \), and deduced the “Biot-Savart” form,\(^{109}\)

\[
d\mathbf{F} = I \, d\mathbf{l} \times \mathbf{B}, \quad \mathbf{F} = \oint I \, d\mathbf{l} \times \mathbf{B}.
\]  

(155)

\(^{108}\)Maxwell’s equations in Art. 600 do not appear to be fully consistent with this “relativistic” statement, as he noted in Art. 601. That is, his eq. (9), Art. 600, is the equivalent to \( \mathbf{E}' = u' \times \mathbf{B} - \frac{\partial A'}{\partial t'} - \nabla' \left( \Psi + \Psi' \right) \), where \( \Psi \) is the electric scalar potential in the lab frame, and \( \Psi' = -\frac{v}{c^2} \cdot \mathbf{A} \) (a lab-frame quantity) according to Maxwell’s eq. (6), Art. 600. Maxwell did seem to realize that in addition to expressing the electromotive intensity \( \mathbf{E}' \) in the moving frame in terms of moving-frame quantities [our eq. (152)], as well as in terms of lab-frame quantities (our eq. (127), Maxwell’s eq. (10), Art. 599), he had also deduced the relation of the electric scalar potential \( V' \) in the moving frame to the lab-frame quantities \( \Psi + \Psi' \), as in our eq. (153).

\(^{109}\)Maxwell had argued for this in his eqs. (12)-(14), p. 172, of [132] (1861), which is the first statement of the “Biot-Savart” force law in terms of a magnetic field. Biot and Savart [25] discussed the force on a magnetic “pole” due to an electric circuit, and had no concept of the magnetic field.
Some other comments on Arts. 602-603 were given around eqs. (129)-(130) above.

A.24.4.9 Article 619, Quaternion Expressions for the Electromagnetic Equations

Article 619 of [149] is meant as a summary of Maxwell’s theory, but the transcription of material in Arts. 598-603 was awkward. This was noticed by FitzGerald [164], who attempted to improve the story, but perhaps did not succeed. FitzGerald’s comments were incorporated in Art. 619 of the 3rd edition [188] of the Treatise, but some typos were also introduced.

Article 619 mentions that the vector potential is subject to the condition \( \nabla \cdot A = 0 \), which is a choice, not a requirement, and corresponds to the use of the Coulomb gauge by Maxwell.\(^\text{110}\) We have previously remarked how with this choice the scalar potential \( \Psi \) obeys the instantaneous Poisson equation \( \nabla^2 \Psi = -e/\varepsilon \), where \( e \) is the (free) electric-charge density, as acknowledged by Maxwell in Art. 783.

The results of Arts. 598-599, Maxwell’s eq. (B) and our eq. (127), are then reproduced in Art. 619.

However, the next sentence in Art. 619 is problematic: The equations (C) of mechanical force (Art. 603), of which the first is,

\[
X = cv - bw - e \frac{d\Psi}{dx} - m \frac{d\Omega}{dx}, \tag{156}
\]

become,

\[
\mathfrak{J} = \mathfrak{J} \times \mathfrak{B} - e \nabla \Psi - m \nabla \Omega. \tag{157}
\]

\(^{110}\)Maxwell’s preference that \( \nabla \cdot A = 0 \) was indicated already in eq. (57), p. 290 of [133], but without justification. He expressed this preference again in Arts. 616-617, where the context is magnetostatics. \(^{111}\)This is problematic in that there is no mechanical force on the electric-field part of displacement \( D = E + \varepsilon_0 P \), where \( P \) is the volume density of electric dipoles (a concept not recognized by Maxwell, and only introduced much later by Lorentz). \(^{112}\)

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\(^{110}\)In the 3rd edition [188] of the Treatise, the relation \( \mathfrak{J} = \mathfrak{R} + \mathfrak{D} \) was mistyped as \( \mathfrak{C} = \mathfrak{R} + \mathfrak{D} \).

\(^{111}\)Maxwell only regarded the relation between \( D \) and \( E \) as \( D = \varepsilon E \), where \( \varepsilon \) is now called the (relative) dielectric constant and/or the (relative) permittivity. See Art. 111 of [187] for Maxwell’s use of the term polarization.

In 1885, Heaviside introduced the concept of an electret as the electrical analog of a permanent magnet [171], and proposed that the electrical analog of magnetization (density) be called electrization. He did not propose a symbol for this, nor did he write an equation such as \( D = E + \varepsilon_0 P \).

The density of electric dipoles was called the polarization by Lorentz (1892) in sec. 102, p. 465 of [186], and assigned the symbol \( M \).

Larmor (1895), p. 738 of [194], introduced the vector \( (f', g', h') \) for what is now written as the polarization density \( P \), and related it to the electric field \( F = (P, Q, R) \) as \( (f', g', h') = (K - 1)(P, Q, R) / 4\pi \), i.e., \( P = (\varepsilon - 1) E / 4\pi = (D - E) / 4\pi \). Larmor’s notation was mentioned briefly on p. 91 of [198] (1897).

The symbol \( M \) for dielectric polarization was changed to \( P \) by Lorentz on p. 263 of [202] (1902), and a relation equivalent to \( D = E + \varepsilon_0 P \) was given in eq. (22), p. 265. See also p. 224, and eq. (147), p. 240 of [203] (1903), which latter subsequently appeared as eq. (142), p. 155 of the textbook [205] (1904) by Abraham.
Maxwell added a term $-m \nabla \Omega$ to his expression for the mechanical force. From the last sentence in Art. 619, we infer he had in mind the special case of permanent magnetism, described by a volume density $\mathcal{J}$ ($\mathbf{M}$) of magnetic dipoles, with a corresponding volume density $m = \nabla \cdot \mathcal{J} (= \nabla \cdot \mathbf{M})$, such that the magnetic field $\mathbf{H} = -\nabla \Omega$ can be deduced from a magnetic scalar potential $\Omega$. This special case is not strictly compatible with the existence of an electric current density $\mathcal{J}$ in the term $\mathbf{J} \times \mathbf{B}$.

Maxwell also included a term $-e \nabla \Psi$ in his expression for the mechanical force. This is clearly not the general case for electric mechanical forces on an electric charge density $e$, but would apply if the charge density were static, and the corresponding electric field related to a scalar potential, $\mathbf{E} = -\nabla \Psi$. FitzGerald (1883) [164] noted that in general the electric force on charge density $e$ is due to the total electric field $-\nabla \Psi - \partial \mathbf{A} / \partial t$, and so suggested that $-e \nabla \Psi$ be replaced by $e \mathbf{E}$. This would be valid if Maxwell’s $\mathbf{E}$ represented the (lab-frame) electric field. However, the point of Arts. 598-599 was that the symbol $\mathbf{E}$ does not represent the (lab-frame) electric field, but rather the total electric force in case of moving charge. Nonetheless, FitzGerald’s suggestion was implemented in Art. 619 of the 3rd edition of Maxwell’s Treatise [188], which has the effect that the revised expression for the mechanical force includes $2 \mathbf{J} \times \mathbf{B}$.

A.24.4.10 Articles 781-805, Electromagnetic Theory of Light

Article 783

In Art. 783 of [149], Maxwell set the stage for discussion of electromagnetic waves other than plane waves, and made a slight generalization of his discussion of the electromagnetic theory of light in secs. 91-99 of [137], by considering currents in a medium with electrical conductivity $\sigma$. Then, eqs. (1) and (2) of Art. 783 combine to give the second line of our eq. (115) becomes,

$$\mathbf{J}_{\text{total}} = \sigma \mathbf{E} + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} = -\left( \sigma + \epsilon \frac{\partial}{\partial t} \right) \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \Psi \right),$$

(158)

which generalizes the second line of our eq. (115), and our wave equation (119) becomes,

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\mu \left( \sigma + \epsilon \frac{\partial}{\partial t} \right) \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \Psi \right),$$

(159)

which is Maxwell’s eq. (6) of Art. 783, noting that in the Treatise, his symbol $\nabla^2$ is the negative of ours. Maxwell had also deduced this relation, but for the case of a nonconducting medium, as eq. (68), sec. 94 of [137].

In [137], Maxwell then took to the curl of eq. (159) (with $\sigma = 0$) to find a wave equation for the magnetic field, with wavespeed $1/\sqrt{\epsilon \mu}$.

In Art. 783, Maxwell took the divergence of his eq. (6), our eq. (159), to find his eq. (8), which we write as,

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113 The computation of mechanical forces due to magnetism is subject to ambiguities that persist in the literature to this day. For further comments by the author on this topic, see [355].

114 This unfortunate “improvement” must have contributed to the impression in the late 1800’s that Maxwell’s theory of electromagnetism was hard to follow.
Maxwell’s next sentence included the phrase: \( \nabla^2 \Psi \) which is proportional to the volumedensity of free electricity, as holds in electrostatics. Here, Maxwell supposes that even in time-dependent examples, the scalar potential obeys \( \nabla^2 \Psi = -\rho_{\text{free}}/\epsilon_0 \), as he discussed in Art. 77 for the static case. We could say that this assumption presumes use of the Coulomb gauge \( (\nabla \cdot \mathbf{A} = 0) \), but in Art. 783 Maxwell appeared to deduce the Coulomb-gauge condition from his assumption. That is, he stated that in a nonconducting medium any free electricity is at rest, such that \( \nabla^2 \Psi \) in independent of \( t \), and hence \( J \) \( (= \nabla \cdot \mathbf{A}) \) must be a linear function of \( t \), or constant, or zero.\(^{115}\)

Maxwell concluded Art. 783 with the statement: we may leave \( J \) \( (= \nabla \cdot \mathbf{A}) \) and \( \Psi \) out of account when considering periodic disturbances. This claim happens to be true for plane waves, as noted at the end of Appendix A.3.10 above, but is not so in general.\(^{116}\)

In Art. 783, Maxwell appears to wish to show that waves of the vector potential also propagate with speed \( 1/\sqrt{\epsilon \mu} \). In a sense, eq. (159) already shows this, if we rewrite it as,

\[
\nabla^2 \mathbf{A} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla (\nabla \cdot \mathbf{A}) + \epsilon \mu \frac{\partial \nabla \Psi}{\partial t} + \sigma \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \Psi \right),
\]

which linear differential equation indicates that at least part of the vector potentials propagates at speed \( 1/\sqrt{\epsilon \mu} \). But, it seems that Maxwell wished to show that there could be no other part of the vector potential that propagates with a different speed.\(^{117}\)

We now know that this cannot be shown, in that one can adopt the so-called velocity gauge (see, for example, sec. 2.3.1 of [392]) in which the scalar potential \( \Psi \) propagates with any specified speed \( v \), and the corresponding vector potential \( \mathbf{A} \) has a term that propagates at speed \( 1/\sqrt{\epsilon \mu} \), and another of the form \( \nabla \phi \) which propagates at speed \( v \), such that the electric

\(^{115}\)The case of a vector potential that is a linear function of time finds application in electrostatics, where one can set \( \Psi = 0 \) and \( \mathbf{A} = -\mathbf{E} t \), which is the so-called Gibb’s gauge [197, 380]. Of course, there is no wave propagation in this case.

\(^{116}\)See, for example, Prob. 2 of [348], where eqs. (91)-(92) give the Coulomb-gauge potentials for an oscillating (Hertzian) electric dipole \( \mathbf{p} = \mathbf{p}_0 e^{-i\omega t} \) at the origin as,

\[
\Psi = \frac{\mathbf{p}_0 \cdot \hat{\mathbf{r}} e^{-i\omega t}}{4\pi \epsilon_0 r^2} \quad \text{(Coulomb gauge)},
\]

\[
\mathbf{A} = -\frac{i}{k} \mathbf{E} - \frac{i}{k} \nabla \Psi \quad \text{(Coulomb gauge)}
\]

\[
= -ik \hat{\mathbf{r}} \times (\mathbf{p}_0 \times \hat{\mathbf{r}}) \frac{e^{i(kr-\omega t)}}{4\pi \epsilon_0 r^2} + [\mathbf{p}_0 - 3(\mathbf{p}_0 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}] \left[ \frac{e^{i(kr-\omega t)}}{4\pi \epsilon_0 kr^2} + \frac{e^{i(kr-\omega t)} - e^{-i\omega t}}{4\pi \epsilon_0 kr^3} \right].
\]

Note how the (periodic) vector potential (162) consists of a part that propagates with speed \( \omega/k = c \), and a part that propagates “instantaneously”.

The first line of eq. (162) holds for any Coulomb-gauge vector potential of angular frequency \( \omega \).

An unpublished manuscript by Maxwell from 1873 [147], probably inspired by Sellmeier [146], contained the statement: The vibrations of molecules which have definite periods, and which produce emission and absorption of particular kinds of light, are due to forces between the parts of the molecule... Unfortunately, Maxwell did not relate this phenomenon to oscillating electric dipoles in his electromagnetic theory.

\(^{117}\)Apparently, Maxwell did not consider instantaneous action at a distance as wave propagation.
field \( \mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \Psi \) and the magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \) propagate only with speed \( 1/\sqrt{\varepsilon \mu} \).\(^{118}\) Of course, in the velocity gauge, \( \nabla^2 \Psi \) is not proportional to \( \rho_{\text{free}} \), so Maxwell’s apparent assumption in Art. 783 of this relation excluded use of a velocity gauge, except for the Coulomb gauge (with \( v = \infty \)).

Article 784

In Art. 783, Maxwell’s eq. (9) gives the wave equation for the vector potential in the Coulomb gauge (or the physically trivial variants) and in a nonconducting medium.

Article 785

Article 785 is interesting in that Maxwell considered spherical waves from a localized source, and noted that a distant observer detects wave associated with earlier behavior at the source. However, Maxwell did not relate this behavior to the retarded potentials of Lorenz [141] (1867), to which Maxwell was averse.\(^{119}\)

Articles 790-791

In sec. 95 of [137], Maxwell may have left the reader with the impression that only the magnetic field, and not the electric field, participates in waves. This possible misimpression was corrected in Arts. 790-791 of [149], which included the Fig. 66 below, illustrating the in-phase oscillations of \( \mathbf{E} \) and \( \mathbf{B} \) for a linearly polarized plane wave (in Gaussian units, where \( E = B \) for a plane wave in vacuum).

Article 798

In Art. 798, Maxwell considered a conducting medium, but (tacitly) with no free charge. Then, (in the Coulomb gauge) \( \nabla \cdot \mathbf{A} = 0 \), and \( \partial \Psi / \partial t = 0 \), so Maxwell’s eq. (6), Art. 783, our eq. (159) becomes eq. (2) of Art. 798,

\[
\nabla^2 \mathbf{A} - \varepsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu \sigma \frac{\partial \mathbf{A}}{\partial t} = 0,
\]

(164)

Maxwell noted that this wave equation implies waves that are exponentially damped in space over a characteristic distance \( 1/p \), now called the skin depth.

A.24.5 In Note on the Electromagnetic Theory of Light [139]

In 1868, Maxwell published a paper whose second part was titled Note on the Electromagnetic Theory of Light [139]. Although this paper did not bear on the issue of special relativity, we

\(^{118}\)The Lorenz gauge, where \( \nabla \cdot \mathbf{A} = -\varepsilon \mu \partial \Psi / \partial t \), is the velocity gauge with \( v = 1/\sqrt{\varepsilon \mu} \).

\(^{119}\)See, for example, [368], and Appendix A.24.5.1 below.
include a few remarks for completeness.\footnote{Maxwell also wrote An Elementary Treatise on Electricity, published posthumously in 1881 [178], which considered only electro- and magnetostatics. For discussion of how this work illustrates Maxwell’s thinking on electromagnetism, see [278].}

\section*{A.24.5.1 Retarded Potentials}

The 1868 paper is the only place where Maxwell mentioned the retarded potentials of Riemann [140] and Lorenz [141], to which he objected that they lead to violations of Newton’s third law in electromagnetism (as does Maxwell’s theory as well; see, for example, [235]), and also to nonconservation of energy. The latter objection was based on a misunderstanding, as reviewed by the author in [368].

\section*{A.24.5.2 Displacement Current}

In 1882, FitzGerald closed his paper [162] with: It may be worth while remarking, that no effect except light has ever yet been traced to the displacement-currents assumed by Maxwell in order to be able to assume all currents to flow in closed circuits. It has not, as far as I am aware, been ever actually demonstrated that open circuits, such as Leyden-jar discharges, produce exactly the same effects as closed circuits; and until some such effect of displacement-currents is observed, the whole theory of them will be open to question.

Indeed, in [137, 149, 134], Maxwell discussed his concept of displacement current primarily in relation to his theory of light, so it is noteworthy that in [139] he included mention of its effect in circuits with capacitors:

\begin{quote}
Theorem D.—When the electric displacement increases or diminishes, the effect is equivalent to that of an electric current in the positive or negative direction.

Thus, if the two conductors in the last case are now joined by a wire, there will be a current in the wire from A to B.

At the same time since the electric displacement in the dielectric is diminishing, there will be an action electromagnetically equivalent to that of an electric current from B to A through the dielectric.

According to this view, the current produced in discharging a condenser is a complete circuit, and might be traced within the dielectric itself by a galvanometer properly constructed. I am not aware that this has been done, so that this part of the theory, though apparently a natural consequence of the former, has not been verified by direct experiment. The experiment would certainly be a very delicate and difficult one.
\end{quote}

\section*{A.24.5.3 Wave Equations}

Perhaps his discussion of perceived difficulties with retarded potentials sensitized Maxwell to the desirability of a deduction of a wave equation for electromagnetism that did not invoke potentials. This was provided for the magnetic field $\mathbf{B}$ in the latter part of [142].
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Mr. Arago orally communicates the results of some experiments that he has conducted on the influence that metals and many other substances exert on a magnetic needle, which has the effect of rapidly reducing the amplitude of the oscillations without altering significantly their duration. He promises, on this subject, a detailed memoir.


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