

Diamagnetic Levitation

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1 Problem

Discuss the stability of levitation of a small diamagnetic sphere of radius b , mass m and permeability $\mu < 1$ in the magnetic field due to a horizontal disk of radius a and height $h \ll a$ that has uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$ in the vertical direction. “Small” means that $b \ll a$.

Find the range of equilibrium heights z_0 above the plane of the magnetized disc for which the motion is stable against small perturbations.

Note that if the sphere were paramagnetic, *i.e.*, $\mu > 1$, then it could be levitated (suspended) below the disk.

2 Solution

This problem is closely related to that of the LevitronTM, a well-known science toy [1, 2, 3, 4, 5]. The 2000 IgNobel Prize in Physics was awarded to Berry and Geim for their study of diamagnetic levitation [6].

2.1 The Magnetic Field On the Axis of the Disk

We recall that the uniform magnetization \mathbf{M} of the disk can be thought of as due to internal current loops whose net current density is zero everywhere inside the disk, but which leads to a surface current density $c\mathbf{M} \times d\mathbf{S}$ on surface area element $d\mathbf{S}$ (in Gaussian units). Since \mathbf{M} is vertical, the equivalent surface currents exist only on the vertical sides of the disk, and the total current circulating around the sides, of height h is

$$I = cMh. \quad (1)$$

Since height h is much less than the radius a of the disk, this current can be thought of as a line current when calculating the magnetic field.

The magnetic field on the axis of the disk is readily calculated using the Biot-Savart law,

$$B_z(0, z) = \frac{I}{c} \oint \frac{(d\mathbf{l} \times \mathbf{r})|_z}{r^3} = \frac{2\pi a I}{c} \frac{a}{r^3} = \frac{2\pi a^2 h M}{(a^2 + z^2)^{3/2}}. \quad (2)$$

2.2 The Induced Magnetic Moment of the Sphere

In the presence of an external magnetic field the diamagnetic sphere takes on a magnetic moment $\boldsymbol{\mu}$. In the present problem the radius of the sphere b is small compared to that of the characteristic spatial extent a of the magnetic field, so we take the external field to be uniform over the sphere.

We recall that the magnetization density \mathbf{m} of a permeable sphere is uniform when the sphere is placed in a uniform external magnetic field. The magnetic moment is related to the magnetization density by

$$\boldsymbol{\mu} = \frac{4\pi}{3}b^3\mathbf{m}. \quad (3)$$

We also recall that the self magnetic field inside a uniform magnetized sphere is uniform, and that the self magnetic field outside the sphere is just that of the dipole $\boldsymbol{\mu}$.

One way to relate these quantities is to imagine the uniform magnetization \mathbf{m} as causing (or being caused by) a surface density $\sigma = \mathbf{m} \cdot \hat{\mathbf{n}}$ of magnetic poles. Then if we consider a Gaussian pillbox that encloses a small “polar cap” on the magnetized sphere, we learn that the self field \mathbf{H} obeys

$$H_{\text{out}}(\text{pole}) - H_{\text{in}} = 4\pi\sigma = 4\pi m = \frac{3}{b^3}|\boldsymbol{\mu}|. \quad (4)$$

The self field outside the sphere is the dipole form

$$\mathbf{H}_{\text{out}} = \frac{3(\boldsymbol{\mu} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}}{r^3}, \quad (5)$$

so the self field just above the pole is

$$H_{\text{out}}(\text{pole}) = \frac{2|\boldsymbol{\mu}|}{b^3}. \quad (6)$$

Hence, eq. (4) tells us that

$$\mathbf{H}_{\text{in}} = -\frac{\boldsymbol{\mu}}{b^3}. \quad (7)$$

The internal self magnetic field \mathbf{B} is also uniform, and is related by (in Gaussian units)

$$\mathbf{B}_{\text{in}} = \mathbf{H}_{\text{in}} + 4\pi\mathbf{m} = \mathbf{H}_{\text{in}} + \frac{3\boldsymbol{\mu}}{b^3} = \frac{2\boldsymbol{\mu}}{b^3}. \quad (8)$$

The total fields inside the sphere are the sum of the internal fields and the external fields,

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{in}} = \mathbf{B}_{\text{ext}} + \frac{2\boldsymbol{\mu}}{b^3}, \quad \mathbf{H} = \mathbf{H}_{\text{ext}} + \mathbf{H}_{\text{in}} = \mathbf{B}_{\text{ext}} - \frac{\boldsymbol{\mu}}{b^3}, \quad (9)$$

since the external field obeys $\mathbf{B}_{\text{ext}} = \mathbf{H}_{\text{ext}}$. Inside the permeable sphere we have $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability, so

$$\mathbf{B}_{\text{ext}} + \frac{2\boldsymbol{\mu}}{b^3} = \mu \left(\mathbf{B}_{\text{ext}} - \frac{\boldsymbol{\mu}}{b^3} \right). \quad (10)$$

Thus,

$$\boldsymbol{\mu} = \frac{\mu - 1}{\mu + 2}b^3\mathbf{B}_{\text{ext}}. \quad (11)$$

For a diamagnetic sphere, $\mu < 1$, so the induced magnetic moment is opposite in direction to that of the external field, as is to be expected from Lenz’ law.

2.3 Conditions for Stability

To discuss the center of mass motion, we construct a potential and require that the second spatial derivatives be positive when the first derivatives vanish.

The gravitational potential energy is just mgz , taking the z -axis as vertically upwards. The potential energy of a magnetic dipole $\boldsymbol{\mu}$ in a magnetic field \mathbf{B} is $-\boldsymbol{\mu} \cdot \mathbf{B}$. In the present case, the moment is induced by the field according to eq. (11), so

$$U(r, z) = mgz - \boldsymbol{\mu} \cdot \mathbf{B}(r, z) = mgz - \frac{\mu - 1}{\mu + 2} B^2(r, z). \quad (12)$$

For a circularly symmetric field, $\mathbf{B}(r, z)$, the equilibrium points will be on the z -axis of symmetry. Then, the condition that $(0, z_0)$ be an equilibrium point is

$$F_z = -\frac{\partial U(0, z_0)}{\partial z} = 0 = -mg + \frac{\mu - 1}{\mu + 2} \frac{\partial B^2(0, z_0)}{\partial z}, \quad (13)$$

$$F_r = -\frac{\partial U(0, z_0)}{\partial r} = 0 = \frac{\mu - 1}{\mu + 2} \frac{\partial B^2(0, z_0)}{\partial r}. \quad (14)$$

The conditions that this equilibrium be stable are

$$\frac{\partial^2 U(0, z_0)}{\partial z^2} = -\frac{\mu - 1}{\mu + 2} \frac{\partial^2 B^2(0, z_0)}{\partial z^2} > 0, \quad (15)$$

$$\frac{\partial^2 U(0, z_0)}{\partial r^2} = -\frac{\mu - 1}{\mu + 2} \frac{\partial^2 B^2(0, z_0)}{\partial r^2} > 0. \quad (16)$$

Since $\mu < 1$, the stability conditions are simply

$$\frac{\partial^2 B^2(0, z_0)}{\partial z^2} > 0, \quad (17)$$

$$\frac{\partial^2 B^2(0, z_0)}{\partial r^2} > 0. \quad (18)$$

2.4 Evaluation of the Field Derivatives

To complete the problem, we express the magnitude of the field B in terms of only its z -component, $B_z(0, z)$ from eq. (2). The approach is to use Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$, to relate B_r to B_z . From the above, we see that we will use only the first and second derivatives of B , so it suffices to use a series expansion to second order in r and z . Say,

$$B_z(r, z) = B_0 + B_1(z - z_0) + B_2(z - z_0)^2 + B_3r + B_4r^2 + B_5r(z - z_0), \quad (19)$$

and

$$B_r(r, z) = C_0 + C_1(z - z_0) + C_2(z - z_0)^2 + C_3r + C_4r^2 + C_5r(z - z_0). \quad (20)$$

In cylindrical coordinates we have

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0, \quad (21)$$

and

$$(\nabla \times \mathbf{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0. \quad (22)$$

From eq. (21),

$$\begin{aligned} & [C_0 + C_1(z - z_0) + C_2(z - z_0)^2] / r + \\ & 2C_3 + 3C_4r + 2C_5(z - z_0) + \\ & B_1 + 2B_2(z - z_0) + B_5r = 0, \end{aligned} \quad (23)$$

and so

$$C_0 = C_1 = C_2 = 0, \quad C_3 = -\frac{B_1}{2}, \quad C_4 = -\frac{B_5}{3}, \quad C_5 = -B_2. \quad (24)$$

That is,

$$B_r(r, z) = -\frac{B_1 r}{2} - \frac{B_5 r^2}{3} - B_2 r(z - z_0). \quad (25)$$

Then, from eq. (22)

$$-B_2 r - B_3 - 2B_4 r - B_5(z - z_0) = 0, \quad (26)$$

and hence,

$$B_3 = B_5 = 0, \quad B_4 = -\frac{B_2}{2}. \quad (27)$$

Altogether,

$$B_z(r, z) = B_0 + B_1(z - z_0) + B_2(z - z_0)^2 - \frac{B_2 r^2}{2}, \quad (28)$$

and

$$B_r(r, z) = -\frac{B_1 r}{2} - B_2 r(z - z_0), \quad (29)$$

accurate to second order in r and z .

To evaluate the equilibrium conditions (13)-(14) and the stability conditions (17)-(18) we need B^2 accurate to second order,

$$B^2 = B_z^2 + B_r^2 = B_0^2 + 2B_0B_1(z - z_0) + (B_1^2 + 2B_0B_2)(z - z_0)^2 + \left(\frac{B_1^2}{4} - B_0B_2\right)r^2. \quad (30)$$

The axial magnetic field is given by eq. (2),

$$B_z(0, z) = \frac{2\pi a^2 h M}{(a^2 + z^2)^{3/2}}. \quad (31)$$

Thus,

$$B_0 = B_z(0, z_0) = \frac{2\pi a^2 h M}{(a^2 + z_0^2)^{3/2}}, \quad (32)$$

$$B_1 = \frac{\partial B_z(0, z_0)}{\partial z} = -\frac{6\pi a^2 h M z_0}{(a^2 + z_0^2)^{5/2}}, \quad (33)$$

and

$$B_2 = \frac{1}{2} \frac{\partial^2 B_z(0, z_0)}{\partial z^2} = \frac{3\pi a^2 h M (4z_0^2 - a^2)}{(a^2 + z_0^2)^{7/2}}. \quad (34)$$

The condition (13) for vertical equilibrium is that

$$-mg\frac{\mu+2}{\mu-1} = -\frac{\partial B^2(0, z_0)}{\partial z} = -2B_0B_1 = \frac{24\pi^2 a^4 h^2 M^2 z_0}{(a^2 + z_0^2)^4}, \quad (35)$$

The righthand side of eq. (35) reaches a maximum for $z_0 = a/\sqrt{7}$, so vertical equilibrium exists only if

$$-mg\frac{\mu+2}{\mu-1} < \frac{1029\sqrt{7}\pi^2 h^2 M^2}{512a^3}. \quad (36)$$

When the equilibrium exists, there are always two solutions, one with $z_0 < a/\sqrt{7}$ and the other with $z_0 > a/\sqrt{7}$. We find below that only the solution with $z_0 > a/\sqrt{7}$ is stable.

The condition (14) for radial equilibrium is trivially satisfied at $r = 0$.

The condition (18) that the radial equilibrium be stable is that

$$2\frac{\partial^2 B^2(0, z_0)}{\partial z^2} = B_1^2 - 4B_0B_2 = \frac{12\pi^2 a^4 h^2 M^2 (2a^2 - 5z_0^2)}{(a^2 + z_0^2)^5} > 0, \quad (37)$$

which requires that $z_0 < a\sqrt{2/5}$.

The condition (17) that the vertical equilibrium be stable is that

$$\frac{1}{2}\frac{\partial^2 B^2(0, z_0)}{\partial z^2} = B_1^2 + 2B_0B_2 = \frac{12\pi^2 a^4 h^2 M^2 (7z_0^2 - a^2)}{(a^2 + z_0^2)^5} > 0, \quad (38)$$

which requires that $z_0 > a/\sqrt{7}$.

In sum, diamagnetic levitation is possible provided condition (36) is satisfied. The equilibrium radius is, of course, zero, and the equilibrium height z_0 is given by eq. (35). The equilibrium is stable against both radial and vertical perturbations provided $a/\sqrt{7} = 0.38a < z_0 < \sqrt{2/5}a = 0.63a$. Surprisingly, in many cases there are two stable equilibria, one above $a/\sqrt{3} = 0.58a$ and one below.

3 References

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