

# Diamagnetic Levitation

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## 1 Problem

Discuss the stability of levitation of a small diamagnetic sphere of radius  $b$ , mass  $m$  and permeability  $\mu < 1$  in the magnetic field due to a horizontal disk of radius  $a$  and height  $h \ll a$  that has uniform magnetization  $\mathbf{M} = M\hat{\mathbf{z}}$  in the vertical direction. “Small” means that  $b \ll a$ .

Find the range of equilibrium heights  $z_0$  above the plane of the magnetized disc for which the motion is stable against small perturbations.

Note that if the sphere were paramagnetic, *i.e.*,  $\mu > 1$ , then it could be levitated (suspended) below the disk.

## 2 Solution

This problem is closely related to that of the Levitron<sup>TM</sup>, a well-known science toy [1, 2, 3, 4, 5].<sup>1</sup> The 2000 IgNobel Prize in Physics was awarded to Berry and Geim for their study of diamagnetic levitation [6].

### 2.1 The Magnetic Field On the Axis of the Disk

We recall that the uniform magnetization  $\mathbf{M}$  of the disk can be thought of as due to internal current loops whose net current density is zero everywhere inside the disk, but which leads to a surface current density  $c\mathbf{M} \times d\mathbf{S}$  on surface area element  $d\mathbf{S}$  (in Gaussian units). Since  $\mathbf{M}$  is vertical, the equivalent surface currents exist only on the vertical sides of the disk, and the total current circulating around the sides, of height  $h$  is

$$I = cMh. \tag{1}$$

Since height  $h$  is much less than the radius  $a$  of the disk, this current can be thought of as a line current when calculating the magnetic field.

The magnetic field on the axis of the disk is readily calculated using the Biot-Savart law,

$$B_z(0, z) = \frac{I}{c} \oint \frac{(d\mathbf{l} \times \mathbf{r})|_z}{r^3} = \frac{2\pi a I}{c} \frac{a}{r^3} = \frac{2\pi a^2 h M}{(a^2 + z^2)^{3/2}}. \tag{2}$$

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<sup>1</sup>A variant in which a magnetized disk is levitated between a pair of diamagnetic sheets is marketed at <http://www.kjmagnetics.com/proddetail.asp?prod=LEV2>

## 2.2 The Induced Magnetic Moment of the Sphere

In the presence of an external magnetic field the diamagnetic sphere takes on a magnetic moment  $\boldsymbol{\mu}$ . In the present problem the radius of the sphere  $b$  is small compared to that of the characteristic spatial extent  $a$  of the magnetic field, so we take the external field to be uniform over the sphere.

We recall that the magnetization density  $\mathbf{m}$  of a permeable sphere is uniform when the sphere is placed in a uniform external magnetic field. The magnetic moment is related to the magnetization density by

$$\boldsymbol{\mu} = \frac{4\pi}{3} b^3 \mathbf{m}. \quad (3)$$

We also recall that the self magnetic field inside a uniform magnetized sphere is uniform, and that the self magnetic field outside the sphere is just that of the dipole  $\boldsymbol{\mu}$ .

One way to relate these quantities is to imagine the uniform magnetization  $\mathbf{m}$  as causing (or being caused by) a surface density  $\sigma = \mathbf{m} \cdot \hat{\mathbf{n}}$  of magnetic poles. Then if we consider a Gaussian pillbox that encloses a small ‘‘polar cap’’ on the magnetized sphere, we learn that the self field  $\mathbf{H}$  obeys

$$H_{\text{out}}(\text{pole}) - H_{\text{in}} = 4\pi\sigma = 4\pi m = \frac{3}{b^3} |\boldsymbol{\mu}|. \quad (4)$$

The self field outside the sphere is the dipole form

$$\mathbf{H}_{\text{out}} = \frac{3(\boldsymbol{\mu} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}}{r^3}, \quad (5)$$

so the self field just above the pole is

$$H_{\text{out}}(\text{pole}) = \frac{2|\boldsymbol{\mu}|}{b^3}. \quad (6)$$

Hence, eq. (4) tells us that

$$\mathbf{H}_{\text{in}} = -\frac{\boldsymbol{\mu}}{b^3}. \quad (7)$$

The internal self magnetic field  $\mathbf{B}$  is also uniform, and is related by (in Gaussian units)

$$\mathbf{B}_{\text{in}} = \mathbf{H}_{\text{in}} + 4\pi\mathbf{m} = \mathbf{H}_{\text{in}} + \frac{3\boldsymbol{\mu}}{b^3} = \frac{2\boldsymbol{\mu}}{b^3}. \quad (8)$$

The total fields inside the sphere are the sum of the internal fields and the external fields,

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{in}} = \mathbf{B}_{\text{ext}} + \frac{2\boldsymbol{\mu}}{b^3}, \quad \mathbf{H} = \mathbf{H}_{\text{ext}} + \mathbf{H}_{\text{in}} = \mathbf{B}_{\text{ext}} - \frac{\boldsymbol{\mu}}{b^3}, \quad (9)$$

since the external field obeys  $\mathbf{B}_{\text{ext}} = \mathbf{H}_{\text{ext}}$ . Inside the permeable sphere we have  $\mathbf{B} = \mu\mathbf{H}$ , where  $\mu$  is the magnetic permeability, so

$$\mathbf{B}_{\text{ext}} + \frac{2\boldsymbol{\mu}}{b^3} = \mu \left( \mathbf{B}_{\text{ext}} - \frac{\boldsymbol{\mu}}{b^3} \right). \quad (10)$$

Thus,

$$\boldsymbol{\mu} = \frac{\mu - 1}{\mu + 2} b^3 \mathbf{B}_{\text{ext}}. \quad (11)$$

For a diamagnetic sphere,  $\mu < 1$ , so the induced magnetic moment is opposite in direction to that of the external field, as is to be expected from Lenz’ law.

## 2.3 Conditions for Stability

To discuss the center of mass motion, we construct a potential and require that the second spatial derivatives be positive when the first derivatives vanish.

The gravitational potential energy is just  $mgz$ , taking the  $z$ -axis as vertically upwards. The potential energy of a magnetic dipole  $\boldsymbol{\mu}$  in a magnetic field  $\mathbf{B}$  is  $-\boldsymbol{\mu} \cdot \mathbf{B}$ . In the present case, the moment is induced by the field according to eq. (11), so

$$U(r, z) = mgz - \boldsymbol{\mu} \cdot \mathbf{B}(r, z) = mgz - \frac{\mu - 1}{\mu + 2} B^2(r, z). \quad (12)$$

For a circularly symmetric field,  $\mathbf{B}(r, z)$ , the equilibrium points will be on the  $z$ -axis of symmetry. Then, the condition that  $(0, z_0)$  be an equilibrium point is

$$F_z = -\frac{\partial U(0, z_0)}{\partial z} = 0 = -mg + \frac{\mu - 1}{\mu + 2} \frac{\partial B^2(0, z_0)}{\partial z}, \quad (13)$$

$$F_r = -\frac{\partial U(0, z_0)}{\partial r} = 0 = \frac{\mu - 1}{\mu + 2} \frac{\partial B^2(0, z_0)}{\partial r}. \quad (14)$$

The conditions that this equilibrium be stable are

$$\frac{\partial^2 U(0, z_0)}{\partial z^2} = -\frac{\mu - 1}{\mu + 2} \frac{\partial^2 B^2(0, z_0)}{\partial z^2} > 0, \quad (15)$$

$$\frac{\partial^2 U(0, z_0)}{\partial r^2} = -\frac{\mu - 1}{\mu + 2} \frac{\partial^2 B^2(0, z_0)}{\partial r^2} > 0. \quad (16)$$

Since  $\mu < 1$ , the stability conditions are simply

$$\frac{\partial^2 B^2(0, z_0)}{\partial z^2} > 0, \quad (17)$$

$$\frac{\partial^2 B^2(0, z_0)}{\partial r^2} > 0. \quad (18)$$

## 2.4 Evaluation of the Field Derivatives

To complete the problem, we express the magnitude of the field  $B$  in terms of only its  $z$ -component,  $B_z(0, z)$  from eq. (2). The approach is to use Maxwell's equations,  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} = 0$ , to relate  $B_r$  to  $B_z$ . From the above, we see that we will use only the first and second derivatives of  $B$ , so it suffices to use a series expansion to second order in  $r$  and  $z$ . Say,

$$B_z(r, z) = B_0 + B_1(z - z_0) + B_2(z - z_0)^2 + B_3r + B_4r^2 + B_5r(z - z_0), \quad (19)$$

and

$$B_r(r, z) = C_0 + C_1(z - z_0) + C_2(z - z_0)^2 + C_3r + C_4r^2 + C_5r(z - z_0). \quad (20)$$

In cylindrical coordinates we have

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0, \quad (21)$$

and

$$(\nabla \times \mathbf{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0. \quad (22)$$

From eq. (21),

$$\begin{aligned} & [C_0 + C_1(z - z_0) + C_2(z - z_0)^2] / r + \\ & 2C_3 + 3C_4r + 2C_5(z - z_0) + \\ & B_1 + 2B_2(z - z_0) + B_5r = 0, \end{aligned} \quad (23)$$

and so

$$C_0 = C_1 = C_2 = 0, \quad C_3 = -\frac{B_1}{2}, \quad C_4 = -\frac{B_5}{3}, \quad C_5 = -B_2. \quad (24)$$

That is,

$$B_r(r, z) = -\frac{B_1 r}{2} - \frac{B_5 r^2}{3} - B_2 r(z - z_0). \quad (25)$$

Then, from eq. (22)

$$-B_2 r - B_3 - 2B_4 r - B_5(z - z_0) = 0, \quad (26)$$

and hence,

$$B_3 = B_5 = 0, \quad B_4 = -\frac{B_2}{2}. \quad (27)$$

Altogether,

$$B_z(r, z) = B_0 + B_1(z - z_0) + B_2(z - z_0)^2 - \frac{B_2 r^2}{2}, \quad (28)$$

and

$$B_r(r, z) = -\frac{B_1 r}{2} - B_2 r(z - z_0), \quad (29)$$

accurate to second order in  $r$  and  $z$ .

To evaluate the equilibrium conditions (13)-(14) and the stability conditions (17)-(18) we need  $B^2$  accurate to second order,

$$B^2 = B_z^2 + B_r^2 = B_0^2 + 2B_0B_1(z - z_0) + (B_1^2 + 2B_0B_2)(z - z_0)^2 + \left(\frac{B_1^2}{4} - B_0B_2\right)r^2. \quad (30)$$

The axial magnetic field is given by eq. (2),

$$B_z(0, z) = \frac{2\pi a^2 h M}{(a^2 + z^2)^{3/2}}. \quad (31)$$

Thus,

$$B_0 = B_z(0, z_0) = \frac{2\pi a^2 h M}{(a^2 + z_0^2)^{3/2}}, \quad (32)$$

$$B_1 = \frac{\partial B_z(0, z_0)}{\partial z} = -\frac{6\pi a^2 h M z_0}{(a^2 + z_0^2)^{5/2}}, \quad (33)$$

and

$$B_2 = \frac{1}{2} \frac{\partial^2 B_z(0, z_0)}{\partial z^2} = \frac{3\pi a^2 h M (4z_0^2 - a^2)}{(a^2 + z_0^2)^{7/2}}. \quad (34)$$

The condition (13) for vertical equilibrium is that

$$-mg\frac{\mu+2}{\mu-1} = -\frac{\partial B^2(0, z_0)}{\partial z} = -2B_0B_1 = \frac{24\pi^2 a^4 h^2 M^2 z_0}{(a^2 + z_0^2)^4}, \quad (35)$$

The righthand side of eq. (35) reaches a maximum for  $z_0 = a/\sqrt{7}$ , so vertical equilibrium exists only if

$$-mg\frac{\mu+2}{\mu-1} < \frac{1029\sqrt{7}\pi^2 h^2 M^2}{512a^3}. \quad (36)$$

When the equilibrium exists, there are always two solutions, one with  $z_0 < a/\sqrt{7}$  and the other with  $z_0 > a/\sqrt{7}$ . We find below that only the solution with  $z_0 > a/\sqrt{7}$  is stable.

The condition (14) for radial equilibrium is trivially satisfied at  $r = 0$ .

The condition (18) that the radial equilibrium be stable is that

$$2\frac{\partial^2 B^2(0, z_0)}{\partial z^2} = B_1^2 - 4B_0B_2 = \frac{12\pi^2 a^4 h^2 M^2 (2a^2 - 5z_0^2)}{(a^2 + z_0^2)^5} > 0, \quad (37)$$

which requires that  $z_0 < a\sqrt{2/5}$ .

The condition (17) that the vertical equilibrium be stable is that

$$\frac{1}{2}\frac{\partial^2 B^2(0, z_0)}{\partial z^2} = B_1^2 + 2B_0B_2 = \frac{12\pi^2 a^4 h^2 M^2 (7z_0^2 - a^2)}{(a^2 + z_0^2)^5} > 0, \quad (38)$$

which requires that  $z_0 > a/\sqrt{7}$ .

In sum, diamagnetic levitation is possible provided condition (36) is satisfied. The equilibrium radius is, of course, zero, and the equilibrium height  $z_0$  is given by eq. (35). The equilibrium is stable against both radial and vertical perturbations provided  $a/\sqrt{7} = 0.38a < z_0 < \sqrt{2/5}a = 0.63a$ . Surprisingly, in many cases there are two stable equilibria, one above  $a/\sqrt{3} = 0.58a$  and one below.

## References

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