A Damped Oscillator as a Hamiltonian System

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1 Problem

It is generally considered that systems with friction are not part of Hamiltonian dynamics, but this is not always the case. Show that a (nonrelativistic) damped harmonic oscillator can be described by a Hamiltonian (and by a Lagrangian), with the implication that Liouville's theorem applies here.

Consider motion in coordinate x of a particle of mass m with equation of motion,

$$m\ddot{x} + \beta\dot{x} + kx = 0, \qquad \text{or} \qquad \ddot{x} + \alpha\dot{x} + \omega_0^2 x = 0, \tag{1}$$

where $\alpha = \beta/m$ and $\omega_0^2 = k/m$.

Comment on the root-mean square emittance of a "bunch" of noninteracting particles each of which obeys eq. (1).

Deduce two independent constants of the motion for a single particle.

Hint: Consider first the case of zero spring constant k.

2 Solution

2.1 $k = 0$

When the spring constant k is zero there is no potential energy, so the spirit of Lagrange and Hamilton is to consider the kinetic energy $T = m\dot{x}^2/2$. The equation of motion (1) can be written in the manner of Lagrange as[1]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} + \alpha\frac{\partial T}{\partial \dot{x}} = 0\,. \tag{2}$$

This form can be written more compactly as

$$\frac{d}{dt}\frac{\partial T^\star}{\partial \dot{x}} = 0\,, \qquad \text{where} \qquad T^\star = T\,e^{\alpha t}. \tag{3}$$

Hence, a Lagrangian for this case is $\mathcal{L} = T^\star$, and the canonical momentum p conjugate to coordinate x is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\,e^{\alpha t} = p_{\rm mech}\,e^{\alpha t}, \tag{4}$$

where $p_{\rm mech} = m\dot{x}$ is the ordinary mechanical momentum.

The Hamiltonian for this case is then

$$H = \dot{x}p - \mathcal{L} = \mathcal{L} = \frac{m\dot{x}^2}{2}\,e^{\alpha t}. \tag{5}$$

[1] See sec. 2a of [1].

2.2 Nonzero k

When the spring constant k is nonzero we consider the potential energy $V = kx^2/2$, and the equation of motion (1) can be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} + \alpha\frac{\partial T}{\partial \dot{x}} = \frac{\partial(-V)}{\partial x}\,. \qquad (6)$$

This form can be written more compactly as

$$\frac{d}{dt}\frac{\partial T^\star}{\partial \dot{x}} = \frac{\partial(-V^\star)}{\partial x}\,, \qquad \text{where} \qquad T^\star = T\,e^{\alpha t}, \quad V^\star = V\,e^{\alpha t}. \qquad (7)$$

Hence, a Lagrangian for this case is $\mathcal{L} = T^\star - V^\star$. The momentum conjugate to coordinate x is again

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\,e^{\alpha t} = p_{\rm mech}\,e^{\alpha t}, \qquad (8)$$

so the Hamiltonian for this case is[2]

$$H = \dot{x}p - \mathcal{L} = T^\star + V^\star = (T+V)\,e^{\alpha t} = \frac{p^2}{2m}\,e^{-\alpha t} + V\,e^{\alpha t} = \frac{m\dot{x}^2 + kx^2}{2}\,e^{\alpha t} = U\,e^{\alpha t}, \qquad (9)$$

where U is the mechanical energy of the system.[3]

2.3 Liouville

This section is based on sec. 3.8 of [5].

The motion of a system governed by eq. (1) can be written as

$$x(t) = x_1\,e^{(\alpha'-\alpha)t/2} + x_2\,e^{-(\alpha'+\alpha)t/2}, \qquad (10)$$

where $\alpha' = \sqrt{\alpha^2 - 4\omega_0^2}$ is possibly imaginary (as may be x_1 and x_2). The canonical momentum (8) is

$$p(t) = mx_1\frac{\alpha'-\alpha}{2}\,e^{(\alpha'+\alpha)t/2} - mx_2\frac{\alpha'+\alpha}{2}\,e^{-(\alpha'-\alpha)t/2}. \qquad (11)$$

The initial conditions (at time $t = 0$) are

$$x(0) \equiv x_0 = x_1 + x_2, \qquad p(0) \equiv p_0 = mx_1\frac{\alpha'-\alpha}{2} - mx_2\frac{\alpha'+\alpha}{2}. \qquad (12)$$

$$x_1 = x_0\frac{\alpha'+\alpha}{2\alpha'} + \frac{p_0}{m\alpha'}\,, \qquad x_2 = x_0\frac{\alpha'-\alpha}{2\alpha'} - \frac{p_0}{m\alpha'}\,. \qquad (13)$$

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[2]The Hamiltonian (9) appears to have first been given via the present argument in [2]. Impressively, this procedure for identifying Hamiltonians can be extended to certain nonlinear examples with dissipation [3]. The Lagrangian $\mathcal{L} = (T-V)e^{\alpha t}$ was deduced by Bateman in 1931 [4], top of p. 817, as a special case of his "dual-Lagrangian".

[3]Clearly, the Lagrangian $\mathcal{L} = (T-V)e^{\alpha t}$ and the Hamiltonian $H = (T+V)e^{\alpha t}$ apply to any particle subject to velocity-dependent damping as well as a force derivable from a potential V.

Thus,

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} A &= e^{-\alpha t/2} \left(\frac{\alpha' + \alpha}{2\alpha'} e^{\alpha' t/2} + \frac{\alpha' - \alpha}{2\alpha'} e^{-\alpha' t/2} \right), & (15) \\ B &= e^{-\alpha t/2} \frac{e^{\alpha' t/2} - e^{-\alpha' t/2}}{m\alpha'}, & (16) \\ C &= m\, e^{\alpha t/2} \frac{\alpha'^2 - \alpha^2}{4\alpha'} \left(e^{\alpha' t/2} - e^{-\alpha'_t/2} \right), & (17) \\ D &= e^{\alpha t/2} \left(\frac{\alpha' - \alpha}{2\alpha'} e^{\alpha' t/2} + \frac{\alpha' + \alpha}{2\alpha'} e^{-\alpha' t/2} \right). & (18) \end{aligned}$$

The determinant of the transformation matrix is

$$\begin{aligned} \Delta &= AD - BC \\ &= \frac{\alpha'^2 - \alpha^2}{4\alpha'^2} e^{\alpha' t} + \frac{\alpha'^2 + \alpha^2}{2\alpha'^2} + \frac{\alpha'^2 - \alpha^2}{4\alpha'^2} e^{-\alpha' t} - \frac{\alpha'^2 - \alpha^2}{4\alpha'^2} \left(e^{\alpha' t} - 2 + e^{-\alpha' t} \right) = 1. \end{aligned} \tag{19}$$

Thus, the linear (canonical) transformation (14) preserves area in the x-p (phase) space, which verifies that Liouville's theorem [6, 7, 8] holds for this Hamiltonian system.

2.4 RMS Emittance

When considering a "bunch" of particles, a practical measure of their extent in phase space is the root-mean-square emittance, which for a 2-dimensional phase space as in the present example can be defined as

$$\epsilon_{\rm canonical}(t) = \sqrt{\langle x^2(t) \rangle \langle p^2(t) \rangle - \langle x(t) p(t) \rangle^2}. \tag{20}$$

As noted in [9], this rms emittance is invariant under linear canonical transformations, so ϵ of eq. (20) is a constant of the motion for a "bunch" of (noninteracting) particles each of which obeys eq. (1).

However, in most applications of the rms emittance concept, people use the mechanical momentum $p_{\rm mech}$ rather than the canonical momentum p. In the present example, $p_{\rm mech} = e^{-\alpha t} p$ according to eq. (8), such that the rms mechanical emittance decreases exponentially with time,

$$\epsilon_{\rm mech}(t) = \sqrt{\langle x^2(t) \rangle \langle p^2_{\rm mech}(t) \rangle - \langle x(t) p_{\rm mech}(t) \rangle^2} = e^{-\alpha t} \epsilon_{\rm canonical}(t). \tag{21}$$

The rms mechanical emittance (21) is "cooled" by the damping, whereas the rms canonical emittance (20) is not.[4]

[4]This contrasts with a theorem due to Swann [10] that if a particle of electric charge e is acted upon only by an electromagnetic field with vector potential $\mathbf{A}$ then the volume in $(\mathbf{x}, \mathbf{p})$ phase space of a "bunch" of such particles is the same as that in $(\mathbf{x}, \mathbf{p}_{\rm mech})$ phase space, where $\mathbf{p} = \mathbf{p}_{\rm mech} + e\mathbf{A}/c$.

2.5 Relativistic Damped Harmonic Oscillator

In accelerator physics the particles of interest typically have velocities near the speed c of light in vacuum, so we also give a relativistic version of the preceeding analysis.

If the force on the particle (of rest mass m) can be deduced from a potential V, a relativistic Hamiltonian is

$$H(x, p_{\rm mech}) = E_{\rm mech} + V = c\sqrt{m^2c^2 + p_{\rm mech}^2} + V, \tag{22}$$

where $p_{\rm mech} = \gamma m\dot{x}$, $E_{\rm mech} = \gamma mc^2 = c\sqrt{m^2c^2 + p_{\rm mech}^2}$ and $\gamma = 1/\sqrt{1 - \dot{x}^2/c^2}$. Hamilton's equations of motion for this system are

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_{\rm mech}} = \frac{p_{\rm mech}}{E_{\rm mech}} = \dot{x}, \tag{23}$$

$$\frac{dp_{\rm mech}}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} = F. \tag{24}$$

If the force includes the velocity-dependent term $-\beta\dot{x} = -\alpha m\dot{x}$, we follow sec. 2.2 to consider the canonical momentum

$$p = p_{\rm mech}\, e^{\alpha t}, \tag{25}$$

and the Hamiltonian

$$H(x, p, t) = e^{\alpha t}\,(E_{\rm mech} + V) = e^{\alpha t}\left(c\sqrt{m^2c^2 + p_{\rm mech}^2} + V\right) = e^{\alpha t}\left(c\sqrt{m^2c^2 + p^2\, e^{-2\alpha t}} + V\right). \tag{26}$$

Hamilton's equations of motion for this system are

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p\, e^{-\alpha t}}{E_{\rm mech}} = \frac{p_{\rm mech}}{E_{\rm mech}} = \dot{x}, \tag{27}$$

$$\frac{dp}{dt} = \frac{d(p_{\rm mech}\, e^{\alpha t})}{dt} = e^{\alpha t}\left(\frac{dp_{\rm mech}}{dt} + \alpha p_{\rm mech}\right) = -\frac{\partial H}{\partial x} = -e^{\alpha t}\frac{\partial V}{\partial x}\,, \tag{28}$$

The last equation can be rewritten as

$$\frac{dp_{\rm mech}}{dt} + \alpha p_{\rm mech} + \frac{\partial V}{\partial x} = \gamma^3(\dot{x})m\ddot{x} + \beta\dot{x} + \frac{\partial V}{\partial x} = 0, \tag{29}$$

which is the equation of motion (nonlinear in $\dot{x}$) of a relativistic particle subject to velocity-dependent damping and another force that is derivable from the potential V.

In particular, we see that the relativistic, damped harmonic oscillator is a Hamiltonian system, and a "bunch" of such (noninteracting) particles obeys Liouville's theorem.

If we characterize the extent of the "bunch" in phase space by an rms emittance, we must note that the canonical transformation $(x_0, p_0) \to (x(t), p(t))$ is not linear (unlike the non-relativistic case), with the implication that the rms canonical emittance (20) actually grows with time [11]. The rms mechanical emittance (21) is again $e^{-\alpha t}$ times the rms canonical emittance, and the exponential damping factor is stronger than the emittance growth due to the nonlinear relativistic time evolution, such that the rms mechanical emittance is "cooled" with time.

2.6 Constants of the Motion and Alternative Hamiltonians

The second-order differential equation (1) has two independent constants of integration, implying that motion governed by this equation has two independent constants of the motion. Of course, the physical significance of these constants will be unclear in systems where energy is not conserved, so this section is largely a mathematical exercise.

If the Hamiltonian of the system were time independent, it would be such a constant of the motion, so one method of identifying the desired constants of the motion is to seek alternative Hamiltonians that are independent of time. This was first done for the damped harmonic oscillator by Havas [12], and next by Leach [13] (and by Lemos [14][5]), each of whom displayed one time-independent Hamiltonian (which happen to be "inequivalent"[6]).

The analysis of Lemos is much simpler than that of Havas, so we first illustrate the former. Inspection of the Hamiltonian (9) indicates that the change of variables,[7]

$$X = x\, e^{\alpha t/2}, \qquad P = p\, e^{-\alpha t/2}, \tag{30}$$

leads to the formally time-independent form $h(X,P) = P^2/2m + kX^2/2$. However, h is not the Hamiltonian in terms of the coordinates X and P. For that, we need the so-called generating function Φ of the canonical transformation (30),[8] which obeys $p = \partial\Phi/\partial x$ and $X = \partial\Phi/\partial P$. In the present case, $\Phi = xP\, e^{\alpha t/2}$, and the transformed Hamiltonian is

$$H(X,P) = H(x,p,t) + \frac{\partial\Phi}{\partial t} = \frac{P^2}{2m} + \frac{\alpha XP}{2} + \frac{kX^2}{2}\,, \tag{31}$$

which is independent of time, and hence a constant of motion of the system. We rewrite $H(X,P)$ in terms of x and p or $\dot{x}$ (using eqs. (8) and (30)) as the constant function

$$F(x,p) = \frac{p^2\, e^{-\alpha t}}{2m} + \frac{\alpha x p}{2} + \frac{kx^2\, e^{\alpha t}}{2} = e^{\alpha t}\,\frac{m}{2}\left(\dot{x}^2 + \alpha x\dot{x} + \omega_0^2 x^2\right)\,, \tag{32}$$

which reverts to the mechanical energy when $\alpha = 0$.

Havas' approach [12] does not involve a change of variables, but seeks an "integrating function" $f(x,\dot{x},t)$ such that an equation of motion $g(x,\dot{x},\ddot{x},t) = 0$ can be related to a Lagrangian $\mathcal{L}$ according to[9]

$$fg = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x}\,. \tag{33}$$

Then, if the corresponding Hamiltonian $H = \dot{x}p - \mathcal{L}$ is independent of time, where $p = \partial\mathcal{L}/\partial\dot{x}$, it will be a constant of the motion. For a damped oscillator with $g = m\ddot{x} + \beta\dot{x} + kx$,

[5] Leach analyzed a damped harmonic oscillator with time-dependent friction and spring constant, and Lemos gave a simplified analysis analysis for time-independent parameters.

[6] Given two "inequivalent", time-independent Hamiltonians for a systems, one can generate an infinite set of alternative Hamiltonians, as discussed in [15]. See also [16] and references therein.

[7] This change of variables was considered in sec. 11 of [17].

[8] See, for example, sec. 45 of [8].

[9] Havas' method differs slightly from that used in secs. 2.1-2 where the nominal Lagrangian $\mathcal{L}_0 = T - V$ was multiplied by $e^{\alpha t}$ to give a Lagrangian $\mathcal{L} = e^{\alpha t}\mathcal{L}_0$ from which the equation of motion can be deduced.

Havas found (p. 387 of [12]) that $f = (m\dot{x}^2 + \beta x\dot{x} + kx^2)^{-1}$, and for $4km > \beta^2$,

$$\mathcal{L} = \frac{2m\dot{x} + \beta x}{x\sqrt{4km - \beta^2}} \tan^{-1}\left(\frac{2m\dot{x} + \beta x}{x\sqrt{4km - \beta^2}}\right) - \frac{1}{2}\ln(m\dot{x}^2 + \beta x\dot{x} + kx^2), \tag{34}$$

$$H = -\frac{\beta}{\sqrt{4km - \beta^2}} \tan^{-1}\left(\frac{2m\dot{x} + \beta x}{x\sqrt{4km - \beta^2}}\right) + \frac{1}{2}\ln(m\dot{x}^2 + \beta x\dot{x} + kx^2). \tag{35}$$

Other constants of the motion can be found by a clever approach to integrating eq. (1) [18]. In this, the parameters α and ω_0^2 are replaced by λ_1 and λ_2 according to

$$\alpha = \lambda_1 + \lambda_2, \qquad \omega_0^2 = \lambda_1\lambda_2, \qquad \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\omega_0^2}}{2}, \tag{36}$$

such that the equation of motion becomes

$$\ddot{x} + (\lambda_1 + \lambda_2)\dot{x} + \lambda_1\lambda_2 x = \ddot{x} + \lambda_1\dot{x} + \lambda_2(\dot{x} + \lambda_1 x) = 0. \tag{37}$$

This is a first-order differential equation in the variable $\dot{x} + \lambda_1 x$, which can be integrated to give

$$e^{\lambda_2 t}(\dot{x} + \lambda_1 x) = \text{const} = D_1. \tag{38}$$

Interchanging λ_1 and λ_2 we obtain a second constant of the motion,

$$e^{\lambda_1 t}(\dot{x} + \lambda_2 x) = \text{const} = D_2. \tag{39}$$

These two constants of the motion can be combined to give other constants of the motion, such as

$$F = \frac{m}{2}D_1 D_2 = e^{(\lambda_1+\lambda_2)t}\frac{m}{2}(\dot{x}^2 + (\lambda_1 + \lambda_2)x\dot{x} + \lambda_1\lambda_2 x^2) = e^{\alpha t}\frac{m}{2}(\dot{x}^2 + \alpha x\dot{x} + \omega_0^2 x^2), \tag{40}$$

and

$$B = \frac{D_1^{\lambda_1}}{D_2^{\lambda_2}} = \frac{(\dot{x} + \lambda_1 x)^{\lambda_1}}{(\dot{x} + \lambda_2 x)^{\lambda_2}}. \tag{41}$$

The constant F of eq. (40) is the same as that found in eq. (32), while the constant B of eq. (41) appears to be different from H of eq. (35). The symbol B is used in honor of Bohlin, who apparently discussed constants of this form in 1908 [18].

Interest in Hamiltonians for the damped oscillator arises mainly in the context of quantum theory, where there seems to be an ongoing debate as to which of the many "inequivalent" Hamiltonians is "best". A review as of 1981 is [19], and an example of recent commentary is [20].

Another issue illustrated by the present example is that equations of motion for a system with damping can be deduced from a variational principle in some cases, as perhaps first noted by Bateman [4]. This issue relates to the origin of conservation laws, as time invariance of a Hamiltonian (such as eq. (35)) apparently does not necessarily imply that this constant Hamiltonian is the system energy). That is, Noether's theorem [21, 22] applies to (classical and quantum) field theory but not to classical mechanics. The ongoing debate on this theme is reviewed in [23]; a recent comment is at [24].

2.7 Time-Dependent Forces

In the application of the present example to accelerator physics we have tacitly imagined performing the analysis in the rest frame of a reference particle near the center of a "bunch" of (noninteracting) particles. In practice, as the "bunch" propagates it will encounter different external forces at different times, which leads to an interest in a Hamiltonian description for time-dependent forces.

Studies of an undamped harmonic oscillator with a time-dependent spring constant began with Lecornu (1895) [25] and Rayleigh (1902) [26] in classical contexts. In a famous discussion at the 1911 Solvay Conference [27], Lorentz posed the problem of a time-dependent quantum oscillator, and Einstein argued (briefly) that the ratio of the instantaneous energy $U(t) = (m\dot{x}^2 + k(t)x^2)/2$ to the instantaneous angular frequency $\omega(t)$ would be constant (which notion seems clearer in a quantum view than classically). Einstein's prescient remark lead to the development by Ehrenfest (1916) [28] of the concept of adiabatic invariance in systems with "slow" time dependence.[10] Kruskal (1962) [30] suggested a method for analysis when the time dependence is somewhat more rapid than that for which adiabatic invariance holds, and Lewis (1967) [31, 32] extended this method to arbitrary time dependence of the spring constant.[11] Leach (1978) [13] extended Lewis' analysis to include time-dependent damping proportional to the velocity.

Following Leach [13], we consider a damped harmonic oscillator with time-dependent forces,

$$m\ddot{x} + \beta(t)\dot{x} + k(t)x = 0, \qquad \ddot{x} + \alpha(t)\dot{x} + \omega(t)^2 x = 0. \tag{42}$$

Multiplying this differential equation by

$$f(t) = \int_0^t \alpha(t')\,dt', \tag{43}$$

it can be rewritten as

$$\frac{d}{dt}(m\dot{x}\,e^f) + \frac{\partial}{\partial x}\left(\frac{k\,e^f\,x^2}{2}\right) = 0. \tag{44}$$

This equation of motion can be deduced from the Lagrangian

$$\mathcal{L} = \frac{m\dot{x}^2 - kx^2}{2}\,e^f, \tag{45}$$

for which the canonical momentum is

$$p = m\dot{x}\,e^f, \tag{46}$$

and the Hamiltonian is[12]

$$H = \dot{x}p - \mathcal{L} = \frac{p^2}{2}\,e^{-f} + \frac{kx^2}{2}\,e^f = m\frac{\dot{x}^2 + \omega^2 x^2}{2}\,e^f. \tag{47}$$

[10] Certain mathematical subtleties related to adiabatic invariance in Lorentz' example are considered to have been clarified only in 1963 [29].

[11] For a perspective on the methods of Kruskal and Lewis, see [33].

[12] Similarly, replacing αt by f in eq. (26) provides a Hamiltonian for the relativistic, damped harmonic oscillator with time-dependent forces.

Thus, the time-dependent damped harmonic oscillator is also a Hamiltonian system, and a "bunch" of particles that each obey eq. (42) also obey Liouville's theorem.

If, for example, the damping is constant until time t_1, and zero thereafter,

$$f(t) = \begin{cases} e^{\alpha t} & (0 < t < t_1), \\ e^{\alpha t_1} & (t > t_1). \end{cases} \tag{48}$$

When the damping goes to zero at time t_1, the canonical momentum does not revert to the mechanical momentum $m\dot{x}$, but to this multiplied by the (large) constant $e^{\alpha t_1}$. The rms canonical emittance (20) is e^f times the rms mechanical emittance (21) at all times, so again the former remains constant in time while the latter is exponentially damped.

The literature is much concerned with the invariants of the time-dependent damped harmonic oscillator, although only one such invariant has been found so far.[13] Lewis [31, 32] deduced an invariant for the undamped, time-dependent harmonic oscillator, which turns out to have been anticipated in a little-known work by Ermakov in 1880 [34]. The generalization of this invariant for the case of time-dependent damping was given by Leach [13].[14] As might be expected, the physical significance of these invariants is obscure.

References

[1] P. Caldirola, *Forze Non Conservative nella Meccanica Quantistica*, Nuovo Cim. **18**, 393 (1941), http://physics.princeton.edu/~mcdonald/examples/QM/caldirola_nc_18_393_41.pdf

[2] E. Kanai, *On the Quantization of the Dissipative Systems*, Prog. Theor. Phys. **3**, 440 (1949), http://physics.princeton.edu/~mcdonald/examples/QM/kanai_ptp_3_440_49.pdf

[3] V.K. Chandrasekar *et al.*, *A simple and unified approach to identify integrable nonlinear oscillators and systems*, J. Math. Phys. **47**, 023508 (2006), http://physics.princeton.edu/~mcdonald/examples/mechanics/chandrasekar_jmp_47_023508_06.pdf

[4] H. Bateman, *On Dissipative Systems and Related Variational Principles*, Phys. Rev. **38**, 815 (1931), http://physics.princeton.edu/~mcdonald/examples/mechanics/bateman_pr_38_815_31.pdf

[5] G.J. Sussman and J. Wisdom, *Structure and Interpretation of Classical Mechanics*, 2nd ed. (MIT Press, 2014), http://physics.princeton.edu/~mcdonald/examples/mechanics/sussman_mechanics_sec3.8.pdf

[6] J. Liouville, *Note sur la Théorie de la Variation des constantes arbitraires*, J. Math. Pures Appl. **3**, 342 (1838), http://physics.princeton.edu/~mcdonald/examples/mechanics/liouville_jmpa_3_342_38.pdf

[13]Replacing αt by f in eq. (30) leads to a canonical transformation with generating function $\Phi = xP\, e^{f/2}$, so the transformed Hamiltonian again has the form (31), but this is no longer independent of time as $\alpha = \alpha(t)$, and eq. (32) (with f for αt) is not an invariant.

[14]Leach's invariant can be deduced by seeking a (generalized) canonical transformation in the extended phase space $(x, p, t, -U)$ such that the Hamiltonian $H'(x', p')$ is independent of t' (and so $H' = U'$ is constant). See sec. 3.3 of [35].

[7] D.D. Nolte, *The Tangled Tale of Phase Space*, Phys. Today **63**, 4, 32 (Apr. 2010), http://physics.princeton.edu/~mcdonald/examples/mechanics/nolte_pt_63_4_32_10.pdf

[8] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976).

[9] E.D. Courant and H.S. Snyder, *Theory of the Alternating-Gradient Synchrotron*, Ann. Phys. (NY) **3**, 1 (1958), http://physics.princeton.edu/~mcdonald/examples/accel/courant_ap_3_1_58.pdf

[10] W.F.G. Swann, *Application of Liouville's Theorem to Electron Orbits in the Earth's Magnetic Field*, Phys. Rev. **44**, 224 (1933), http://physics.princeton.edu/~mcdonald/examples/accel/swann_pr_44_224_33.pdf

[11] K.T. McDonald, *Emittance Growth from Weak Relativistic Effects* (Mar. 21, 2011), http://physics.princeton.edu/~mcdonald/examples/growth.pdf

[12] P. Havas, *The Range of Application of the Lagrange Formalism*, Suppl. Nuovo Cim. **5**, 363 (1957), http://physics.princeton.edu/~mcdonald/examples/mechanics/havas_nc_5s_363_57.pdf

[13] P.G.L. Leach, *On a Direct Method for the Determination of an Exact Invariant for the Time-Dependent Harmonic Oscillator*, J. Austral. Math. Soc. **209**, 97 (1977), http://physics.princeton.edu/~mcdonald/examples/mechanics/leach_jams_20_97_77.pdf

[14] N.A. Lemos, *Canonical approach to the damped harmonic oscillator*, Am. J. Phys. **47**, 857 (1979), http://physics.princeton.edu/~mcdonald/examples/mechanics/lemos_ajp_47_857_79.pdf

[15] C. Leubner, *Inequivalent Lagrangians from Constants of the Motion*, Phys. Lett. **86A**, 68 (1981), http://physics.princeton.edu/~mcdonald/examples/mechanics/leubner_pl_86a_68_81.pdf

[16] A. Choudhuri *et al.*, *Symmetries and conservation laws of the damped harmonic oscillator*, Pramana **70**, 657 (2008), http://physics.princeton.edu/~mcdonald/examples/mechanics/choudhuri_pramana_70_657_08.pdf

[17] H. Lamb, *The Dynamical Theory of Sound* (Arnold, London, 1910), http://physics.princeton.edu/~mcdonald/examples/mechanics/lamb_sound_10.pdf

[18] E.W. Gettys *et al.*, *Bohlin's and other integrals for the damped harmonic oscillator*, Am. J. Phys. **42**, 162 (1981), http://physics.princeton.edu/~mcdonald/examples/mechanics/gettys_ajp_49_162_81.pdf

[19] H. Dekker, *Classical and Quantum Mechanics of the Damped Harmonic Oscillator*, Phys. Rep. **80**, 1 (1981), http://physics.princeton.edu/~mcdonald/examples/QM/dekker_pr_80_1_81.pdf

[20] D. Schuch *et al.*, *Interrelations between different canonical descriptions of dissipative systems*, Phys. Scr. **90**, 045209 (2015), http://physics.princeton.edu/~mcdonald/examples/mechanics/schuch_ps_90_045209_15.pdf

[21] E. Noether, *Invariante Variationenprobleme*, Nachr. Kgl. Ges. Wiss. Göttingen, 235 (1918), http://physics.princeton.edu/~mcdonald/examples/mechanics/noether_nkwg_235_18.pdf
English translation: *Invariant Variation Problems*, Trans. Theory Stat. Phys. **1**, 186 (1971), http://physics.princeton.edu/~mcdonald/examples/mechanics/noether_ttsp_3_186_71.pdf

[22] E.L. Hill, *Hamilton's Principle and the Conservation Theorems of Mathematical Physics*, Rev. Mod. Phys. **23**, 253 (1951),
http://physics.princeton.edu/~mcdonald/examples/mechanics/hill_rmp_23_253_51.pdf

[23] S.R. Smith, *Symmetries and the explanation of conservation laws in the light of the inverse problem in Lagrangian mechanics*, Stud. Hist. Phil. Mod. Phys. Scr. **39**, 325 (2008), http://physics.princeton.edu/~mcdonald/examples/mechanics/smith_shpmp_39_325_08.pdf

[24] H Qin, J.W. Burby and R.C. Davidson, *Field theory and weak Euler-Lagrange equation for classical particle-field systems*, Phys. Rev. E **90**, 032102 (2014),
http://physics.princeton.edu/~mcdonald/examples/mechanics/qin_pre_90_043102_14.pdf

[25] L. Lecornu, *Mémoire sur la Pendule de Longeur Variable*, Acta Math. **19**, 201 (1895),
http://physics.princeton.edu/~mcdonald/examples/mechanics/lecornu_am_19_201_95.pdf

[26] Lord Rayleigh, *The Pressure of Vibrations*, Phil. Mag. **3**, 338 (1902),
http://physics.princeton.edu/~mcdonald/examples/mechanics/rayleigh_pm_3_338_02.pdf

[27] H.A. Lorentz and A. Einstein, *Discussion at the 1911 Solvay Conference*, *La Théorie du Rayonnement et les Quanta* (Gauthier-Villars, Paris, 1912), p. 450,
http://physics.princeton.edu/~mcdonald/examples/QM/solvay_11.pdf

[28] P. Ehrenfest, *Adiabatische Invarianten und Quantentheorie*, Ann. d. Phys. **356**, 327 (1916), http://physics.princeton.edu/~mcdonald/examples/QM/ehrenfest_ap_356_327_16.pdf

[29] J.E. Littlewood, *Lorentz's Pendulum Problem*, Ann. Phys. **21**, 233 (1963),
http://physics.princeton.edu/~mcdonald/examples/mechanics/littlewood_ap_21_233_63.pdf

[30] M. Kruskal, *Asymptotic Theory of Hamiltonian and other Systems with all Solutions Nearly Periodic*, J. Math. Phys. **3**, 806 (1962),
http://physics.princeton.edu/~mcdonald/examples/mechanics/kruskal_jmp_3_806_62.pdf

[31] H.R. Lewis, Jr, *Classical and Quantum Systems with Time-Dependent Harmonic-Oscillator-Type Hamiltonians*, Phys. Rev. Lett. **18**, 510 (1967),
http://physics.princeton.edu/~mcdonald/examples/mechanics/lewis_prl_18_510_67.pdf

[32] H.R. Lewis, Jr, *Class of Exact Invariants for Classical and Quantum Time-Dependent Harmonic Oscillators*, J. Math. Phys. **9**, 1976 (1968),
http://physics.princeton.edu/~mcdonald/examples/mechanics/lewis_jmp_9_1976_68.pdf

[33] W. Sarlet, *Class of Hamiltonians with One Degree-of-Freedom Allowing Application of Kruskal's Asymptotic Theory in Closed Form. I & II*, Ann. Phys. **92** 232,248 (1975),
http://physics.princeton.edu/~mcdonald/examples/mechanics/sarlet_ap_92_232_75.pdf

[34] V.P. Ermakov, U. Izvestia Kiev, Ser. III 9 (1880), English translation: *Second-order differential equations. Conditions of complete integrability*, Appl. Anal. Discrete Math. **2**, 123 (2008), http://physics.princeton.edu/~mcdonald/examples/mechanics/ermakov_aadm_2_123_08.pdf

[35] J. Struckmeier, *Hamiltonian dynamics on the symplectic extended phase space for autonomous and non-autonomous systems*, J. Phys. Math. A **38**, 1257 (2005), http://physics.princeton.edu/~mcdonald/examples/mechanics/struckmeier_jpa_38_1257_05.pdf