Rotational Stability of a Diamagnetic Rod

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1 Problem

When Faraday identified/named diamagnetism (relative permeability $\mu < 1$) in 1845, he reported in Arts. 2253-2260 of [1] that a glass rod would align itself at right angles to the lines of magnetic field of his laboratory electromagnet.[1] He also noted in Art. 2269 that such object moved "into the positions of weakest magnetic action".

Faraday's test objects were suspended from a string, but W. Thomson [5, 6] (at age 23) soon noted that it should be possible to levitate diamagnetic objects, although he speculated that the magnetic force would be too weak for such levitation to be observed in practice.

Diamagnetic levitation was first observed in 1939 [7]. See also [8, 9, 10, 11, 12, 13]. A superconductor can be regarded as an extreme form of diamagnetism, with relative permeability $\mu = 0$; levitation of a superconductor was first observed in 1945 [14, 15].

A recent example of diamagnetic levitation of a graphite rod has been given in [16, 17], in which the magnetic field was provided by a pair of cylindrical magnets, each with horizontal (diametric) magnetization, perpendicular to the axes of the cylinders, as illustrated below (PDL = parallel dipole line).

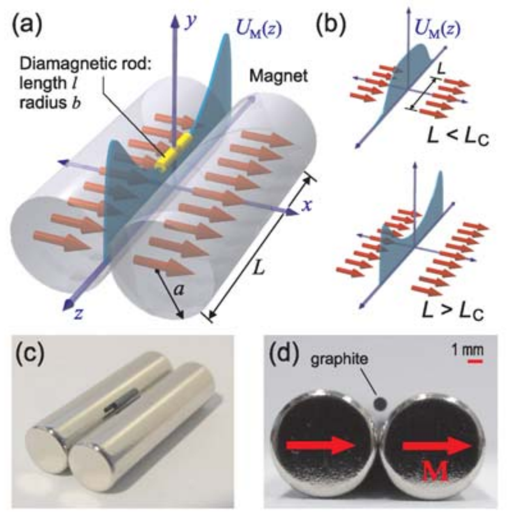

FIG. 1. (a) The PDL trap with a diamagnetic rod at the center and the magnetic camelback potential (U_M) along the longitudinal axis. (b) The camelback potential only appears when the dipole line length exceeds certain critical value "L_C". (c) Experimental setup using a pair of diametric magnets (PDM) and a graphite rod. (d) Cross section image.

[1] In a footnote on p. 21 of [1], Faraday remarked that Coulomb had observed such behavior for a cocoon of silk in 1802 [2], and Becquerel confirmed this behavior for a wooden needle in 1827 [3]. The earliest report of a diamagnetic effect may be due to Brugmans (1778) [4] who observed that bismuth was repelled by a magnet.

For long rods, the equilibrium orientation of the rod is transverse to the magnetic fields lines as show above, consistent with Faraday's observations. However, sufficiently short rods align themselves with the field lines. Explain this behavior.

2 Solution

2.1 Diamagnetic Rod in a Uniform Magnetic Field

In his early papers on diamagnetism, Thomson [5, 6, 18, 19] argued that a diamagnetic rod, like a paramagnetic/ferromagnetic rod, would align itself with a uniform magnetic field, while Faraday's observations of transverse alignment were due to the nonuniformity of the laboratory magnetic field. See also [20].

A uniform, permeable sphere has magnetization density $\mathbf{M}$ parallel to the external magnetic field $\mathbf{B}_0$,[2] so the magnetic torque density $\boldsymbol{\tau} = \mathbf{M} \times \mathbf{B}_0 = 0$, and the sphere does not rotate. If a diamagnetic rod had magnetization parallel to $\mathbf{B}_0$, it also would not rotate or align itself perpendicular to the magnetic field as observed by Faraday [1]. Hence, we need to consider departures from the approximation that $\mathbf{M} = M\,\hat{\mathbf{B}}_0$ to understand the rotational behavior of a diamagnetic rod, although this approximation is sufficient for an understanding of the translational stability of the rod (see the Supplementary Material to [16]).

The experiments [16, 17] were performed with a diamagnetic cylinder, for which there is no simple analytic relation between $\mathbf{M}$ and $\mathbf{B}_0$,[3] so we will instead consider a diamagnetic spheroid with uniform permeability μ, for which such a relation does exist, as first deduced by Poisson [21].[4]

2.1.1 Spheroidal Rod with Uniform Permeability in a Uniform External Field

Thomson [19] gave a qualitative discussion of the torque on an ellipsoid with uniform permeability in a uniform external field, and Maxwell gave a more quantitative discussion in Art. 438 of [28]. Here, we restrict ourselves to a prolate spheroid of semimajor axis r_1 and semiminor axes $r_2 = \sqrt{1-\epsilon^2}\, r_1$, where $\epsilon < 1$ is called the eccentricity.

In a coordinate system (x, y, z) based on the principal axes of the spheroid, with the x principal axis being the longest, the magnetic field $\mathbf{H}$ inside it can be written (in Gaussian units) as

$$H_i = H_{0,i} - 4\pi N_i M_i, \quad \text{with} \quad N_x = N = \frac{1-\epsilon^2}{2\epsilon^3}\left(\ln\frac{1+\epsilon}{1-\epsilon} - 2\epsilon\right), \quad N_y = N_z = \frac{1-N}{2}, \tag{1}$$

for the case of a uniform external magnetic field $\mathbf{B}_0 = \mathbf{H}_0$, as given in Art. 438 of [28].[5] Also,

[2]This result was apparently first deduced by Poisson [21].

[3]Approximate models of the magnetization of objects other than ellipsoids began with Green, p. 106 [22], are were continued by Kirchhoff [23]. For summaries of more recent work, see [24, 25].

[4]Early rederivations of Poisson's result include Neumann [26], a footnote by Thomson on p. 471 of [27], and Arts. 437-438 of Maxwell's *Treatise* [28]. See also secs. 4, 8 and 29 of [29].

[5]The quantities $N_x = N$, N_y and N_z obey $N_x + N_y + N_z = 1$, and are called the demagnetization factors, perhaps following [31].

$\mathbf{B} = \mu\mathbf{H} = \mathbf{H} + 4\pi\mathbf{M}$, so that $\mathbf{M} = (\mu - 1)\mathbf{H}/4\pi = \chi\mathbf{H}/4\pi$, and

$$M_i = \frac{\chi H_{0,i}}{4\pi(1+\chi N_i)} \approx \frac{\chi H_{0,i}}{4\pi}(1 - \chi N_i)\,, \qquad H_i = \frac{H_{0,i}}{1+\chi N_i} \approx H_{0,i}(1 - \chi N_i), \tag{2}$$

where $\chi = \mu - 1$ is the magnetic susceptibility, and the approximation holds for small $|\chi|$. Thus, the magnetization of the spheroid is uniform, although not parallel to $\mathbf{H}_0 = \mathbf{B}_0$, unless the external field is parallel to a principal axis of the spheroid. The total magnetic moment of the spheroid is $\mathbf{m} = \mathbf{M}$ Vol, where Vol $= 4\pi r_1 r_2^2/3$ is the volume of the spheroid.

For a sphere ($\epsilon = 0$), $N = 1/3$, $\mathbf{M} = \chi\mathbf{H_0}/4\pi(1 + \chi/3) = 3(\mu - 1)\mathbf{H_0}/4\pi(\mu + 2)$,[6] while for a very long needle ($\epsilon \to 1$), $N \to 0$ and $\mathbf{M} \to (\mu - 1)\mathbf{H}_0/4\pi$.

There is no net force on a permeable spheroid in a uniform external field, while the torque on it is, taking the field to lie in the x-y plane[7]

$$\begin{aligned}\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}_0 &= (m_x B_{0,y} - m_y B_{0,x})\,\hat{\mathbf{z}} = \frac{\chi B_{0,x}B_{0,y}\mathrm{Vol}}{4\pi}\left[\frac{1}{1+\chi N} - \frac{1}{1+\chi\frac{1-N}{2}}\right]\hat{\mathbf{z}} \\ &= \frac{\chi^2 B_{0,x}B_{0,y}\mathrm{Vol}}{8\pi}\frac{1-3N}{(1+\chi N)(1+\chi\frac{1-N}{2})}\,\hat{\mathbf{z}} \approx \frac{\chi^2 B_{0,x}B_{0,y}\mathrm{Vol}}{8\pi}(1-3N)\,\hat{\mathbf{z}}. \end{aligned}\tag{3}$$

The factor $1 - 3N$ indicates that the torque vanishes for a sphere, as expected.

The sign of the torque is independent of the sign of χ (except for a certain range of $\chi < -1$, $\mu < 0$, which is not possible for "ordinary" materials), *i.e.*, the sign is the same for both diamagnetic and paramagnetic objects. Since $0 < N < 1/3$ for a prolate spheroid, the sign of the torque is that same as that of the product $B_{0,x}B_{0,y}$, which implies that the torque tends to align the long axis (x) of the spheroid with the external magnetic field (as anticipated by Thomson [5], but first explicitly demonstrated in Art. 438 of [28], which eq. (3) transcribes).[8]

The torque (3) is proportional to χ^2 for small $|\chi|$, rather than to the susceptibility χ, which latter behavior we might naïvely have expected from the form $\mathbf{M} = \chi\mathbf{H}/4\pi$. However, if the magnetization $\mathbf{M}$ were strictly proportional to $\mathbf{H}_0 = \mathbf{B}_0$, then the torque would vanish. So, the existence of a nonzero torque on the permeable spheroid is related to the existence of the quadratic, correction term in the expression (2) for the magnetization.

2.1.2 Torque via an Energy Method

The torque $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}_0$ on a magnetic dipole $\mathbf{m}$ in an external magnetic field $\mathbf{B}_0$ (that is uniform over the dipole) can be deduced from an interaction energy

$$U = -\mathbf{m}\cdot\mathbf{B}_0 = -mB_0\cos\alpha \qquad (|\mathbf{m}|\ \text{fixed}), \tag{4}$$

Only for ellipsoids with uniform permeability, and in a uniform external field, are the demagnetization factors independent of the permeability and of position (contrary to the assumption in [32]).

Until 1945, the demagnetization factors were defined to be 4π times those now used (which latter follow the convention advocated in [33].

[6]Compare sec. 5.11 of [30].

[7]Note that $B_{0,x}B_{0,y} = (1/2)B_0^2 \sin 2\alpha$, where α is the angle of $\mathbf{B}_0$ to either the x-axis = long axis of the spheroid. The torque (3) acts to decrease angle α, so if we replace $B_{0,x}B_{0,y}$ by $(1/2)B_0^2 \sin 2\alpha$, we should give the revised equation an overall minus sign. See also eq. (10).

[8]A bismuth bar (of unspecified shape) was reported in [32] to align itself with the field of a strong magnet.

for a dipole of fixed magnitude, where α is the angle between $\mathbf{m}$ and $\mathbf{B}_0$, via the relation

$$\tau = -\frac{\partial U}{\partial \alpha} = -mB_0 \sin\alpha, \qquad \tau = \mathbf{m} \times \mathbf{B}_0 \qquad (|\mathbf{m}| \text{ fixed}), \tag{5}$$

noting that the minus sign in $-mB_0 \sin\alpha$ means that the torque acts to decrease angle α, *i.e.*, to align $\mathbf{m}$ with $\mathbf{B}_0$, The force

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}_0), \tag{6}$$

on the magnetic dipole is agreeably deduced as $\mathbf{F} = -\boldsymbol{\nabla} U$.

If the magnetic dipole moment is not of fixed magnitude, as for a permeable object, then in the approximation that $\mathbf{m} = k\mathbf{B}_0$, we recover the force law (6) if we take the interaction energy to be $U = -\mathbf{m} \cdot \mathbf{B}_0/2$. This suggests that for permeable media where $\mathbf{m} = \chi\mathbf{H}/4\pi$, we take the interaction energy to be

$$U = -\int \frac{\chi \mathbf{H} \cdot \mathbf{B}_0}{8\pi}\, d\text{Vol} \qquad \text{(permeable object)}. \tag{7}$$

More complete justifications of eq. (7) are given in sec. 29 of [29] and in [34], where U is called the free energy.

For the example above of a permeable, prolate spheroid whose long (x) axis makes angle α to uniform magnetic field $\mathbf{B}_0 = B_0(\cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}})$, the $\mathbf{H}$ field inside the spheroid is

$$\mathbf{H} = \frac{B_0 \cos\alpha}{1 + \chi N}\,\hat{\mathbf{x}} + \frac{B_0 \sin\alpha}{1 + \chi \frac{1-N}{2}}\,\hat{\mathbf{y}}, \tag{8}$$

according to eq. (2). Then, the interaction energy (7) is

$$U = -\frac{\chi B_0^2 \text{Vol}}{8\pi} \left[\frac{\cos^2\alpha}{1 + \chi N} + \frac{\sin^2\alpha}{1 + \chi \frac{1-N}{2}} \right], \tag{9}$$

and the torque (in the z-direction) is

$$\begin{aligned} \tau &= -\frac{\partial U}{\partial \alpha} = \frac{\chi B_0^2 \sin 2\alpha \text{Vol}}{8\pi} \left[\frac{-1}{1 + \chi N} + \frac{1}{1 + \chi \frac{(1-N}{2}} \right] \\ &= -\frac{\chi B_0^2 \sin 2\alpha \text{Vol}}{16\pi} \frac{1 - 3N}{(1 + \chi N)(1 + \chi \frac{1-N}{2})}\,. \end{aligned} \tag{10}$$

This agrees with eq. (3), which was obtained by direct computation of $\mathbf{m} \times \mathbf{B}_0$, noting the sign convention mentioned in footnote 7, that the overall minus sign in eq. (10) means the torque reduces angle α and aligns the long axis of the prolate spheroid with $\mathbf{B}_0$.

2.2 Spheroidal Rod in the Field of Two Long Diametric Magnets

We now turn to the configuration of the experiment [16], as shown in the figure on p. 1. In the folowing, we approximate the graphite cylinder as a prolate spheroid, and also take the limit of very long magnets.

2.2.1 The Field of a Pair of Diametric Magnets

We now use the coordinate system of the figure on p. 1 (which is not the coordinate system based on the axes of the spheroid that was used in sec. 2.1). Now, the magnetic field is in the x-direction, and a levitated rod has its axis parallel to the z-axis. The y-axis is vertical.

The two diametric magnets of radius a each have internal field $\mathbf{B}_{\rm int} = B_M\,\hat{\mathbf{x}}$, where B_M is a constant of order 10,000 gauss. Assuming these magnets to be very long, their external fields have the form [35]

$$\mathbf{B}_{\rm ext} = \frac{B_M a^2}{r'^4}\left[(x'^2 - y'^2)\,\hat{\mathbf{x}}' + 2x'y'\,\hat{\mathbf{y}}'\right], \tag{11}$$

in a coordinate systems where the axis of the rod is the z'-axis. In the laboratory coordinate system, the rods have axes $\pm a, 0, z)$ to the total external field $\mathbf{B}_0$ has nonzero components

$$B_{0,x} = B_M a^2 \left\{ \frac{(x-a)^2 - y^2}{[(x-a)^2 + y^2]^2} + \frac{(x+a)^2 - y^2}{[(x+a)^2 + y^2]^2} \right\}, \tag{12}$$

$$B_{0,y} = 2B_M a^2 \left\{ \frac{(x-a)\,y}{[(x-a)^2 + y^2]^2} + \frac{(x+a)\,y}{[(x+a)^2 + y^2]^2} \right\}. \tag{13}$$

In the y-z symmetry plane, $B_{0,x}(0,y,z) = 2B_M a^2(a^2 - y^2)/(a^2 + y^2)^2$ and $B_{0,y}(0,y,z) = 0$.

In the experiment [16], levitation was observed for $y \approx a/\sqrt{2}$. In the following, we suppose that the equilibrium position of the center of the spheroid is $y = a/\sqrt{2}$, at which point the external magnetic field is $\mathbf{B}_0(0, a/\sqrt{2}, z) = 4B_M\,\hat{\mathbf{x}}/9$.

We now wish to deduce a condition such that if the spheroid has its long axis in the horizontal plane $y = a/\sqrt{2}$, and makes a small angle to the z-axis, the torque on it will push it towards that axis, leading to alignment transverse to the external magnetic field (in contrast to alignment with the magnetic field as occurs if that field were uniform).

2.2.2 Approximation to the Interaction Energy

As discussed in sec. 2.1,2, the torque on a permeable object in a static, external magnetic field $\mathbf{B}_0$ can be computed from the interaction energy U according to

$$\tau_\alpha = -\frac{\partial U}{\partial \alpha}, \qquad U = -\int \frac{\chi \mathbf{H}\cdot\mathbf{B}_0}{8\pi}\, d\mathrm{Vol} \qquad \text{(permeable object)}, \tag{14}$$

where τ_α is the torque component along an axis perpendicular to the plane in which α is the angle between the magnetic moment $\mathbf{m}$ of the object and the direction of the magnetic field $\mathbf{B}_0(0)$ at its (magnetic) center. The difficulty is that except for ellipsoids in a uniform external field, $\mathbf{H}$ inside the object cannot be expressed in a simple analytic form.

One approximation would be to replace the factor B_0^2 Vol in eq. (9) by $\int B_0^2\, d\mathrm{Vol}$, which would simply increase the result (10) by a small amount. This would have the merit of still predicting zero torque on a sphere in a nonuniform magnetic field, but it would predict that a prolate spheroid (of any eccentricty $\epsilon > 0$) would align its long axis with the magnetic field of the pair of diametric magnets, in contrast to the observation of transverse alignment for $r_1/r_2 \gtrsim 4$ in the experiment [16].

2.2.3 Approximation to the form $\tau = \mathbf{M} \times \mathbf{B}$

The result (3) for the torque on the permeable spheroid in a uniform magnetic field $\mathbf{B}_0$ is noteworthy in that the magnetic force on any element of the spheroid,

$$d\mathbf{F} = -\boldsymbol{\nabla}(\mathbf{M}\, d\mathrm{Vol} \cdot \mathbf{B}_0), \tag{15}$$

is zero, so that the computation $\boldsymbol{\tau} = \int \mathbf{r} \times d\mathbf{F}$ also yields a null result.When the external field is nonuniform, we compute the torque as

$$\tau = \int \mathbf{M}(\mathbf{r}) \times \mathbf{B}_0(\mathbf{r})\, d\mathrm{Vol} \approx \mathbf{M} \times \mathbf{B}_0(0) - \int \mathbf{r} \times \boldsymbol{\nabla}[\mathbf{M}(\mathbf{r}) \cdot \mathbf{B}_0(\mathbf{r})]\, d\mathrm{Vol} \equiv \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2. \tag{16}$$

The Torque Term $\boldsymbol{\tau}_1 \approx \mathbf{M} \times \mathbf{B}_0(0)$

We estimate the first term, $\boldsymbol{\tau}_1$, in the expression (16) for the torque on a diamagnetic prolate spheroid by approximating the external magnetic field throughout the spheroid by its value $\mathbf{B}_0(0, a/\sqrt{2}, z) = 4B_M/9$ at the center of the spheroid, and then approximating the form (2) for the magnetization of the spheroid by

$$\mathbf{M} \approx \frac{\chi \mathbf{H}_0}{4\pi} = \frac{\chi \mathbf{B}_0}{4\pi} \approx \frac{\chi B_M}{9\pi}\, \hat{\mathbf{x}}. \tag{17}$$

noting that $|\chi| \approx 10^{-5}$ for graphite. Then, when the long axis of the spheroid makes small angle α to the z-axis in the lab coordinates, the result (3) (in the coordinate system of the principal axes of the spheroid) becomes

$$\boldsymbol{\tau}_1 \approx \frac{\alpha \chi^2 B_M^2 (1 - 3N)\mathrm{Vol}}{18\pi}\, \hat{\mathbf{y}}. \tag{18}$$

Th torque $\boldsymbol{\tau}_1$ pulls the long axis of the spheroid away from the z-axis and tends to align it with the x-axis = direction of the external magnetic field.

The Torque Term $\boldsymbol{\tau}_2 = -\int \mathbf{r} \times \boldsymbol{\nabla}[\mathbf{M}(\mathbf{r}) \cdot \mathbf{B}_0(\mathbf{r})]\, d\mathrm{Vol}$

We need to compute the y-component of the torque integral $\boldsymbol{\tau}_2$ of eq. (16) to order α = angle of the long axis of the spheroid to the z-axis.

Using the first approximation for the magnetization density of eq. (17), we have that

$$\tau_{2,y} \approx -\frac{\chi}{4\pi} \int z \frac{\partial B_0^2}{\partial x}\, d\mathrm{Vol}. \tag{19}$$

In a further approximation, we replace $\partial B_0^2(x, y)/\partial x$ in a slice of the spheroid at constant z by its value on the axis of the spheroid, $x = \sin\alpha\, z \approx \alpha\, z$, *i.e.*, by $\partial B_0^2(\alpha\, z, a/\sqrt{2})/\partial x$ where $\alpha\, z \ll a$. Now,

$$\frac{\partial B_0^2}{\partial x} = 2B_{0,x} \frac{\partial B_{0,x}}{\partial x} + 2B_{0,y} \frac{\partial B_{0,y}}{\partial x}\, . \tag{20}$$

Since we can approximate $f(x)$ and $f'(x)$ as

$$f(x) = f(0) + x\, f'(0) + \cdots, \qquad f'(x) = f'(0) + x\, f''(0) + \cdots, \tag{21}$$

we also have that

$$f(x)f'(x) = f(0)f'(0) + x[(f'(0))^2 + f(0)f''(0)] + \cdots \tag{22}$$

In the present example, recalling eqs. (12)-(13) (and using Wolfram Alpha),

$$B_{0,x}(0, a/\sqrt{2}) = \frac{4B_M}{9}, \qquad \frac{\partial B_{0,x}(0, a/\sqrt{2})}{\partial x} = 0, \qquad \frac{\partial^2 B_{0,x}(0, a/\sqrt{2})}{\partial x^2} = -\frac{336B_M}{81a^2}, \tag{23}$$

$$B_{0,y}(0, a/\sqrt{2}) = 0, \qquad \frac{\partial B_{0,xy}(0, a/\sqrt{2})}{\partial x} = -\frac{40\sqrt{2}B_M}{27a}, \tag{24}$$

so that

$$\frac{\partial B_0^2(\alpha\, z, a/\sqrt{2})}{\partial x} \approx 2\alpha\, z \left[-\frac{4B_M}{9}\frac{336B_M}{81a^2} + \left(\frac{40\sqrt{2}B_M}{27a}\right)^2 \right] = \frac{3712\,\alpha\, zB_M^2}{729a^2}. \tag{25}$$

Then, the second torque integral (19) is

$$\begin{aligned} \tau_{2,y} &\approx -\frac{\chi}{4\pi}\frac{3712\,\alpha\, B_M^2}{729a^2}\int z^2\, d\mathrm{Vol} \approx -\frac{\chi}{4\pi}\frac{3712\,\alpha\, B_M^2}{729a^2} 2\pi r_2^2 \int_0^{r_1} z^2 \left(1 - \frac{z^2}{r_1^2}\right) dz \\ &\approx -\frac{\alpha\chi\, B_M^2}{4\pi}\frac{(2r_1)^2}{a^2}\,\mathrm{Vol}, \end{aligned} \tag{26}$$

using that the transverse area of the spheroid, for small angle α is $\pi r_2^2(1 - z^2/r_1^2)$ and its volume is $4\pi r_1 r_2^2/3$. The negative sign of $\tau_{2,y}$ implies that it tends to align the long axis of the diamagnetic spheroid with the z-axis, *i.e.*, transverse to the magnetic field lines

The spheroid will align itself transverse to the magnetic field for all lengths $l = 2r_1$ such that τ_y is negative. Comparing eqs. (18) and (26), this implies that $l/a \gtrsim \sqrt{\chi(1 - 3N)}$, which suggests that only rods much shorter than those used in the experiment [16] would align themselves with the field lines, contrary to the observations that rods with $l/a \lesssim 1/3$ aligned themselves with the field.

A general feature of the present model is that the torque $\tau_{y,1}$ which aligns the rod with the field lines scales as χ^2, while the torque $\tau_{y,2}$ which aligns the rod transverse to the field lines scales as χ. Since the susceptibility χ is extremely small for graphite, a transition from transverse to parallel alignment for $l/a \lesssim 1/3$ requires the dimensionless numerical coefficient in the result for $\tau_{y,2}$ to be much smaller than that found in the present approximation.

Also, the present model does not successfully predict that the torque on a permeable sphere would be zero, even when in a nonuniform magnetic field.

It would be interesting to have experimental data on a paramagnetic spheroid/rod, for which the sign of the susceptibility χ is positive (rather than negative as for a diamagnetic object). If the torque in a nonuniform field varied as χ rather than as χ^2, then (I believe) a paramagnetic spheroid/rod would show no stability with alignment transverse to the magnetic field.

The susceptibility of aluminum is about 1.6×10^{-6} (in Gaussian units), which is nearly equal and opposite to that of graphite ($\approx -1 \times 10^{-6}$).

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