

“Hidden” Momentum in an e^+e^- Pair

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1 Problem

Discuss the momentum (mechanical, field, and total) of an e^+e^- pair in the classical limit, supposing the particles move in a circle at opposite ends of a diameter.

Take into account that the electron and positron have rest mass m_e , electric charge of magnitude e and magnetic moments of approximate magnitude $e\hbar/2m_e c$ (in Gaussian units), where \hbar is Planck’s constant divided by 2π , and c is the speed of light in vacuum.^{1,2}

2 Solution

2.1 Thomson on the Field Momentum of a Charge and a Magnet

In 1891, J.J. Thomson [9] first enunciated what has since become the standard expression for the density of momentum stored in electromagnetic fields in vacuum,

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}. \quad (1)$$

In 1904, he argued (p. 348 of [10], see also [11, 12]) that the total field momentum,

$$\mathbf{P}_{\text{EM}} = \int \mathbf{p}_{\text{EM}} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \quad (2)$$

is equal, assuming the magnetic field to be due to electrical currents (and that electromagnetic waves can be neglected), to the form,

$$\mathbf{P}_{\text{EM}}^{(\text{Maxwell})} = \int \frac{\rho \mathbf{A}^{(C)}}{c} d\text{Vol}, \quad (3)$$

¹That electrons have intrinsic angular momentum (spin) and an intrinsic magnetic moment was first proposed by Uhlenbeck and Goudsmit (1925) [1], and was deduced as a consequence of Dirac’s (quantum) theory of the electron (1928) [2]. That theory included nominally negative-energy states, which were reinterpreted as positive-energy “electrons” that have positive charge by Dirac (1931) [3] (in a paper better known for introducing the quantum condition $ep/c = n\hbar/2$ if magnetic monopoles p exist). The experimental discovery of positive electrons was reported by Anderson (1933) [4], who named them **positrons** (apparently following a suggestion by a referee). Quantum e^+e^- states were first discussed by Mohorovičić (1934) [5], given the name **positronium** by Ruark (1945) [6], and first detected by Deutsch (1951) [7].

²This problem on “classical positronium” was considered in Appendix B of [8], with neglect of the magnetic moments.

given by Maxwell [13, 14], who built this concept on Faraday’s electrotonic state [15], where ρ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [16]).^{3,4}

In 1904, Thomson [10, 11, 12] illustrated the concept of electromagnetic field momentum for several examples of systems at rest, including a small magnet with magnetic dipole moment \mathbf{m} together with an electric charge q well outside the magnet. He computed that if the magnetic field is due to electrical currents (Ampèrian magnetic moment), then the field momentum of the system is

$$\mathbf{P}_{\text{EM}} = \frac{q\mathbf{A}}{c} = q\frac{\mathbf{m} \times \hat{\mathbf{r}}}{cr^2} = \frac{\mathbf{E} \times \mathbf{m}}{c}, \quad (8)$$

where $\hat{\mathbf{r}}$ points from the magnetic dipole to the charge, such that $\mathbf{E} = -q\hat{\mathbf{r}}/r^2$ is the electric field at the magnetic dipole due to the electric charge. In contrast, if the magnetic dipole

³Some of Thomson’s thoughts on field momentum are traced in [17].

Thomson [9] argued that a sheet of electric displacement \mathbf{D} (parallel to the surface) which moves perpendicular to its surface with velocity \mathbf{v} must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\nabla \times \mathbf{H} = (1/c)\partial\mathbf{D}/\partial t$. Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [9],

$$\mathbf{p}_{\text{EM}}^{(\text{Thomson})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}. \quad (4)$$

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [18], adding the remark (p. 9) that the momentum density (4) is closely related to the Poynting vector [19, 20],

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \quad (5)$$

Thomson argued, in effect, that the field momentum density (4) is related by $\mathbf{p}_{\text{EM}} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$.

Variants of this argument were given by Heaviside in 1891, sec. 45 of [21], and much later in sec. 18-4 of [22], where it is noted that Faraday’s law, $\nabla \times \mathbf{E} = -(1/c)\partial\mathbf{B}/\partial t$, combined with the Maxwell equation for \mathbf{H} implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted in sec. 265 of [23].

The form (4) was also used by Poincaré in 1900 [24], following Lorentz’ convention [25] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector be $(c/4\pi)\mathbf{D} \times \mathbf{H}$. In 1903 Abraham [26] argued for

$$\mathbf{p}_{\text{EM}}^{(\text{Abraham})} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}, \quad (6)$$

and in 1908 Minkowski [27] advocated the form

$$\mathbf{p}_{\text{EM}}^{(\text{Minkowski})} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}. \quad (7)$$

Minkowski, like Poynting [19], Heaviside [20] and Abraham [26], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [27]. For some remarks on the “perpetual” Abraham-Minkowski debate see [28].

⁴Thomson did not relate the momentum density (2) to Maxwell’s argument that radiation pressure P of light (sec. 792 of [14]) is equal to its energy density u , $P = u = D^2/4\pi = H^2/\pi$, eq. (2), until 1904 (p. 355 of [10]) when he noted that $P = F/A = c p_{\text{EM}} = D^2/4\pi = H^2/4\pi$ for fields moving with speed c in vacuum, for which $D = H$. Possibly, Thomson delayed publishing the relation of radiation pressure to his expression (4) until he could demonstrate its equivalence to Maxwell’s form (3). For other demonstrations of this equivalence, see Appendix B of [38], and [29].

were due to a pair of opposite, true (Gilbertian) magnetic poles, Thomson noted that the field momentum would be zero.

Thomson was somewhat aware that it would be unusual for a system “at rest” to have nonzero total momentum, and commented (p. 348 of [10]) on the ambiguity as to whether the charge or the magnet would move if the magnetic moment were to vanish. This difficulty went largely unnoticed until 1967 [30, 31], leading Shockley [32] to conclude that the system contains a **hidden** momentum that had been overlooked in the (macroscopic) analyses, such that the total momentum of a system “at rest” is indeed zero.⁵

2.2 Field Momentum of an Electron and Positron at Rest

We infer from eq. (8) that if an electron and positron could be at rest in each other’s electromagnetic fields, then their total field momentum would be

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{E} \times (\mathbf{m}_+ + \mathbf{m}_-)}{c} \quad (e^+e^- \text{ at rest}), \quad (9)$$

where \mathbf{E} is the electric field of the electron at the position of the positron (which equals the field of the positron at the position of the electron), and \mathbf{m}_+ and \mathbf{m}_- are the magnetic moments of the positron and electron, respectively.

We then expect that the system contains a “hidden” momentum,

$$\mathbf{P}_{\text{hidden}} = -\mathbf{P}_{\text{EM}} = \frac{(\mathbf{m}_+ + \mathbf{m}_-) \times \mathbf{E}}{c}, \quad (e^+e^- \text{ at rest}), \quad (10)$$

such that $\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{hidden}} + \mathbf{P}_{\text{EM}} = 0$.

The existence of this “hidden” momentum is somewhat disconcerting, in that as far as is known from high-energy scattering experiments, the electron and positron are structureless,⁶ and hence can’t “hide” any momentum inside themselves. Of course, the fact that electrons and positrons have magnetic moments is also disconcerting, given their apparent lack of internal structure.

The magnitude of the field momentum (9) and of the “hidden” momentum (10) is of order $e^2\hbar/m_e c^2 d^2 = \hbar r_e/d^2$, where d is the distance between the electron and positron, and $r_e = e^2/ec^2$ is the so-called classical electron radius. As these momenta are of order $1/c^2$ they are sometimes called “relativistic” effects.

The notion of “hidden” momentum was invoked above to restore to zero the total momentum of an isolated system “at rest”. See [38] for a general definition of “hidden” momentum in a macroscopic description of a subsystem, isolated or not, in terms of quantities of that subsystem only. One consequence of this definition is that the field momentum (2) is also the “hidden” momentum of the subsystem of macroscopic electromagnetic fields in quasistatic examples where electromagnetic waves are neglected.

⁵Variants of Thomson’s example of a magnet and an electric charge in which the (electrically neutral) current is modeled as charges moving inside a nonconducting tube are considered in [30], ex. 12-13 of [33], and [34]. A model of the current as charges fixed on the rim of a rotating disk is considered in [32, 35], and the case of a toroidal permanent magnet is discussed in [36].

⁶A controversial interpretation [37] of the data is that the size of the electron is about 10^{-17} cm, $\approx 1/10,000$ of the size of a proton/neutron. My view here is that this value should be regarded as the present experimental limit on the size of the electron.

2.3 Circular Motion

An electron and positron will not stay at rest, but can be in a quasistatic state of uniform circular motion with radius r and velocity v related by $v^2/c^2 \approx r_e/4r$, if radiation is ignored.⁷ As discussed, for example, in sec. 4.1.4 of [38], the field momentum and “hidden” momentum of quasistatic systems are at order $1/c^2$, and the same as those for the corresponding system at rest. Hence, we anticipate that eqs. (9)-(10) hold for an e^+e^- system in uniform circular motion, where the symbol R is twice the radius r of the circular orbit. Justification of this claim is given to order $1/c^2$ in Appendix A.

Of particular interest is the case that $\mathbf{m}_+ = \mathbf{m}_-$, and both moments are perpendicular to the plane of their orbits.⁸ Then, the field momentum $\mathbf{E} \times (\mathbf{m}_+ + \mathbf{m}_-)/c$ of the system is nonzero,⁹ while its total mechanical momentum is zero. The center of mass/energy of this system is at rest, so the total momentum must be zero. Hence, there must be a “hidden mechanical momentum” associated with the moving electrons/positrons which is equal and opposite to the field momentum.

There is no classical model for this “hidden” momentum, in that there exists no successful classical model of the magnetic moment of an electron. One can say that the “hidden” momentum in an e^+e^- system is a quantum effect that protudes into a classical discussion.

2.4 Comments

“Hidden” momentum can exist in the macroscopic description of (sub)systems not involving permanent magnetism, which was the context of Shockley’s original consideration [32] of this concept. In such cases, macroscopic explanations can be found for the “hidden” momentum, as in [30, 33, 34]. If one takes a microscopic view of these examples, in which intrinsic magnetic moments are neglected and all charges are structureless points that can’t contain internal momentum, one finds [39] no microscopic “hidden” momentum,¹⁰ and that the

⁷See Appendix B for discussion of the velocity, and Appendix C for discussion of the radiation.

⁸The spin angular momentum \mathbf{S}_\pm of an electron or positron has magnitude $\hbar/2$, and is related to its magnetic moment \mathbf{m}_\pm by $\mathbf{S}_\pm = \pm(mc/e)\mathbf{m}_\pm$, so that if $\mathbf{m}_+ = \mathbf{m}_-$ the total spin angular momentum of the pair is zero.

⁹The electromagnetic field momentum associated with the magnetic fields $\mathbf{B}_\pm(\mathbf{v}) = \mathbf{v}_\pm/c \times \mathbf{E}_\pm$ of the moving charges is zero. To see this, we note that the electric fields at point \mathbf{r} are $\mathbf{E}_\pm = \pm e\mathbf{R}_\pm/R_\pm^3 = \pm e(\mathbf{r} - \mathbf{r}_\pm)/R_\pm^3$, where $\mathbf{r}_- = -\mathbf{r}_+$ and $\mathbf{v}_- = -\mathbf{v}_+$. Hence,

$$\begin{aligned}
 \mathbf{P}_{\text{EM}}(\mathbf{v}) &= \int \frac{(\mathbf{E}_+ + \mathbf{E}_-) \times (\mathbf{B}_+ + \mathbf{B}_-)}{4\pi c} d\text{Vol} \\
 &= \int \frac{\mathbf{E}_+ \times \mathbf{B}_+ + \mathbf{E}_- \times \mathbf{B}_-}{4\pi c} d\text{Vol} + \int \frac{\mathbf{E}_+ \times \mathbf{B}_- + \mathbf{E}_- \times \mathbf{B}_+}{4\pi c} d\text{Vol} \\
 &= -e^2 \int \frac{(\mathbf{r} - \mathbf{r}_+) \times [\mathbf{v}_- \times (\mathbf{r} - \mathbf{r}_-)] + (\mathbf{r} - \mathbf{r}_-) \times [\mathbf{v}_+ \times (\mathbf{r} - \mathbf{r}_+)]}{4\pi c^2 R_+^3 R_-^3} d\text{Vol} \\
 &= -e^2 \int \frac{[\mathbf{r} \cdot (\mathbf{v}_- + \mathbf{v}_-)]\mathbf{r} - (\mathbf{r} \cdot \mathbf{v}_-)\mathbf{r}_+ - (\mathbf{r} \cdot \mathbf{v}_+)\mathbf{r}_- - (\mathbf{r}_+ \cdot \mathbf{v}_-)\mathbf{r} - (\mathbf{r}_- \cdot \mathbf{v}_+)\mathbf{r} + (\mathbf{r}_+ \cdot \mathbf{v}_-)\mathbf{r}_- + (\mathbf{r}_- \cdot \mathbf{v}_+)\mathbf{r}_+}{4\pi c^2 R_+^3 R_-^3} d\text{Vol} \\
 &= 0.
 \end{aligned} \tag{11}$$

¹⁰See also sec. 4 of [38].

macroscopic “hidden” momentum of electrical currents is associated with the field momentum of the microscopic electromagnetic fields which were neglected in the macroscopic view. Of course, such microscopic arguments fail for examples involving permanent magnetism, and also fail to account for the stability of macroscopic matter, both of which phenomena are better understood in quantum theory.^{11,12}

A Appendix: Darwin’s Approximation

As “hidden” momentum (including the field momentum) is of order $1/c^2$, we will consider details of circular motion of an e^+e^- system only to this order. A general discussion of the motion of charged particles to order $1/c^2$ was given by Darwin [41] (1920), prior to the understanding [1] that particles can have intrinsic magnetic moments.

A.1 The Darwin Approximation without Magnetic Moments

The Lagrangian for a charge e of mass m that moves with velocity \mathbf{v} in an external electromagnetic field that is described by potentials V and \mathbf{A} can be written (see, for example, sec. 16 of [42])

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - eV + e\frac{\mathbf{v}}{c} \cdot \mathbf{A}. \quad (12)$$

Darwin [41] worked in the Coulomb gauge, and kept terms only to order v^2/c^2 . For a collection of charged particles, the Darwin Lagrangian is

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} e_i V_{ij}^{(C)} + \sum_{i>j} e_i \frac{\mathbf{v}_i}{c} \cdot \mathbf{A}_{ij}^{(C)}, \quad (13)$$

where we ignore the constant sum of the rest energies of the particles, and $V_{ij}^{(C)}$, $\mathbf{A}_{ij}^{(C)}$ are the Coulomb-gauge potentials at particle i due to particle j .

The Lagrangian (13) does not depend explicitly on time, so the corresponding Hamiltonian,

$$\mathcal{H} = \sum_i \mathbf{p}_i \cdot \mathbf{v}_i - \mathcal{L}, \quad (14)$$

is the conserved energy of the system, where

$$\mathbf{p}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_{j \neq i} e_i \frac{\mathbf{A}_{ij}^{(C)}}{c} \quad (15)$$

¹¹One could take that attitude that “classical” electromagnetism excludes quantum effects, and hence excludes intrinsic magnetic moments and permanent magnetism. Then, in the microscopic view of this “classical” electrodynamics, that all electrical charges are structureless points and “point” magnetic dipoles don’t exist, there will be no “hidden” momentum. It remains that “hidden” momentum exists in a macroscopic description of this electrodynamics.

Note that the “classical” electrodynamics of structureless point charges is quite different from Maxwell’s vision of electrodynamics [13], which accommodated permanent magnetism, and did not assume that electric charge is associated with mathematical points.

¹²A quantum calculation of “hidden” momentum in a hydrogen atom, ignoring electron spin, has been given in [40].

is the canonical momentum of particle i . Hence, the Hamiltonian/energy is¹³

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} e_i V_{ij}^{(C)} + \sum_{i>j} e_i \frac{\mathbf{v}_i}{c} \cdot \mathbf{A}_{ij}^{(C)}. \quad (17)$$

The Coulomb-gauge scalar and vector potentials due to a charge e that has velocity \mathbf{v} but no magnetic moment are (see sec. 65 of [42] or sec. 12.6 of [43])

$$V_e^{(C)} = \frac{e}{R}, \quad \mathbf{A}_e^{(C)} = \frac{e[\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR}, \quad (18)$$

where $\hat{\mathbf{n}}$ is directed from the charge to the observer, whose (present) distance is R . Then, the Lagrangian, the canonical momenta and the conserved energy/Hamiltonian for a system of electrical charges without magnetic moments are

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (19)$$

$$\mathbf{p}_i = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_{j \neq i} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})] \quad (20)$$

and

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})], \quad (21)$$

as first given by Darwin [41].

The part of this Hamiltonian/energy associated with electromagnetic interactions is

$$U_{\text{EM}} = \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]. \quad (22)$$

A.2 The Darwin Approximation Including Magnetic Moments

In a frame where a particle with intrinsic electric and magnetic dipole moments \mathbf{p} and \mathbf{m} has velocity \mathbf{v} it appears to have both an electric dipole moment \mathbf{p}' and magnetic moment \mathbf{m}' given by¹⁴

$$\mathbf{p}' = \mathbf{p} + \frac{\mathbf{v}}{c} \times \mathbf{m} - (1 - 1/\gamma)(\hat{\mathbf{v}} \cdot \mathbf{p})\hat{\mathbf{v}}, \quad \mathbf{m}' = \mathbf{m} - \frac{\mathbf{v}}{c} \times \mathbf{p} - (1 - 1/\gamma)(\hat{\mathbf{v}} \cdot \mathbf{m})\hat{\mathbf{v}}. \quad (23)$$

¹³The integral form of eq. (17),

$$U_{\text{EM}} = \frac{1}{2} \int \left(\rho \phi + \frac{\mathbf{J} \cdot \mathbf{A}}{c} \right) d\text{Vol}, \quad (16)$$

shows the possibly surprising result that the electromagnetic energy in the Darwin approximation has the form of that for a system of **quasistatic** charge and current densities ρ and \mathbf{J} (which implies use of the Coulomb gauge; see, for example, sec. 5.16 of [43] or secs. 31 and 33 of [47]).

¹⁴The transformation (23) was first written for macroscopic densities \mathbf{P} and \mathbf{M} of electric and magnetic dipole moments by Lorentz [44], and for an electron by Frenkel [45, 46].

As far as we know, the intrinsic electric dipole moment \mathbf{p} of electrons and positrons is zero, so the moments of a moving electron/positron are approximately,

$$\mathbf{p}' \approx \frac{\mathbf{v}}{c} \times \mathbf{m}, \quad \mathbf{m}' \approx \mathbf{m}. \quad (24)$$

The Coulomb-gauge scalar potential associated with the ‘‘motional’’ electric dipole moment \mathbf{p}' is, to order $1/c^2$,

$$V_m^{(C)} = \frac{\mathbf{p}' \cdot \hat{\mathbf{n}}}{R^2} \approx \frac{\hat{\mathbf{n}} \cdot (\mathbf{v} \times \mathbf{m})}{cR^2}, \quad (25)$$

The vector potential of a magnetic moment at rest is

$$\mathbf{A}_m^{(C)} = \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2}. \quad (26)$$

The vector potential of a moving magnetic dipole differs from this, but with terms that depend on various powers of $1/c$. Since we want the electromagnetic momentum $q\mathbf{A}/c$ to order $1/c^2$, we only need the vector potential to order $1/c$. Then, as the magnetic moment \mathbf{m} is of order $1/c$, we don't need any corrections for the form (26).

We can verify this conclusion by following the procedure of sec. 65 of [42], which first finds the (approximate) retarded potentials in the Lorenz gauge, and then transforms these to the Coulomb gauge.

The retarded (Lorenz-gauge) vector potential is

$$\mathbf{A}_m^{(L)}(t) = \left[\frac{\mathbf{m}' \times \mathbf{R}}{R^3} \right]_{t'=t-R/c} \approx \left. \frac{\mathbf{m} \times \mathbf{R}}{R^3} \right|_t = \left. \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2} \right|_t, \quad (27)$$

since we want the electromagnetic momentum $q\mathbf{A}/c$ to order $1/c^2$, we only need the vector potential to order $1/c$. Then, since the magnetic moment \mathbf{m} is of order $1/c$, we don't need any corrections to these that effects of retardation negligible for \mathbf{A} in the desired approximation. More care is required for the retarded scalar potential, which can be expanded as

$$\begin{aligned} V^{(L)}(\mathbf{r}, t) &= \int \frac{\rho(\mathbf{r}', t' = t - R/c)}{R} d\text{Vol}' \\ &\approx \int \frac{\rho(\mathbf{r}', t)}{R} d\text{Vol}' - \frac{1}{c} \frac{\partial}{\partial t} \int \rho(\mathbf{r}', t) d\text{Vol}' + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int R\rho(\mathbf{r}', t) d\text{Vol}'. \end{aligned} \quad (28)$$

For an electric dipole moment $\mathbf{p}' = q\mathbf{d}'$ consisting of electric charges $\pm q$ separated by (small) distance \mathbf{d}' , we have that $\mathbf{R}_\pm = \mathbf{R}_\pm \pm \mathbf{d}'/2$, $R_\pm \approx R \mp \hat{\mathbf{n}} \cdot \mathbf{d}'/2$, so the Lorenz-gauge scalar potential of the magnetic dipole in the lab frame is

$$V_m^{(L)} \approx \frac{\mathbf{p}' \cdot \hat{\mathbf{n}}}{R^2} - \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \mathbf{p}' \cdot \hat{\mathbf{n}}. \quad (29)$$

We now wish to transform the Lorenz-gauge potentials into the Coulomb gauge via a gauge-transformation function χ ,

$$V_m^{(C)} = V_m^{(L)} - \frac{1}{c} \frac{\partial \chi}{\partial t}, \quad \mathbf{A}_m^{(C)} = \mathbf{A}_m^{(L)} + \nabla \chi. \quad (30)$$

Comparing eqs. (25) and (29), we see that

$$\chi = \frac{1}{2c} \frac{\partial}{\partial t} \mathbf{p}' \cdot \hat{\mathbf{n}}, \quad (31)$$

and hence,

$$\mathbf{A}_m^{(C)} = \mathbf{A}_m^{(L)} + \frac{1}{2c} \frac{\partial}{\partial t} \nabla(\mathbf{p}' \cdot \hat{\mathbf{n}}) \approx \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2} + \frac{1}{2c} \frac{\partial}{\partial t} (\mathbf{p}' \cdot \nabla) \hat{\mathbf{n}} = \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2} + \frac{1}{2c} \frac{\partial}{\partial t} \frac{\mathbf{p}' - (\mathbf{p}' \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}}{R} \quad (32)$$

Now,

$$\begin{aligned} \frac{\partial \mathbf{p}'}{\partial t} &\approx \frac{\mathbf{a}}{c} \times \mathbf{m}, & \frac{\partial \mathbf{R}}{\partial t} &= -\mathbf{v}, & R \frac{\partial R}{\partial t} &= \frac{\partial R^2}{\partial t} \frac{1}{2} = \frac{\partial \mathbf{R}^2}{\partial t} \frac{1}{2} = \mathbf{R} \cdot \frac{\partial \mathbf{R}}{\partial t} = -\mathbf{v} \cdot \mathbf{R}, \\ \frac{\partial R}{\partial t} &= -\mathbf{v} \cdot \hat{\mathbf{n}}, & \frac{\partial}{\partial t} \frac{1}{R} &= \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{R^2}, & \frac{\partial \hat{\mathbf{n}}}{\partial t} &= \frac{\partial}{\partial t} \frac{\mathbf{R}}{R} = \frac{1}{R} \frac{\partial \mathbf{R}}{\partial t} - \frac{\mathbf{R}}{R^2} \frac{\partial R}{\partial t} = \frac{-\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}}{R}, \end{aligned} \quad (33)$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the present acceleration of the dipole. Finally, the Coulomb-gauge vector potential of the moving dipole in the lab frame is

$$\mathbf{A}_m^{(C)} \approx \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2} + \frac{(\mathbf{v} \times \mathbf{m} \cdot \hat{\mathbf{n}})(\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}})}{2c^2 R^2} + \frac{\mathbf{a} \times \mathbf{m} - (\mathbf{a} \times \mathbf{m} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}}{2c^2 R} \approx \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2}. \quad (34)$$

Now that we have the potentials of a moving magnetic dipole to order $1/c^2$ in the Coulomb gauge, we readily obtain the Lagrangian, the canonical momenta and the conserved energy/Hamiltonian for a system of electrical charges with magnetic moments,

$$\begin{aligned} \mathcal{L} &= \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} + \sum_{i>j} e_i \left(-\frac{e_j}{R_{ij}} + \frac{e_j [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]}{2c^2 R_{ij}} \right. \\ &\quad \left. - \frac{\hat{\mathbf{n}}_{ij} \cdot (\mathbf{v}_j \times \mathbf{m}_j)}{c R_{ij}^2} + \frac{\mathbf{m}_j \times \hat{\mathbf{n}}_{ij}}{c R_{ij}^2} \right), \end{aligned} \quad (35)$$

$$\mathbf{p}_i = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + e_i \sum_{j \neq i} \left(\frac{e_j}{2c^2 R_{ij}} [\mathbf{v}_j + \hat{\mathbf{n}}_{ij} (\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})] + \frac{\mathbf{m}_j \times \hat{\mathbf{n}}_{ij}}{c R_{ij}^2} \right), \quad (36)$$

and

$$\begin{aligned} U &= \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \sum_{i>j} e_i \left(\frac{e_j}{R_{ij}} + \frac{e_j [\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})]}{2c^2 R_{ij}} \right. \\ &\quad \left. + \frac{\hat{\mathbf{n}}_{ij} \cdot (\mathbf{v}_j \times \mathbf{m}_j)}{c R_{ij}^2} + \frac{\mathbf{m}_j \times \hat{\mathbf{n}}_{ij}}{c R_{ij}^2} \right). \end{aligned} \quad (37)$$

A.3 Field Momentum for e^+e^- in Circular Motion

After these lengthy preliminaries, we find that the electromagnetic field momentum for an e^+e^- in uniform circular motion has the same form (9) for the charges at rest.

$$\begin{aligned} \mathbf{P}_{\text{EM}} &= \frac{e_+ \mathbf{A}_{+-} + e_- \mathbf{A}_{-+}}{c} \approx -\frac{e^2}{cR^2} [\mathbf{v}_- + \hat{\mathbf{n}}(\mathbf{v}_- \cdot \hat{\mathbf{n}}) \mathbf{v}_+ \hat{\mathbf{n}}(\mathbf{v}_+ \cdot \hat{\mathbf{n}})] + \frac{e(\mathbf{m}_+ + \mathbf{m}_-) \times \hat{\mathbf{n}}}{cR^2} \\ &= \frac{e(\mathbf{m}_+ + \mathbf{m}_-) \times \hat{\mathbf{n}}}{cR^2} = \frac{\mathbf{E} \times (\mathbf{m}_+ + \mathbf{m}_-)}{c}, \end{aligned} \quad (38)$$

where $\hat{\mathbf{n}}$ points from the electron to the positron, and $\mathbf{E} = -e \hat{\mathbf{n}}/R^2$ is the electric field due to the electron at the location of the positron.

B Forces On an e^+e^- Pair in Circular Motion

The potentials of a moving electron/positron in the Darwin approximation are

$$V^{(C)} = \frac{e}{R} + \frac{\hat{\mathbf{n}} \cdot (\mathbf{v} \times \mathbf{m})}{cR^2}, \quad \mathbf{A}^{(C)} = \frac{e[\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR} + \frac{\mathbf{m} \times \hat{\mathbf{n}}}{R^2}, \quad (39)$$

and the corresponding electromagnetic fields are¹⁵

$$\begin{aligned} \mathbf{E} = & -\nabla V^{(C)} - \frac{\partial \mathbf{A}^{(C)}}{\partial ct} = \frac{e \hat{\mathbf{n}}}{R^2} + \frac{3[\hat{\mathbf{n}} \cdot (\mathbf{v} \times \mathbf{m})]\hat{\mathbf{n}} - \mathbf{v} \times \mathbf{m}}{cR^3} \\ & - \frac{e}{2c^2R} \left[\mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \frac{3(\mathbf{v} \cdot \hat{\mathbf{n}})^2 - v^2}{R}\hat{\mathbf{n}} \right] - \frac{\dot{\mathbf{m}} \times \hat{\mathbf{n}}}{cR^2} + \frac{\mathbf{m} \times \mathbf{v}}{cR^3}, \end{aligned} \quad (40)$$

$$\mathbf{B} = \nabla \times \mathbf{A}^{(C)} = \frac{e\mathbf{v} \times \hat{\mathbf{n}}}{cR^2} + \frac{3(\hat{\mathbf{n}} \cdot \mathbf{m})\hat{\mathbf{n}} - \mathbf{m}}{R^3}, \quad (41)$$

In general, the electric field is not along $\hat{\mathbf{n}}$, *i.e.*, not along the line of centers of the electron and positron.

We restrict our attention to electrons and positrons in uniform circular motion with radius $r = r/2$, and with $\mathbf{m}_+ = \mathbf{m}_- \equiv \mathbf{m} \perp \hat{\mathbf{n}}$. Then, the magnetic field at the electron or positron (due to the positron or electron) is parallel to its magnetic moment, such that the torque on this moment is zero, and $\dot{\mathbf{m}}_{\pm} = 0$. Further, $\mathbf{v} \times \mathbf{m}$ is parallel to $\hat{\mathbf{n}}$, so the Lorentz force $e(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$ on the electron or positron is parallel to $\hat{\mathbf{n}}$. Also, the lab-frame electric dipole moment $\mathbf{p}' = \mathbf{v}/c \times \mathbf{m}$ is parallel to $\hat{\mathbf{n}}$, so the force on this moment, $(\mathbf{p}' \cdot \nabla)\mathbf{E}$ is parallel to $\hat{\mathbf{n}}$. Finally, $\mathbf{m} \cdot \mathbf{B} = emv/cR^2 - m^2/r^3$, so the force $\nabla(\mathbf{m} \cdot \mathbf{B})$ on each magnetic dipole is parallel to $\hat{\mathbf{n}}$.

The total force on the electron or positron is parallel to $\hat{\mathbf{n}}$, and to leading order is just the Coulomb force $e^2/R^2 = e^2/4r^2$. All other force terms discussed above are of order $1/c^2$. To a good approximation, the radial force equation is

$$F_r = -\frac{e^2}{4r^2} = -\frac{m_e v^2}{r} \quad \frac{v^2}{c^2} = \frac{e^2}{4m_e c^2 r} = \frac{r_e}{4r}, \quad (42)$$

where m_e is the rest mass of the electron and $r_e = e^2/m_e c^2 = 2.8 \times 10^{-13}$ cm is the so-called classical electron radius. For example, if $r = 1$ cm, then $v/c \approx 2.6 \times 10^{-7}$, and $v \approx 8000$ cm/s.

C Lifetime of the e^+e^- Circular Motion

We have ignored electromagnetic radiation in the preceding. However, the e^+e^- system emits energy, predominantly in the form of electric dipole radiation, at the rate

$$\frac{dU}{dt} = -\frac{2\ddot{d}^2}{3c^3} = -\frac{2d^2\omega^4}{3c^3} = -\frac{8e^3 r^2 v^4}{3c^3 r^4} = -\frac{8e^2 c r_e^2}{3r^4}, \quad (43)$$

¹⁵The general fields of a moving magnetic dipole have been given in [48].

noting that the electric dipole moment of the system is $d = 2er$. The energy of the system is

$$U = -\frac{e^2}{4r}, \quad \frac{dU}{dt} = \frac{e^2}{4r^2} \frac{dr}{dt}, \quad (44)$$

such that

$$r^2 \frac{dr}{dt} = -\frac{2cr_e^2}{3}, \quad r^3 = r_0^3 - 2cr_e^2 t. \quad (45)$$

The radius of the e^+e^- system falls to zero in time

$$t = \frac{r_0^3}{2cr_e^2} = 2 \times 10^{14} r_0^3 \text{ s}, \quad (46)$$

for initial radius r_0 of the orbit in cm. The lifetime is very long for $r_0 = 1$ cm, but only about 10^{-10} s for $r_0 = 1 \text{ \AA}$ (which is roughly the lifetime of the ground state of positronium).

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